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EXPANSIONS WHICH EXPRESS THE MAGNETIC FIELD ON EITHER SIDE OF A PLANE SURFACE IN TERMS OF THE MAGNETIC FIELD ON THE SURFACE, AND THEIR APPLICATION TO THE MARK V, FFAG ACCELERATOR.

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PART I: Notations, definitions and postulates.

Given a region R of space and a plane surface S which separates R into two parts.

$\vec{i}$ and $\vec{j}$ are orthogamal unit vectors parallel to S

$\vec{k}$ is a unit vector normal to S

$\vec{i}\,x + \vec{j}y + \vec{k}z$ is the radius vector of a point in R

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\nabla_t = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$$

$$\Delta = \nabla\cdot\nabla$$

$\vec{H}(x,y,z)$ is a vector function of position (interpreted here as magnet field strength), which satisfies the relations:

$$\nabla \times \vec{H} = o \qquad \nabla\cdot\vec{H} = o \qquad \text{(in R)} \tag{1.1}$$

It follows that there exists a scalar function $\Phi(x,y,z)$ such that

$$H = -\nabla\Phi \text{ and } \Delta\Phi = o$$

$\vec{H} = \vec{H}_t + \vec{k}H_z$ where $\vec{H}_t$ is the component of H parallel to S, and $\vec{k}H_z$ is the component perpendicular to S.

Furthermore, since $\nabla \cdot \vec{H} = o$, there exists a vector function $\vec{A}(x,y,z)$

such that $\vec{H} = \nabla \times \vec{A}$ and $\nabla \cdot \vec{A} = o$ (1.3)

so that $\Delta \vec{A} = o$

The subscript o on any function indicates the value of that function on S,

thus $H_z(x,y,o) = H_{z,o}(x,y) = H_{z,o}$

$A_x(x,y,o) = A_{x,o}(x,y) = A_{x,o}$

$\vec{H}(x,y,o) = \vec{H}_o(x,y) = \vec{H}_o$

$$H_{x,o} = -\frac{\partial \Phi_o}{\partial x}$$

$$H_{y,o} = -\frac{\partial \Phi_o}{\partial y}$$

PART II: The expansions may be derived:

$$\Phi = \Phi_o + \frac{z^2}{2!}\nabla_t \cdot \vec{H}_o - \frac{z^4}{4!}\Delta_t \nabla \cdot \vec{H}_o + \frac{z^6}{6!}\Delta_t\Delta_t\nabla_t \cdot \vec{H}_o$$

$$- z H_{z,o} + \frac{z^3}{3!}\Delta_t H_{z,o} - \frac{z^5}{5!}\Delta_t\Delta_t H_{z,o} \tag{1.4}$$

$$H_x = H_{x,o} - \frac{z^2}{2!}\Delta_t H_{x,o} + \frac{z^4}{4!}\Delta_t\Delta_t H_{x,o} - \frac{z^6}{6!}\Delta_t\Delta_t\Delta_t H_{xo} \tag{1.5}$$

$$+ z\frac{\partial H_{z,o}}{\partial x} - \frac{z^3}{3!}\Delta_t\frac{\partial H_{z,o}}{\partial x} + \frac{z^5}{5!}\Delta_t\Delta_t\frac{\partial H_{z,o}}{\partial x}$$

$$H_y = H_{y,o} - \frac{z^2}{2!}\Delta_t H_{y,o} + \frac{z^4}{4!}\Delta_t\Delta_t H_{y,o} - \frac{z^6}{6!}\Delta_t\Delta_t\Delta_t H_{yo} \tag{1.6}$$

$$+ z\frac{\partial H_{z,o}}{\partial y} - \frac{z^3}{3!}\Delta_t\frac{\partial H_{z,o}}{\partial y} + \frac{z^5}{5!}\Delta_t\Delta_t\frac{\partial H_{z,o}}{\partial y}$$

$$H_z = H_{z,o} - \frac{z^2}{2!}\Delta_t H_{z,o} + \frac{z^4}{4!}\Delta_t\Delta_t H_{z,o} - \frac{z^6}{6!}\Delta_t\Delta_t\Delta_t H_{z,o} \tag{1.7}$$

