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Perturbation Theory of AG Motion with Non-Linear Restoring Forces.*

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In view of the great difficulties involved in finding exact solutions of non-linear differential equations, it is worthwhile to seek approximate solutions as a guide both to qualitative thinking and to numerical integration of the differential equations. Perturbation theory, where the non-linear terms are small, has the advantage of utilizing our knowledge of the solutions of linear differential equations. In the following report a perturbation theory is developed and applied to the one dimensional AG motion.

1. Introduction of New Variables

We wish to consider equations of the form

$$x'' + f(x,\theta) + g(x,\theta) = 0 \tag{1.1}$$

when θ is the independent variable and where the general solution of

$$x'' + f(x,\theta) = 0 \tag{1.2}$$

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is known. This solution will have the form

$$x = x(\theta, a, \varphi) \tag{1.3}$$

where a and φ are arbitrary constants.

Our method is an extension of a method due to E. W. Brown.[1] We write the solution of (1.1) in the form (1.3) with a and φ now functions of θ. Thus (1.3) may be called a transformation equation from the set of canonical variables x and x' = y to the set of variables (not necessarily canonical) a and φ. The second transformation equation is taken from

$$y = x' = \frac{dx}{d\theta} = \frac{\partial x}{\partial \theta} + \frac{\partial x}{\partial a}\frac{da}{d\theta} \quad \frac{\partial x}{\partial \varphi}\frac{d\varphi}{d\theta} \tag{1.4}$$

We choose a special form for (1.4) by assuming

$$\frac{\partial x}{\partial a}\frac{da}{d\theta} + \frac{\partial x}{\partial \varphi}\frac{d\varphi}{d\theta} = 0 \tag{1.5}$$

and thus

$$y = \frac{dx}{d\theta} = \frac{\partial x}{\partial \theta} \tag{1.6}$$

is the second transformation equation.

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1. E. W. Brown, Rice Institute pamphlets, Vol. XIX, Jan. 1932.

Because of the special form of (1.1) (f and g independent of y) the Jacobian of the transformation (1.3) and (1.6) is constant. Thus

$$J = \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial a} - \frac{\partial x}{\partial a}\frac{\partial y}{\partial \varphi} \tag{1.7}$$

$$\frac{dJ}{d\theta} = \frac{d}{d\theta}\left(\frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial a} - \frac{\partial x}{\partial a}\frac{\partial y}{\partial \varphi}\right)$$

$$\frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial a} + \frac{\partial x}{\partial \varphi}\frac{\partial y'}{\partial a} - \frac{\partial y}{\partial a}\frac{\partial y}{\partial \varphi} - \frac{\partial x}{\partial a}\frac{\partial y'}{\partial \varphi}$$

and since

$$y' = f(x,\theta) + g(x,\theta) = F(x,\theta)$$

$$\frac{dJ}{d\theta} = \frac{\partial x}{\partial \varphi}\left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial a}\right) - \frac{\partial x}{\partial a}\left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial \varphi}\right) = 0$$

(This proof follows one given in a seminar by J. L. Powell).

For one degree of freedom the Jacobian (1.7) is directly the Poisson Bracket of x and y with respect to a and φ. Thus if we choose the scale of a and φ so that J = 1, the new variables a and φ are canonical variables.

We note further that, since (1.3) is a solution of (1.2), the equation

$$\frac{\partial^2 x}{\partial \theta^2} + f(x,\theta) = 0 \tag{1.8}$$

where x is given by (1.3), is satisfied identically in a and φ.

