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FFAG MARK I b FORMATION INCLUDING EDGES AND STRAIGHT SECTIONS

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I. Introduction.

It is shown that focussing by sector boundaries is important in Mark Ib FFAG designs, especially for smaller numbers of sectors. An approximate method of including the effects of edges is presented. The transfer matrices for edges and for straight sections are found, and the general resulting σ expressions are given.

II. Approximate Treatment of Edges using Lens Analogies.

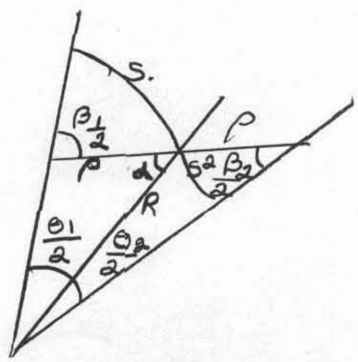
In Mark Ib FFAG the radius of curvature on any equilibrium orbit is the same in radial focussing sectors as in radial defocussing sectors. It is therefore convenient to express all quantities in angle variables as in EDD-HSS-1. Using the notation of Symon[1]:

$$\beta_1 = \frac{S_1}{\rho} \qquad \frac{\beta_1}{\beta_2} = \gamma$$

$$\beta_2 = \frac{S_2}{\rho} \qquad \beta_1 - \beta_2 = \theta_1 + \theta_2 = \frac{2\pi}{N}$$

$$M = \left| \frac{\rho}{H}\frac{dH}{dr} \right| = M_1 = M_2$$

$$\alpha = \frac{(\beta_1 + \beta_2) - (\theta_1 - \theta_2)}{4}$$



1. K. Symon MURA-KRS-6.

The above differs from MURA-KRS-6 in that β_2 is defined as positive here. Following the notation of Hammermesh[2] we consider sectors with subscript 2 vertically focussing for a positive momentum compaction design.

In Mark Ib the edges are vertically focussing and radially defocussing due to scalloping of the equilibrium orbit, with the focal length across one edge pair given by

$$\frac{1}{f} = \frac{2 \tan\alpha}{\rho}$$

The equivalent edge focal length for a complete sector with two edge pairs is one half of the above for very long focal length.

$$\frac{1}{f_e} = \frac{4 \tan\alpha}{\rho}$$

where f_e is the focal length of an equivalent lens which replaces the edges for one complete sector. As shown in the Mark II case,[3] a series of lenses of local length f and spacing T will produce an oscillation of wavelength λ, where

$$\lambda = 2\pi\sqrt{fT}$$

$$\sigma = \frac{2\pi T}{\lambda} = \sqrt{\frac{T}{f}}$$

So, for edges alone, the σ for vertical oscillation is given by

$$\sigma_e = \sqrt{\frac{4T\tan\alpha}{\rho}}$$

For radial oscillations due to edges,

$$\sigma_e = i\sqrt{\frac{4T\tan\alpha}{\rho}}$$

2. Argonne Accelerator Group Progress Report #1.
3. MURA-LWJ-7

To approximate the edge effects, for small angles

$$\tan\alpha \cong \alpha$$

$$C = R/\rho \cong \frac{\beta_1}{\theta_1} \cong \frac{\beta_2}{\theta_2}\,, \quad \theta_1+\theta_2 = \beta_1-\beta_2 = \frac{2\pi}{N}$$

$$\alpha = \frac{\beta_1+\beta_2-(\theta_1-\theta_2)}{4} \cong \frac{C(\theta_1+\theta_2)-\frac{1}{C}(\beta_1-\beta_2)}{4} = \frac{2\pi}{4N}\left(\frac{C^2-1}{C}\right).$$

$$\sigma_e^2 \cong \frac{4T\alpha}{\rho} \cong \frac{4}{\rho}\cdot\frac{2\pi R}{N}\cdot\frac{2\pi}{4N}\left(\frac{C^2-1}{C}\right) = \left(\frac{2\pi}{N}\right)^2(C^2-1)$$

$$\sigma_e \cong \frac{2\pi}{N}\sqrt{C^2-1} \cong \frac{2\pi C}{N}.$$

The edges should improve the stability characteristics of the positive momentum compaction Mark Ib, which has too little vertical and too much radial focussing. The total σ_T, including edges, can be approximately determined from

$$\sigma_T^2 \cong \sigma^2 + \sigma_e^2$$

where σ is found neglecting edges. The above is a good approximation when all σ_s are small, and σ^2 is not equal and opposite in sign to σ_e^2.

For a positive α Mark Ib, neglecting edges a typical set of parameters is

$$C = 5$$

$$\sigma_x = 5\pi/6$$

$$\sigma_z = \pi/6.$$

For 20 sectors and this C,

$$|\sigma_e| \cong \pi/2$$

The resulting σ_T s are much more nearly equal. C could be decreased to 4 and still have radial and vertical stability. This is smaller than would be possible neglecting the edges. Since the edge effects are inversely proportional to the number of sectors, it is most important in small radius, few sector machines.