$$- z\nabla_t \cdot \vec{H}_o + \frac{z^3}{3!}\Delta_t\nabla_t\vec{H}_o - \frac{z^5}{5!}\Delta_t\Delta_t\nabla_t \cdot \vec{H}_o$$

$$A_x = A_{x,o} - \frac{z^2}{2!}\Delta_t A_{x,o} + \frac{z^4}{4!}\Delta_t\Delta_t A_{x,o} - \frac{z^6}{6!}\Delta_t\Delta_t\Delta_t A_{x,o} + - - \tag{1.8}$$

$$+ z\left[\frac{\partial A_{z,o}}{\partial x} + H_{y,o}\right] - \frac{z^3}{3!}\Delta_t\left[\frac{\partial A_{z,o}}{\partial x} + H_{y,o}\right] + \frac{z^5}{5!}\Delta_t\Delta_t\left[\frac{\partial A_{z,o}}{\partial x} + H_{y,o}\right] - - -$$

$$A_y = A_{y,o} - \frac{z^2}{2!}\Delta_t A_{y,o} + \frac{z^4}{4!}\Delta_t\Delta_t A_{y,o} - \frac{z^6}{6!}\Delta_t\Delta_t\Delta_t A_{y,o} + - - \tag{1.9}$$

$$+ z\left[\frac{\partial A_{z,o}}{\partial y} - H_{x,o}\right] - \frac{z^3}{3!}\Delta_t\left[\frac{\partial A_{z,o}}{\partial y} - H_{x,o}\right] + - -$$

$$A_z = A_{z,o} - \frac{z^2}{2!}\Delta_t A_{z,o} + \frac{z^4}{4!}\Delta_t\Delta_t A_{z,o} - - - - \tag{1.10}$$

$$- z\nabla_t \cdot \vec{A}_o + \frac{z^3}{3!}\Delta_t\nabla_t \cdot \vec{A}_o - - -$$

If one introduces the operators:

$$\text{Cos } A = 1 - \frac{A^2}{2!} + \frac{A^4}{4!} - - -$$

$$\text{Sin } A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - - - \tag{1.11}$$

the above expansions can be written thus:

$$H_x = \left[\cos\left(z\sqrt{\Delta_t}\right)\right] H_{x,o} + \left[\frac{\sin\left(z\sqrt{\Delta_t}\right)}{\sqrt{\Delta_t}}\right]\frac{\partial H_{z,o}}{\partial x} \tag{1.12}$$

$$H_y = \left[\cos\left(z\sqrt{\Delta_t}\right)\right] H_{y,o} + \frac{\sin\left(z\sqrt{\Delta_t}\right)}{\sqrt{\Delta_t}}\frac{\partial H_{z,o}}{\partial y} \tag{1.13}$$

$$H_z = \left[\cos\left(z\sqrt{\Delta_t}\right)\right] H_{z,o} - \frac{\sin\left(z\sqrt{\Delta_t}\right)}{\sqrt{\Delta_t}}\nabla_t \cdot \vec{H}_o \tag{1.14}$$

$$A_x = \left[\cos\left(z\sqrt{\Delta_t}\right)\right] A_{x,o} + \frac{\sin\left(z\sqrt{\Delta_t}\right)}{\sqrt{\Delta_t}}\left[\frac{\partial A_{z,o}}{\partial x} + H_{y,o}\right] \tag{1.15}$$

$$A_y = \left[\cos z\sqrt{\Delta_t}\right] A_{y,o} + \frac{\sin z\sqrt{\Delta_t}}{\sqrt{\Delta_t}}\left[\frac{\partial A_{z,o}}{\partial y} - H_{x,o}\right] \tag{1.16}$$

$$A_z = \left[\cos z\sqrt{\Delta_t}\right] A_{z,o} - \left[\frac{\sin z\sqrt{\Delta_t}}{\sqrt{\Delta_t}}\right]\nabla_t \cdot \vec{A}_o \tag{1.17}$$

Expansions (1.12) and (1.13) for H_x and H_y can be replaced by:

$$\vec{H}_t = \left[\cos z\sqrt{\Delta}\right]\vec{H}_{t,o} + \left[\frac{\sin z\sqrt{\Delta}}{\sqrt{\Delta}}\right]\nabla \vec{H}_{z,o}$$