Differentiating (1.6)

$$x'' = \frac{d^2 x}{d\theta^2} = \frac{\partial^2 x}{\partial \theta^2} + \frac{\partial^2 x}{\partial a \partial \theta}\frac{da}{d\theta} + \frac{\partial^2 x}{\partial \varphi \partial \theta}\frac{d\varphi}{d\theta} \tag{1.9}$$

and substituting in (1.1) and using (1.8),

$$\frac{\partial^2 x}{\partial a \partial \varphi}\frac{da}{d\theta} + \frac{\partial^2 x}{\partial \varphi \partial \theta}\frac{d\varphi}{d\theta} + g(x,\theta) = 0 \tag{1.10}$$

(1.10) and (1.5) may be solved together for $\frac{d\varphi}{d\theta}$ and $\frac{da}{d\theta}$. Thus

$$\left[-\frac{\partial x}{\partial \varphi}\frac{\partial^2 x}{\partial a \partial \varphi} + \frac{\partial^2 x}{\partial \varphi \partial \theta}\frac{\partial x}{\partial a}\right]\frac{d\varphi}{d\theta} + \frac{\partial x}{\partial a}\, g(x,\theta) = 0$$

(This proof follows one given in a seminar by L. L. Powell)

or, using (1.6)

$$-\left[\frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial a} - \frac{\partial x}{\partial a}\frac{\partial y}{\partial \varphi}\right]\frac{d\varphi}{d\theta} = -\frac{\partial x}{\partial a}\, g(x,\theta)$$

The bracket is just the Jacobian (1.7) which we choose equal to unity.

Then

$$\frac{d\varphi}{d\theta} = \frac{\partial x}{\partial a} g(x,\theta) \tag{1.11}$$

and similarly,

$$\frac{da}{d\theta} = -\frac{\partial x}{\partial \varphi} g(x,\theta) \tag{1.12}$$

The assumption (1.5) thus gives first order equations in a and φ.

If we define $h(x,\theta)$ and $j(x,\theta)$ by

$$\begin{cases} \dfrac{\partial h}{\partial x} = g(x,\theta) \\[2ex] \dfrac{\partial j}{\partial x} = f(x,\theta) \end{cases} \tag{1.13}$$

then the original differential equation (1.1) is derived from the Hamiltonian

$$H(x,y,\theta) = \tfrac{1}{2} y^2 + j(x,\theta) + h(x,\theta) \tag{1.14}$$

and the new differential equations (1.11) and (1.12) from the Hamiltonian

$$\bar{H}(a,\varphi,\theta) = h[x(\theta,a,\varphi),\theta] \tag{1.15}$$

and therefore the canonical transformation from (x,y) to (φ,a) is generated by the $S = S(x,a,\theta)$ which is a solution of the Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 + f(x,\theta) + \frac{\partial S}{\partial \theta} = 0 \tag{1.16}$$

with

$$\begin{cases} y = \dfrac{\partial S}{\partial x} \\ \varphi = \dfrac{\partial S}{\partial a} \end{cases} \tag{1.17}$$

2. <u>Perturbation Theory</u>

We suppose that $g(x,\theta)$ is small compared to $f(x,\theta)$ for all x and θ, and introduce a coefficient λ by writing λg for g . We seek a solution of the form

$$\begin{cases} \varphi = \sum_n \lambda^n \varphi_n(\theta) \\ a = \sum_n \lambda^n a_n(\theta) \end{cases} \tag{2.1}$$

Substituting in (1.1) gives

$$\begin{aligned} \sum_n \lambda^n \frac{d\varphi_n}{d\theta} &= \lambda \frac{\partial x}{\partial a}\left(\theta, \sum_n \lambda^n \varphi_n, \sum_n \lambda^n a_n\right) g\left[x\left(\sum_n \lambda^n \varphi_n, \sum_n \lambda^n a_n, \theta\right), \theta\right] \\ &= \lambda\left\{\left(\frac{\partial x}{\partial a}\right)_0 + \left(\frac{\partial^2 x}{\partial \varphi \partial a}\right)_0 \lambda\varphi_1 + \left(\frac{\partial^2 x}{\partial a^2}\right)_0 \lambda a_1 + \dots\right\} \times \\ &\quad \times \left\{ g_0(\theta) + \left(\frac{\partial g}{\partial x}\right)_0 \left\{\left(\frac{\partial x}{\partial a}\right)_0 \lambda a_1 + \left(\frac{\partial x}{\partial \varphi}\right)_0 \lambda\varphi_1\right\} + \dots \right\} \end{aligned} \tag{2.2}$$