The above expression for σ_e checks the exact matrix solution of the following section in the small angle approximation.

III. Matrix Formulation of Edge Effects.

For the geometry defined above, we define the following:

$$\psi_1 = \sqrt{m}\,\beta_1 \qquad \phi_1 = \sqrt{M+1}\,\beta_1 \qquad K_1 = \sqrt{M+1}$$

$$\psi_2 = \sqrt{n}\,\beta_2 \qquad \phi_2 = \sqrt{M-1}\,\beta_2 \qquad K_2 = \sqrt{M-1}$$

$$K = \sqrt{M}$$

If the magnet edges are considered as thin lenses, the matrix elements for transformations across the edges are given by:

$$\begin{pmatrix} 1 & 0 \\ 2\tan\alpha & 1 \end{pmatrix} \quad \text{for radial motion,}$$

$$\begin{pmatrix} 1 & 0 \\ -2\tan\alpha & 1 \end{pmatrix} \quad \text{for vertical motion.}$$

In Mark Ib, the edges are radially defocussing and vertically focussing, hence a positive momentum compaction design (where the radial focussing is already stronger and the vertical focussing weaker than considered optimum) is more desirable than a negative momentum compaction. In the following only positive momentum compaction is considered.

The cosine σ expressions are given below.

$$\cos\sigma_x = \cos\phi_1 \cosh\phi_2 - \frac{1}{K_1K_2}\sin\phi_1 \sinh\phi_2 + 2\tan\alpha\left[\frac{\sin\phi_1\cosh\phi_2}{K_1} + \frac{\cos\phi_1 \sinh\phi_2}{K_2}\right] + \frac{2\tan^2\alpha}{K_1K_2}\sin\phi_1\sinh\phi_2$$

$$\cos\sigma_z = \cos\psi_2\cosh\psi_1 - \frac{2\tan\alpha}{K}\left[\sin\psi_2\cosh\psi_1 + \cos\psi_2\sinh\psi_1\right] + \frac{2\tan^2\alpha}{K^2}\sin\psi_2\sinh\psi_1$$

For a particular design, N and γ determine <u>all</u> <u>geometric</u> parameters, using the following:

$$\beta_2 = \frac{2\pi}{N}\left(\frac{1}{\gamma-1}\right) \qquad \text{if } \sin\frac{\theta_1}{2} \cong \frac{\theta_1}{2} \text{ and } \sin\frac{\theta_2}{2} \cong \frac{\theta_2}{2}$$

$$\beta_1 = \frac{2\pi}{N} + \beta_2 \qquad \frac{N}{\pi}\left(\sin\frac{\beta_1}{2} + \sin\frac{\beta_2}{2}\right) = \frac{R}{\rho} = C.$$

The small angle approximation for $\theta/2$ is good to three significant figures for the 20 sector design below, although the values of β_1, β_2 are quite large.

A particular Mark Ib design is therefore completely specified by N, γ, and m.

IV. Straight Sections.

The length of a straight section, ℓ, can be specified in units of the radius of curvature, ρ, by $S = \ell/\rho$. The matrix for transformation across a straight section, horizontally or vertically, is given by:

$$M_\delta = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}$$

This must be inserted between two edge matrices (e.g. radial), each of the form

$$M_\alpha = \begin{pmatrix} 1 & 0 \\ \tan\alpha & 1 \end{pmatrix}$$

in finding the transformation matrix for an entire sector, The expressions for cosine σ ,radial and vertical, are given below, where the abbreviation $t = \tan\alpha$ is used.

$$\cos\sigma_x = \cos\phi_1 \cosh\phi_2 (1+4t\delta+2t^2\delta^2) + \sin\phi_1 \sinh\phi_2 \left[\tfrac{1}{2}\left(\frac{K_2}{K_1}-\frac{K_1}{K_2}\right)(1+t\delta)^2 + \frac{(2t+t^2\delta)^2}{2K_1K_2} - K_1K_2\frac{\delta^2}{2}\right] + \sin\phi_1 \cosh\phi_2 (1+t\delta)\left[\frac{2t+t^2\delta}{K_1} - K_1\delta\right] + \cos\phi_1 \sinh\phi_2 (1+t\delta)\left[\frac{2t+t^2\delta}{K_2} + K_2\delta\right]$$

$$\cos\sigma_z = \cos\psi_2 \cosh\psi_1 (1-4t\delta+2t^2\delta^2) + \sin\psi_2 \sinh\psi_1 \left[\frac{(-2t+t^2\delta)^2}{2K^2} - K^2\frac{\delta^2}{2}\right] + \sin\psi_2 \cosh\psi_1 (1-t\delta)\left[\frac{-2t+t^2\delta}{K} - K\delta\right] + \cos\psi_2 \sinh\psi_1 (1-t\delta)\left[\frac{-2t+t^2\delta}{K} + K\delta\right]$$