Likewise, expansions (1.15) and (1.16) for A_x and A_y give:

$$\vec{A}_t = \left[\cos z\sqrt{\Delta}\right]\vec{A}_{t,o} + \left[\frac{\sin z\sqrt{\Delta}}{\sqrt{\Delta}}\right]\left[\nabla A_{z,o} + \vec{k} \times \vec{H}_{t,o}\right]$$

The expansions for $\vec{H}$ and $\vec{A}$ are now in a form invariant under coordinate transformations in the (x,y) plane. One can therefore use any system of curvilinear coordinates in this plane together with the z coordinate perpendicular to this plane.

The methods for deriving these expansions will now be indicated. Let us consider H_z. Expand it in a Taylor series in z:

$$H_z = H_{z,o} + z\left(\frac{\partial H_z}{\partial z}\right)_{z=o} + \frac{z^2}{2!}\left(\frac{\partial H_z}{\partial z^2}\right)_{z=o} + - - - \tag{1.18}$$

$$\left(\frac{\partial^{2n} H_z}{\partial z^{2n}}\right)_{z=o} = \left[\frac{\partial^{2n-2}}{\partial z^{2n-2}}\frac{\partial^2 H_z}{\partial z^2}\right]_{z=o} = -\left[\frac{\partial^{2n-2}}{\partial z^{2n-2}}\Delta_t H_z\right]_{z=o} = -\Delta_t\left[\frac{\partial^{2n-2}}{\partial z^{2n-2}} H_z\right]_{z=o} =$$

$$= \left[(-1)^n (\Delta_t)^n H_z\right]_{z=o} = (-1)^n (\Delta_t)^n H_{z,o} \tag{1.19}$$

$$\left(\frac{\partial^{2n-1} H_z}{\partial z^{2n-1}}\right)_{z=o} = \left[\frac{\partial^{2n-3}}{\partial z^{2n-3}}\frac{\partial^2 H_z}{\partial z^2}\right]_{z=o} = -\left[\frac{\partial^{2n-3}}{\partial z^{2n-3}}\Delta_t H_z\right]_{z=o} = -\Delta_t\left[\frac{\partial^{2n-3}}{\partial z^{2n-3}} H_z\right]_{z=o} =$$

$$= (-1)^{n-1}(\Delta_t)^{n-1}\left(\frac{\partial H_z}{\partial z}\right)_{z=o} = (-1)^n(\Delta_t)^{n-1}\nabla_t \cdot \vec{H}_o \tag{1.20}$$

If (1.19) and (1.20) are substituted in (1.18), one obtains the expansion (1.14) for H_z. The expansions for H_x, H_y, A_x, A_y, A_z are obtained in a similar way. In the case of (A_x, A_y, A_z), the following relations are made use of:

$$\begin{aligned} H_{x,o} &= \frac{\partial A_{z,o}}{\partial y} - \left(\frac{\partial A_y}{\partial z}\right)_{z=o} \\ H_{y,o} &= \left(\frac{\partial A_x}{\partial z}\right)_{z=o} - \frac{\partial A_{z,o}}{\partial x} \\ \left(\frac{\partial A_z}{\partial z}\right)_{z=o} &= -\left(\frac{\partial A_{xo}}{\partial x} + \frac{\partial A_{yo}}{\partial y}\right) = -\nabla_t \cdot \vec{A}_o \end{aligned} \tag{1.21}$$

The functions (H_x, H_y, H_z) are determined uniquely by these expansions when the values of $\vec{H}_o$ and H_{oz} are given. The function $\vec{A}_1$ is also uniquely determined except for an additive constant.

However, (A_x, A_y, A_y) are not determined uniquely under these curcumstances. Uniqueness is attained if, in addition to $\vec{H}_o$ and H_{oz}, $(A_{x,o}, A_{yo}, A_{z,o})$ are also given. The only condition on these quantities for a given H_o and H_{oz} is

$$H_{z,o} = \frac{\partial A_{y,o}}{\partial x} - \frac{\partial A_{x,o}}{\partial y} \tag{1.22}$$

In particular, $A_{z,o}$ can be chosen in a quite arbitrary manner. For instance, one could set $A_{x,o} = A_{z,o} = o$. If, in addition $A_{y,o}$ is determined in a manner consistent with (1.22), then the (A_x, A_y, A_y) are determined uniquely.