where the subscript 0 means the expression is evaluated at $a = a_0$ and $\varphi = \varphi_0$. We equate like powers of λ to find

$$\begin{cases} \dfrac{d\varphi_0}{d\theta} = 0 \\[2ex] \dfrac{d\varphi_1}{d\theta} = \left(\dfrac{\partial x}{\partial a}\right)_0 g_0(\theta) \\[2ex] \dfrac{d\varphi_2}{d\theta} = g_0(\theta)\left[\left(\dfrac{\partial^2 x}{\partial\varphi\partial a}\right)_0 \varphi_1 + \left(\dfrac{\partial^2 x}{\partial a^2}\right)_0 a_1\right] + \\ \qquad + \left(\dfrac{\partial x}{\partial a}\right)_0 \left(\dfrac{\partial g}{\partial x}\right)_0 \left\{\left(\dfrac{\partial x}{\partial a}\right)_0 a_1 + \left(\dfrac{\partial x}{\partial \varphi}\right)_0 \varphi_1\right\} \end{cases} \tag{2.3}$$

Similarly, from (1.12) we find

$$\begin{cases} \dfrac{da_0}{d\theta} = 0 \\[2ex] \dfrac{da_1}{d\theta} = \left(\dfrac{\partial x}{\partial \varphi}\right)_0 g_0(\theta) \\[2ex] \dfrac{da_2}{d\theta} = g_0(\theta)\left[\left(\dfrac{\partial^2 x}{\partial\varphi^2}\right)_0 \varphi_1 + \left(\dfrac{\partial^2 x}{\partial a\partial \varphi}\right)_0 a_1\right] + \\ \qquad + \left(\dfrac{\partial x}{\partial \varphi}\right)_0 \left(\dfrac{\partial g}{\partial x}\right)_0 \left\{\left(\dfrac{\partial x}{\partial a}\right)_0 a_1 + \left(\dfrac{\partial x}{\partial \varphi}\right)_0 \varphi_1\right\} \end{cases} \tag{2.4}$$

a_o and φ_o are of course the arbitrary constants of the solution of (1.2) and are determined by the initial conditions. Then $a_i(0) = \varphi_i(0) = 0$ for $i > 0$, so that

$$\begin{cases} \varphi_1(\theta) = \displaystyle\int_0^\theta \frac{\partial x}{\partial a}(\theta, a_0, \varphi_0)\, g(x(\theta, a_0, \varphi_0), \theta)\, d\theta \\ a_1(\theta) = \displaystyle\int_0^\theta \frac{\partial x}{\partial \varphi}(\theta, a_0, \varphi_0)\, g(x(\theta, a_0, \varphi_0), \theta)\, d\theta \end{cases} \tag{2.5}$$

and so on.

3. <u>One Dimensional AG Motion without Inhomogeneities</u>.

We take the equation of motion

$$x'' + n(\theta)\, x + \frac{e(\theta)}{3} x^3 = 0 \tag{3.1}$$

where

$$n(\theta) = \begin{cases} n_1 : & 0 < \theta < \frac{\tau}{2} \\ -n_2 : & \frac{\tau}{2} < \theta < \tau \\ n(\theta + \tau) & \end{cases} \tag{3.2}$$

and

$$e(\theta) = \begin{cases} e_1 : & 0 < \theta < \frac{\tau}{2} \\ -e_2 : & \frac{\tau}{2} < \theta < \tau \\ e(\theta+\tau) & \end{cases} \tag{3.3}$$

In order to apply the perturbation theory above, we take

$$\begin{cases} f(x,\theta) = n(\theta)\, x \\ g(x,\theta) = \frac{1}{3}\, e(\theta)\, x^3 \end{cases} \tag{3.4}$$