When effects of edges may be neglected (as in large N designs where $\tan\alpha \rightarrow 0$) the above expressions may be reduced to

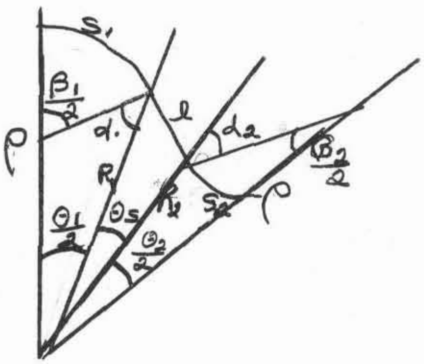
$$\cos\sigma_x = \cos\phi_1 \cosh\phi_2 + \sin\phi_1 \sinh\phi_2 \left[\tfrac{1}{2}\left(\frac{K_2}{K_1} - \frac{K_1}{K_2}\right) - K_1K_2\delta^2\right] - K_1\delta \sin\phi_1 \cosh\phi_2 + K_2\delta \cos\phi_1 \sinh\phi_2$$

$$\cos\sigma_z = \cos\psi_2 \cosh\psi_1 + \sin\psi_2 \sinh\psi_1 \left[-K^2\delta^2\right] + K\delta\left[\cos\psi_2 \sinh\psi_1 - \sin\psi_2 \cosh\psi_1\right].$$

When straight sections are introduced, the geometrical relationships are altered somewhat.

$$\Theta_1 + \Theta_2 + 2\,\Theta_S = \frac{2\pi}{N}$$

$$\Theta \cong \frac{2\,\ell \cos \alpha}{R_1 + R_2}$$



If ℓ is not short compared to S_1 and S_2, then $\alpha_1 \neq \alpha_2$ and the above expression for Θ_S is less valid. However for $\ell << R$, the expression for α given previously is a very good approximation, and the values of β will remain unchanged.

V. Approximate Effects of Straight Sections.

In order to gain a first order "feel" for the effects of straight sections, the procedure used in MURA-LWJ-7 is useful. If S is the path length of particle in a magnet and ℓ is the length of a straight section,

$$S_1 + S_2 + 2\,\ell = \frac{2\pi R}{N} = S$$

The first integral of the motion of the particle in a sector is:

$$\frac{dx}{ds} = -\int_0^S \frac{H'}{p}\,x\,ds \cong -\frac{x}{\rho}\int_0^S H'\,ds$$

where x may be factored out of the integral for small σ.

When straight sections are included,

$$\int_0^S H' ds = H_1' S_1 + H_2' S_2 \quad .$$

If the region with gradient extended across the straight sections,

$$\int_0^S H' ds = H_{1e}' (S_1 + \ell) + H_{2e}' (S_2 + \ell),$$

where the subscript e refers to effective values.

Thus:

$$H_{1e}' = H_1' \frac{S_1}{S_1 + \ell}$$

$$H_{2e}' = H_2' \frac{S_2}{S_2 + \ell}$$

Introduction of straight sections reduces the effective gradients from their local values within sectors.

For the same turning around the machine,

$$\int_0^S H ds = \frac{2\pi R}{N}$$

With straight sections,

$$\int_0^S H ds = H_1 S_1 + H_2 S_2$$

For no straight sections but the same R,

$$\int_0^S H ds = H_{1e} (S_1 + \ell) + H_{2e} (S_2 + \ell)$$

$$H_{1e} = H_1 \frac{S_1}{S_1 + \ell} \qquad H_{2e} = H_2 \frac{S_2}{S_2 + \ell}$$

$$M = \left| \frac{\rho H'}{H^2} \right| , \quad M_{effective} = \left| \frac{\rho H'_e}{H_e^2} \right|$$

$$M_e = M \left(\frac{S_1 + S_2 + 2\ell}{S_1 + S_2} \right)$$

Thus introducting straight sections increases the effective n over n within the sector. For a straight section design, a first approximation calculation would be to find a design neglecting straight sections and then to reduce the n within the sectors by $\frac{S_1 + S_2}{S_1 + S_2 + 2\ell}$, keeping the same γ and N . This new n would then be tried in the expressions of Section III to find the σ s . The circumference factor C is increased by straight sections. Without straight sections,

$$C_o = \frac{S_1 + S_2}{S_1 - S_2}$$

with straight sections,

$$C = \frac{S_1 + \ell + S_2 + \ell}{S_1 - S_2} = C_o \frac{S_1 + S_2 + 2\ell}{S_1 + S_2}$$

The smooth approximation indicates that, for fixed N and σs , the product MC should be invariant. From the above,

$$M_e C_o = MC$$