It should be noted that when the expansions for (A_x, A_y, A_z) are substituted in the equations:

$$H_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$H_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \qquad (1.23)$$

$$H_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_z}{\partial y}$$

and makes use of the relations (1.21), one obtains the expansions (1.5) (1.6) and (1.7) for (H_x, H_y, H_z)

PART III: Special Solutions.

We shall now consider certain magnetic fields which may have applications to accelerators. In each case, the magnetic field will be given in the median plane, and , H_x, H_y, H_z, A_x, A_y, A_z will be determined on both sides of the plane. In the cases considered here, all the expansions can be summed.

(a) $H_{z,o} = e^{\alpha x}$; $H_{x,o} = H_{y,o} = o$

$$\Delta_t H_{z,o} = \alpha^2 e^{\alpha x} \; ; \; (\Delta_t)^n H_{z,o} = \alpha^{2n} H_{z,o}$$

$$\frac{\partial H_{z,o}}{\partial x} = \alpha e^{\alpha x} \; ; \; (\Delta_t)^n \frac{\partial H_{z,o}}{\partial x} = \alpha^{2n+1} e^{\alpha x}$$

The expansion for Φ gives:

$$\Phi = -\frac{e^{\alpha x}}{\alpha} \sin \alpha z$$

so that

$$H_x = e^{\alpha x} \sin \alpha z, \; H_y = o, \; H_z = e^{\alpha x} \cos \alpha z$$

set $A_{x,o} = A_{z,o} = o$, then $e^{\alpha x} = \frac{\partial A_{y,o}}{\partial x}$

Let $A_{y,o} = \frac{e^{\alpha x}}{\alpha}$

Then the expansions give:

$$A_x = o, \quad A_y = \frac{e^{\alpha x}}{\alpha} \cos \alpha z, \quad A_z = o$$

(b) $H_{z,o} = e^{(\alpha+i\beta)x-i\theta} = e^{\alpha+i(\beta x-\theta)}$ where α, β, and θ are real.

$$H_{x,o} = H_{y,o} = o$$

$$H_{z,o} = (\alpha+i\beta)^2 H_{z,o} \quad \left[H_{z,o}\right]^n = (\alpha+i\beta)^{2n} H_{z,o}$$

The expansion for $\vec{\Phi}$ gives

$$= \frac{-e^{\alpha x+i(\beta x-\theta)}}{\alpha+i\beta} \sin (\alpha+i\beta)Z$$

and therefore

$$H_x = e^{\alpha x+i(\beta x-\theta)} \sin (\alpha+i\beta)Z$$

$$H_y = o$$

$$H_z = e^{\alpha x+i(\beta x-\theta)} \cos (\alpha+i\beta)Z$$

Set $A_{x,o} = A_{z,o}$ Then $H_{z,o} = \frac{\partial A_{y,o}}{\partial x}$

Let $A_{y,o} = \frac{e^{\alpha x+i(\beta x-\theta)}}{\alpha+i\beta}$

The expansions give

$$A_x = o, \quad A_y = \frac{e^{\alpha x+i(\beta x-\theta)}}{\alpha+i\beta} \cos(\alpha+i\beta)z, \quad A_z = o$$

The real and pure imaginary parts of this solution will represent possible solutions. These are given in (c) and (d) respectively.

(c) $H_{z,o} = e^{\alpha x} \cos (\beta x-\theta)$ $\quad H_{x,o} = H_{y,o} = o$

$$\Phi = -\frac{e^{\alpha x}}{\alpha^2+\beta^2}\left\{\cos(\beta x-\theta)\left[\alpha \sin\alpha z \cosh\beta z + \beta\cos\alpha z \sinh\beta z\right] + \sin(\beta x-\theta)\left[-\alpha\cos\alpha z \sinh\beta x + \beta\sin\alpha z\cosh\beta z\right]\right\}$$