We write the solutions of $x'' + f = 0$ separately in regions of $n > 0$ and $n < 0$ and thus define arbitrary constants for each half-sector. Thus, calling $\sqrt{n_1} = \rho$, $\sqrt{n_2} = \delta$

$$\begin{cases} x = \sqrt{\frac{2a^k}{\rho}} \cos(\rho\theta + \varphi^k) \\ x' = -\sqrt{2\rho a^k} \sin(\rho\theta + \varphi^k) \end{cases} \tag{3.5}$$

for $k\tau < \theta < (k+\frac{1}{2})\tau$ and

$$\begin{cases} x = \sqrt{\frac{2b^k}{\delta}} \cosh(\delta\theta + \xi^k) \\ x' = \sqrt{2b^k\delta} \sinh(\delta\theta + \xi^k) \end{cases} \tag{3.6}$$

for $(k+\frac{1}{2})\tau < \theta < (k+1)\tau$

The a^k, b^k, φ^k and ξ^k are not independent, since the solutions join smoothly at the boundaries. Thus, with $\theta = (k+\frac{1}{2})T = \zeta$

$$\begin{cases} b^k = a^k \left\{ \frac{\delta}{\rho} \cos^2(\rho\zeta + \varphi^k) - \frac{\rho}{\delta} \sin^2(\rho\zeta + \varphi^k) \right\} \\ \tan(\rho\zeta + \varphi^k) = -\frac{\delta}{\rho} \tanh(\delta\zeta + \xi^k) \end{cases} \tag{3.7}$$

and at $\theta = (k+1)T = \eta$

$$a^{k+1} = b^k \left\{ \frac{\rho}{\delta} \cosh^2(\delta\eta + \xi^k) + \frac{\delta}{\rho} \sinh^2(\delta\eta + \xi^k) \right\}$$

$$\tan(\rho\eta + \varphi^{k+1}) = -\frac{\delta}{\rho} \tanh(\delta\eta + \xi^k) \tag{3.8}$$

The a^k, b^k, φ^k and ξ^k have been chosen so that they are canonical variables within each half-sector. We can then apply the method of paragraph 2 above to each half sector and join the results using (3.7) and (3.8), which are still valid with the a^k, b^k, φ^k and ξ^k evaluated at $\theta = \zeta$ in (3.7) and $\theta = \eta$ in (3.8).

Thus in the kth positive cell, using (2.5),

$$\begin{cases} \varphi_1^k((k+\frac{1}{2})\tau) = \frac{e_1}{3}\int_{k\tau}^{(k+\frac{1}{2})\tau}\left(\frac{\partial x}{\partial a^k}\right)_0 x_0^3(\theta)\,d\theta \\ a_1^k((k+\frac{1}{2})\tau) = \frac{e_1}{3}\int_{k\tau}^{(k+\frac{1}{2})\tau}\left(\frac{\partial x}{\partial \varphi^k}\right)_0 x_0^3(\theta)\,d\theta \end{cases} \tag{3.9}$$

$$\begin{aligned} \varphi_1^k((k+\tfrac{1}{2})\tau) &= \frac{e_1}{3}\cdot\frac{2a_0^k}{\rho^2}\int_{k\tau}^{(k+\frac{1}{2})\tau}\cos^4(\rho\theta+\varphi_0^k)\,d\theta \\ &= \frac{e_1}{3}\,\frac{2a_0^k}{\rho^2}\left\{\frac{3\tau}{16}+\frac{1}{4\rho}\left[\sin 2(\rho\tau+\varphi_0^k)-\sin 2(\rho k\tau+\varphi_0^k)\right]+\right. \\ &\qquad \left.+\frac{1}{32\rho}\left[\sin 4(\rho\tau+\varphi_0^k)-\sin 4(\rho k\tau+\varphi_0^k)\right]\right\} \end{aligned} \tag{3.10}$$

$$a_1^k((k+\tfrac{1}{2})\tau) = \frac{e_1}{3}\left(\frac{2a_0^k}{\rho}\right)^2\frac{1}{4\rho}\left[\cos^4(\rho\tau+\varphi_0^k)-\cos^4(\rho k\tau+\varphi_0^k)\right] \tag{3.11}$$

These are the change in φ^k and a^k in the kth positive cell due to the non-linearity.

In the kth negative cell,

$$\begin{cases} \xi_1^k((k+1)\tau) = \bar{\xi}_0^k((k+\frac{1}{2})\tau)-\xi_0^k((k+\frac{1}{2})\tau)-\frac{e_2}{3}\int_{(k+\frac{1}{2})\tau}^{(k+1)\tau}\left(\frac{\partial x}{\partial b^k}\right)_0\cdot x_0^3(\theta)\,d\theta \\ b_1^k((k+1)\tau) = \bar{b}_0^k((k+\frac{1}{2})\tau)-b_0^k((k+\frac{1}{2})\tau)-\frac{e_2}{3}\int_{(k+\frac{1}{2})\tau}^{(k+1)\tau}\left(\frac{\partial x}{\partial \xi^k}\right)_0 x_0^3(\theta)\,d\theta \end{cases} \tag{3.12}$$

where the values of $\xi_0^k((k+\frac{1}{2})\tau)$ and $b_0^k((k+\frac{1}{2})\tau)$ in the integrals contain a contribution from (3.10) and (3.11).

$$\xi_1^k((k+1)\tau) = \bar{\xi}_0((k+\tfrac{1}{2})\tau) - \xi_0((k+\tfrac{1}{2})\tau - \frac{e_2}{3}\cdot\frac{2\bar{b}_0^k}{\delta^2}\Big\{\frac{3}{16}\tau +$$

$$+\frac{1}{4\delta}\left[\sinh 2(\delta\eta + \bar{\xi}_0^k) - \sinh 2(\delta\rho + \bar{\xi}_0^k)\right] +$$

$$+\frac{1}{32\delta}\left[\sinh 4(\delta\eta + \bar{\xi}_0^k) - \sinh 4(\delta\rho + \bar{\xi}_0^k)\right]\Big\}$$

$$b_1^k((k+1)\tau) = \bar{b}_1^k((k+\tfrac{1}{2})\tau) - b_0^k((k+\tfrac{1}{2})\tau)$$

$$-\frac{e_2}{12\delta}\left(\frac{2\bar{b}_0^k}{\delta}\right)^2\left[\cosh^4(\delta\eta + \bar{\xi}_0^k) - \cosh^4(\delta\rho + \bar{\xi}_0^k)\right] \tag{3.13}$$

(3.14)

where $\bar{b}_0^k$ and $\bar{\xi}_0^k$ have added the contributions from the positive cell.

We can use (3.7) and (3.8) to calculate the change in a^{k+1} and φ^{k+1} due to the non-linear term. From (3.7), ξ^k and b^k are functions of a^k and φ^k. Then

$$\begin{cases} \bar{\xi}_0^k = \xi_0^k + \left(\dfrac{\partial\xi^k}{\partial\varphi^k}\right)_0 \varphi_1^k + \left(\dfrac{\partial\xi^k}{\partial a^k}\right)_0 a_1^k \\ \bar{b}_0^k = b_0^k + \left(\dfrac{\partial b^k}{\partial\varphi^k}\right)_0 \varphi_1^k + \left(\dfrac{\partial b^k}{\partial a^k}\right)_0 a_1^k \end{cases} \tag{3.15}$$

But from the second of (3.7)

$$\xi_k^k = -\delta\zeta - \tanh^{-1}\left(\frac{f}{\delta}\tan(\rho\zeta+\varphi^k)\right)$$

is independent of a^k, so that

$$\bar{\xi}_k = \xi_0^k - \varphi_1^k\left[\frac{f}{\delta}\,\frac{\sec^2(\rho\zeta+\varphi_0^k)}{\operatorname{sech}^2(\delta\zeta+\xi_0^k)}\right] \tag{3.16}$$

$$\bar{b}_0^k = b_0^k - \left\{\left(\frac{\delta}{\rho}+\frac{f}{\delta}\right)a_0^k\sin 2(\rho\zeta+\varphi_0^k)\right\}\varphi_1^k +$$

$$+\left\{\frac{\delta}{\rho}\cos^2(\rho\zeta+\varphi_0^k) - \frac{f}{\delta}\sin^2(\rho\zeta+\varphi_0^k)\right\}a_1^k \tag{3.17}$$