$$H_x = e^{\alpha x}\left\{\cos(\beta x-\theta)\sin\alpha z\cosh\beta z - \sin(\beta x-\theta)\cos\alpha z\sinh\beta z\right\}$$

$$H_y = o$$

$$H_z = e^{\alpha x}\left\{\cos(\beta x-\theta)\cos\alpha z\cosh\beta z + \sin(\beta x-\theta)\sin\alpha z\sinh\beta z\right\}$$

$$A_x = o$$

$$A_y = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left\{\cos(\beta x-\theta)\left[\alpha\cos\alpha z\cosh\beta z - \beta\sin\alpha z\sinh\beta z\right] + \sin(\beta x-\theta)\left[\alpha\sin\alpha z\sinh\beta z + \beta\cos\alpha z\cosh\beta z\right]\right\}$$

$$A_z = o$$

(d) $H_{z,o} = e^{\alpha x}\sin(\beta x-\theta)$; $H_{x,o} = H_{y,o} = o$

$$= -\frac{e^{\alpha x}}{\alpha^2+\beta^2}\left\{\cos(\beta x-\theta)\left[\alpha\cos\alpha z\sinh\beta z - \beta\sin\alpha z\cosh\beta z\right] + \sin(\beta x-\theta)\left[+\alpha\sin\alpha z\cosh\beta z + \beta\cos\alpha z\sinh\beta z\right]\right\}$$

$$H_x = e^{\alpha x}\left\{\cos(\beta x-\theta)\cos\alpha z\sinh\beta z + \sin(\beta x-\theta)\sin\alpha z\cosh\beta z\right.$$

$$H_y = o$$

$$H_z = e^{\alpha x}\left\{-\cos(\beta x-\theta)\sin\alpha z\,\mathrm{sinch}\,\beta z + \sin(\beta x-\theta)\cos\alpha z\cosh\beta z\right\}$$

$$A_x = o$$

$$A_y = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\Big\{ -\cos(\beta x-\theta)\left[\alpha \sin \alpha z \sinh \beta z + \beta \cos \alpha z \cosh \beta z\right] + \sin(\beta x-\theta)\left[\alpha \cos \alpha z \cosh \beta z - \beta \sin \alpha z \sinh \beta z\right]\Big\}$$

$$A_z = 0$$

(e) $H_{z,o} = e^{\alpha x + i(\beta x - \gamma y)}$; $H_{x,o} = H_{y,o} = o$

From expansion (1.4), one finds:

$$= -\frac{e^{\alpha x + i(\beta x-\gamma y)}}{\alpha^1 + i\beta^1} \sin(\alpha^1 + i\beta^1)z$$

where $\alpha^1 = \sqrt{\dfrac{\sqrt{(\alpha^2-\beta^2-\gamma^2)^2 + 4\alpha^2\beta^2} + (\alpha^2-\beta^2-\gamma^2)}{2}}$

$\beta^1 = \sqrt{\dfrac{\sqrt{(\alpha^2-\beta^2-\gamma^2)^2 + 4\alpha^2\beta^2} - (\alpha^2-\beta^2-\gamma^2)}{2}}$

By differentiating Φ , one finds:

$$H_x = \frac{\alpha+i\beta}{\alpha^1+i\beta^1} e^{\alpha x+i(\beta x-\gamma y)} \sin(\alpha^1+i\beta^1)z$$

$$H_y = -\frac{i\gamma}{\alpha^1+i\beta^1} e^{\alpha x+i(\beta x-\gamma y)} \sin(\alpha^1+i\beta^1)z$$

$$H_z = e^{\alpha x+i(\beta x-\gamma y)} \cos(\alpha^1+i\beta^1)z$$

Let $A_{x,o} = A_{z,o} = o$, then $\dfrac{\partial A_{y,o}}{\partial x} = H_{z,o}$

Choose $A_{y,o} = \dfrac{e^{\alpha x+i(\beta x-\gamma y)}}{\alpha + i\beta}$

Then the expansions (1.8), (1.9) and (1.10) give:

$$A_x = o$$

$$A_y = \frac{e^{\alpha x+i(\beta x-\gamma y)}}{(\alpha+i\beta)} \cos(\alpha^1+i\beta)z$$

$$A_z = \frac{i\gamma e^{\alpha x+i(\beta x-\gamma y)}}{(\alpha+i\beta)(\alpha^1+i\beta^1)} \cos(\alpha^1 + i\beta^1)z$$

This solution is of importance in the Mark V FFAG accelerator, in which case γ is small. In this case it is convenient to expand $\alpha^1+i\beta^1$ in a power series in γ^2, retaining only the first two terms of the expansion. This approximate solution is given in the next section.