Similarly, from (3.8)

$$\begin{cases} \Delta\varphi^{k+1} = \dfrac{\partial\varphi^{k+1}}{\partial\xi^k}\xi_1^k + \dfrac{\partial\varphi^{k+1}}{\partial b^k}b_1^k \\ \Delta a^{k+1} = \dfrac{\partial a^{k+1}}{\partial\xi^k}\xi_1^k + \dfrac{\partial a^{k+1}}{\partial b^k}b_1^k \end{cases} \tag{3.18}$$

$$\Delta\varphi^{k+1} = -\frac{\delta}{\rho}\,\frac{\operatorname{sech}^2(\delta\eta+\bar{\xi}_0^k)}{\sec^2(\rho\eta+\varphi_0^{k+1})}\,\bar{\xi}_1^k$$

$$\Delta a^{k+1} = \left\{\frac{f}{\delta}\cosh^2(\delta\eta+\bar{\xi}_0^k) + \frac{\delta}{\rho}\sinh^2(\delta\eta+\bar{\xi}_0^k)\right\}b_1^k + \tag{3.19}$$

$$+\bar{b}_1^k\left\{\left(\frac{f}{\delta}+\frac{\delta}{\rho}\right)\sinh 2(\delta\eta+\bar{\xi}_0^k)\right\}\xi_1^k$$

These can be written more neatly by noting from (3.5), (3.6), (3.7) and (3.8) that

$$\frac{b_k}{a_k} = \frac{\delta}{f}\,\frac{\operatorname{sech}^2(\delta\xi + \xi^k)}{\sec^2(\rho\xi + \xi^k)} = \frac{\delta}{\rho}\cos^2(\rho\xi + \varphi^k) - f\sin^2(\rho\xi + \varphi^k)$$

$$= \mathfrak{z}_1(k) \tag{3.20}$$

$$\frac{a^{k+1}}{b^k} = \left[\frac{\delta}{\rho}\cos^2(\rho\eta + \varphi^{k+1}) - f\sin^2(\rho\eta + \varphi^{k+1})\right]^{-1}$$

$$= \frac{\rho}{\delta}\,\frac{\sec^2(\rho\eta + \varphi^{k+1})}{\operatorname{sech}^2(\delta\eta + \xi^k)}$$

$$= \frac{1}{\mathfrak{z}_2(k+1)} \tag{3.21}$$

$$= \left[f\cosh^2(\delta\eta + \xi^k) + \frac{\delta}{\rho}\sinh^2(\delta\eta + \xi^k)\right]$$

Then

$$\Delta\varphi^{k+1} = -\mathfrak{z}_2(k+1)\,\xi_1^k$$

$$\Delta a^{k+1} = \frac{1}{\mathfrak{z}_2(k+1)}\,b_1^k + \bar{b}_0^k\left\{\left(\frac{f}{\delta} + \frac{\delta}{\rho}\right)\sinh 2(\delta\eta + \bar{\xi}_0^k)\right\} \times \tag{3.22}$$

$$\times\ \xi_1^k$$

$$\Delta\varphi^{k+1} = -z_2(k+1)\left\{-\varphi_1^{k}\frac{1}{z_1(k)} - \frac{e_2}{3}\frac{2\bar{b}_0^{k}}{\delta^2}\left[\frac{3}{16}\tau + \frac{1}{4\delta}\left[\sinh 2(\delta\eta + \bar{\xi}_0^{k}) - \right.\right.\right.$$

$$\left.\left.\left. - \sinh 2(\delta\zeta + \bar{\xi}_0^{k})\right] + \frac{1}{32\delta}\left[\sinh 4(\delta\eta + \bar{\xi}_0^{k}) - \sinh 4(\delta\zeta + \bar{\xi}_0^{k})\right]\right]\right\}$$

$$= \frac{2a_0^{k}}{\rho^2} z_2(k+1)\left\{\frac{e_1}{3}\frac{1}{z_1(k)}\left[\frac{3}{16}\tau + \frac{1}{4\rho}\left[\sin 2(\rho\zeta + \varphi_0^{k}) - \sin 2(\rho k\tau + \varphi_0^{k})\right] + \right.\right.$$