(f) $$\alpha^1+i\beta^1 \simeq \left[1 - \frac{\gamma^2}{2(\alpha^2+\beta^2)}\right](\alpha+i\beta)$$

then one finds

$$H_x \simeq \left[1 + \frac{\gamma^2}{2(\alpha^2+\beta^2)}\right] e^{\alpha x+i(\beta x-\gamma y)} \sin\,(\alpha^1+i\beta^1)z$$

$$H_z \simeq e^{\alpha x+i(\beta x-\gamma y)} \cos(\alpha^1+i\beta^1)z$$

$$H_y \simeq -\frac{i\gamma}{\alpha+i\beta} H_x$$

$$A_x = 0$$

$$A_y \simeq \frac{H_z}{\alpha+i\beta}$$

$$A_z \simeq + \frac{i\gamma H_x}{(\alpha+i\beta)^2}$$

The imaginary part of this solution is of particular interest for the Mark V accelerator, and is given in the next section.

(g) $$H_{z,o} = e^{\alpha x} \sin(\beta x-\gamma y) \quad ; \; H_{x,o} = H_{y,o} = 0$$

$$\Phi \simeq -\frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[1 + \frac{\gamma^2}{2(\alpha^2+\beta^2)}\right]\Big\{\cos\,(\beta x-\gamma y)\Big[\alpha\cos\alpha^1 z \sinh \beta^1 z-\beta \sin \alpha^1 z \cosh \beta^1 z\Big]$$

$$+ \sin\,(\beta x-\gamma y)\Big[\alpha \sin \alpha^1 z \cosh \beta^1 z+\beta \cos \alpha^1 z \sinh \beta^1 z\Big]\Big\}$$

$$H_x \simeq \left[1 + \frac{\gamma^2}{2(\alpha^2+\beta^2)}\right] e^{\alpha x} \Big\{ \cos(\beta x-\gamma y) \cos \alpha^1 z \sinh \beta^1 z + \sin(\beta x-\gamma y) \sin \alpha^1 z \cosh \beta^1 z \Big\}$$

$$H_y \simeq -\frac{\gamma e^{\alpha x}}{\alpha^2+\beta^2}\left[1 + \frac{\gamma^2}{\alpha^2+\beta^2}\right]\Big\{\cos(\beta x-\gamma y)\Big[\alpha \sin \alpha^1 z \cosh \beta^1 z + \beta \cos \alpha^1 z \sinh \beta^1 z\Big]$$

$$+ \sin(\beta x-\gamma y)\Big[-\alpha \cos \alpha^1 z \sinh \beta^1 z + \beta \sin \alpha^1 z \cosh \beta^1 z\Big]\Big\}$$

$$H_z \simeq e^{\alpha x}\Big\{-\cos(\beta x-\gamma y) \sin \alpha^1 z \sinh \beta^1 z + \sin(\beta x-\gamma y) \cos \alpha^1 z \cosh \beta^1 z\Big\}$$

$$A_x = 0$$

$$A_y \simeq \frac{e^{\alpha x}}{\alpha^2+\beta^2}\Big\{-\cos(\beta x-\gamma y)\, \alpha \sin \alpha^1 z \sinh \beta^1 z + \beta \cos \alpha^1 z \cosh \beta^1 z$$

$$+ \sin(\beta x-\gamma y)\Big[\alpha \cos \alpha^1 z \operatorname{soch} \beta^1 z - \beta \sin \alpha^1 z \sinh \beta^1 z\Big]\Big\}$$

$$A_z = \frac{e^{\alpha x}}{(\alpha^2+\beta^2)^2}\gamma\left[1 + \frac{\gamma^2}{2(\alpha^2+\beta^2)}\right]\cos(\beta x-\gamma y)\Big[(\alpha^2-\beta^2) \sin \alpha z \cosh \beta z + 2\alpha\beta \cos \alpha z \sinh \beta z\Big]$$

$$+ \sin(\beta x - \gamma y)\Big[-(\alpha^2-\beta^2) \cos \alpha z \sinh \beta z + 2\alpha\beta \sin \alpha z \cosh \beta z\Big]\Big\}$$

It should be noted for small γ, that this represents a small perturbation of the special solution (d). One might suggest that (g) to a certain approximation could also be obtained from (d) simply by setting $\theta = \gamma y$ in that solution. Since the solution g is rather complicated, the solution thus obtained might be quite useful. However, it represents a poor approximation, not being correct in the first power of γ.

PART IV. Applications of the "special solutions" to the Mark V FFAG accelerator.

Two different magnetic fields in the median plane have been proposed for this accelerator, namely

$$H^{I}_{z,o} = H_o\left(\frac{r}{r_o}\right)^k\left[1 + f\sin\left(\frac{r-r_o}{\lambda} - N\theta\right)\right] \tag{4.1}$$

$$H^{II}_{z,o} = H_o\left(\frac{r}{r_o}\right)^k\left[1 + f\sin\left(\frac{1}{w}\ln\frac{r}{r_o} - N\theta\right)\right] \tag{4.2}$$

If one writes $r = r_o + x$, $r_o\theta = y$, the two fields may be written in the form:

$$H^{I}_{z,o} = H_o\left(1 + \frac{x}{r_o}\right)^k\left[1 + f\sin\left(\frac{x}{\lambda} - \frac{N}{r_o}y\right)\right] \tag{4.3}$$

$$H^{II}_{z,o} = H_o\left(1 + \frac{x}{r_o}\right)^k\left[1 + f\sin\left(\frac{1}{w}\ln\left(1 + \frac{x}{r_o}\right) - \frac{N}{r_o}y\right)\right] \tag{4.4}$$

$$\text{Now } \left(1 + \frac{x}{r_o}\right)^k = \left(1 + \frac{xk}{r_o k}\right)^k \simeq e^{\frac{xk}{r_o}} \text{ (for k large and } \frac{xk}{r_o} \text{ small)} \tag{4.5}$$

In this approximation, one obtains:

$$H^{I}_{z,o} \simeq H_o\left\{e^{\frac{k}{r_o}x} + f\, I_m\left[e^{\frac{k}{r_o}x + i\left(\frac{x}{\lambda} - \frac{N}{r_o}y\right)}\right]\right\} \tag{4.6}$$

$$H^{II}_{z,o} \simeq H_o\left\{e^{\frac{k}{r_o}x} + f\, I_m\, e^{x\left[\frac{k}{r_o} - \frac{i}{wr_o}\right] - \frac{N}{r_o}y}\right\} \tag{4.7}$$

Where $I_m(A+iB) = B$ if A and B are real.

The $(r_o + x, y)$ are actually polar coordinates in the median plane. If, however, (x,y) are small compared with r_o, they may as a first approximation be treated as rectangular coordinates in this plane. In this case, the fields $H^{I}_{z,o}$ and $H^{II}_{z,o}$ are linear combinations of the fields

(a) and (e) of part III, and the values of $\vec{H}$ and $\vec{A}$ can at once be written down from the results of part III.

Two approximations are involved here, (1) that contained in equation (4.6) and (2) the treatment of (x,y) as rectangular coordinates. It would seem likely that these approximations are sufficiently valid for most considerations. If, however, more accurate solutions are required, those obtained from a perturbation method would probably be more convenient to use than the corresponding exact solutions, which possibly might be obtainable by the methods used here.

These results may be used to find (1) the magnetic pole shapes required to produce the prescribed fields and (2) the vector potential of this field in the neighborhood of the median plane in order to study questions of orbit stability. In the first case, Φ as obtained from part III, is required. In the second case, it is probably more convenient to use the first few terms of power expansion in z for (A_x, A_y, A_z), which can easily be found from the expansions (1.8), (1.9) and (1.10). One should notice that the vector potential is not uniquely determined, and often it may be more convenient to make a different choice than was done in this report.