$$\left. + \frac{1}{32\rho}\left[\sin 4(\rho\zeta + \varphi_0^{k}) - \sin 4(\rho k\tau + \varphi_0^{k})\right]\right] +$$

$$+ \frac{e_2}{3}\frac{\rho^2}{\delta^2} z_1(k)\left[\frac{3}{16}\tau + \frac{1}{4\delta}\left[\sinh 2(\delta\eta + \xi_0^{k}) - \sinh 2(\delta\zeta + \xi_0^{k})\right] + \right.$$

$$\left.\left. + \frac{1}{32\delta}\left[\sinh 4(\delta\eta + \xi_0^{k}) - \sinh 4(\delta\zeta + \xi_0^{k})\right]\right]\right\} \tag{3.23}$$

where we have dropped the hi her order co:rections due to $\bar{\xi}_0^{k}$ and $\bar{b}_0^{k}$.

This complicated expression is the change of phase in traversing the kth sector due to the non-linearity. It

constant terms increase. Thus we can neglect the oscillating terms. The terms in $\Delta\varphi^{k+1}$ with hyperbolic functions are finite also, because the motion is stable. Then the only terms which contribute to the sum over many sectors are the constant terms in z_1 and z_2 and the terms proportional to τ in $\Delta\varphi^{k+1}$

Then the phase change due to the non-linear term for m sectors is

$$\Delta\varphi(m\tau) = m\Delta\sigma \cong \frac{3}{16}\left(\sqrt{n_1}\,x_0^2 + \frac{x_0'^2}{\sqrt{n_1}}\right)\left[\frac{e_1}{3n_1} + \frac{e_2}{3n_2}\,\frac{1}{4}\left(\sqrt{\frac{n_2}{n_1}} + \sqrt{\frac{n_1}{n_2}}\right)^2\right] m\tau \tag{3.26}$$

(The rigorous proof of (3.26) follows the sketch above and is laborious).

The same argument applied to Δa^{k+1} shows that the amplitude of the motion does not increase with θ, since there are only sinusoidal terms in the sum. Thus the "secular term" difficulties of perturbation theory do not arise in this method and the perturbation theory can be applied over long intervals. This is closely connected with the fact that a and φ are canonical coordinates and therefore satisfy Liouville's Theorem.

is thus a correction to the σ of linear theory.

We can reduce (3.23) to a simpler approximate expression by adding up the phase changes over many sectors. To do this, note that

$$\frac{a^{k+1}}{a^k} = \frac{z_1(k)}{z_2(k+1)}$$

and by iterating this

$$a_0^k = a_0^0 \frac{z_1(k-1)\, z_1(k-2) \cdots z_1(0)}{z_2(k)\, z_2(k-1) \cdots z_2(1)} \tag{3.24}$$

so that we can substitute for a_0^k in (3.24). From (3.5) we see that

$$a_0^0 = \frac{1}{2}\rho x_0^2 + \frac{1}{2\rho} x_0'^2 \tag{3.25}$$

so that a_0^0 is given in terms of the initial conditions. Now, calling $\lambda_k = \rho\xi + \varphi^k$, $\nu^k = \rho\eta + \varphi^{k+1}$,

$$z_1(k) = \frac{1}{2}\left(\frac{\delta}{\rho} + \frac{\rho}{\delta}\right) + \frac{1}{2}\left(\frac{\delta}{\rho} - \frac{\rho}{\delta}\right)\cos 2\lambda_k$$

$$z_2(k+1) = \frac{1}{2}\left(\frac{\delta}{\rho} + \frac{\rho}{\delta}\right) + \frac{1}{2}\left(\frac{\delta}{\rho} - \frac{\rho}{\delta}\right)\cos 2\nu_k$$

When the linear motion is stable, the phase change per focusing sector is always less than π, so that $\lambda_k < \pi$ and $\nu_k < \pi$. Then the sinusoidal terms in z_1 and z_2 will give an oscillating contribution over many sectors, while the