Fermilab Library
0 1160 0038407 7

MURA-49 (Pt. 1)

ZGS LIBRARY MURA-RW/JLP-4

NON-LINEARITIES IN A.G. SYNCHROTRONS. PART I. A SURVEY OF RESULTS OBTAINED BY USE OF FAST DIGITAL COMPUTER.

J. L. Powell and R. S. Wright

University of Wisconsin and Midwestern Universities Research Association

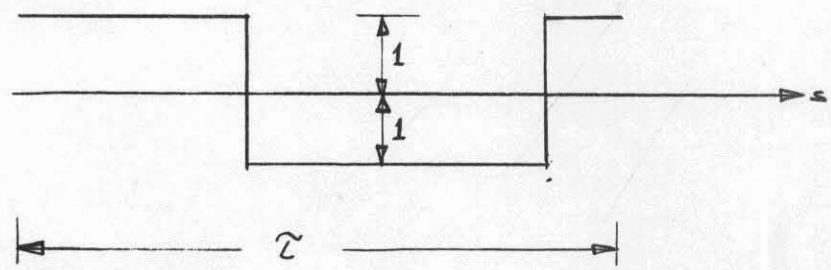
Introduction:

This is a report summarizing the results which have been obtained in the study of the uncoupled differential equations,

$$x'' = \left[mx - \tfrac{1}{3}ex^3\right]U(s)$$

$$y'' = \left[-my - \tfrac{1}{3}ey^3\right]U(s)$$

$U(s) = U(s+\tau) =$



A preliminary report (MAC-JLP-3) describes the methods which were used and it is noted there that the equations may be scaled to the "standard form",

$$\ddot{\xi} = N\left[\xi - \tfrac{1}{3}\xi^3\right]U(at)$$

$$\ddot{\eta} = N\left[-\eta - \tfrac{1}{3}\eta^3\right]U(at)$$

where

$$s = at \qquad x = S\xi \qquad y = S\eta \qquad x' = \frac{S}{a}\dot{\xi} \qquad y' = \frac{S}{a}\dot{\eta}$$

$$T = \frac{1}{10} \qquad a = 200h \qquad S = \sqrt{\frac{m}{e}} \qquad N = (200h)^2 m$$

The results are given in terms of the ξ and η coordinates and

can be applied to problems with machine parameters n, e, and τ by means of the above relations.

The expressions " ξ -motion" and " η -motion" are used to denote those motions resulting when the cubic term in the force respectively opposes and aids the linear term.

The problem has been considered as one in which the differential equation induces in the phase plane a transformation resulting from passage through one focussing-defocussing sector pair. Unless otherwise stated all results apply to the case where the initial point is at the middle of a focussing sector for the particular type of motion being investigated. The reasons for this choice are that there is then the additional symmetry under reflection in the coordinate axis, and that for a purely linear motion the maximum displacements occur here.

The ξ- and η - motions have been investigated as a function of initial displacement for the N values 36, 200, 300, 400, 600, 800, 1000, 1200, and 1440. These motions are represented by a series of phase curves on one phase plot for each N value for each type of motion. It is felt that a reproduction of all of these curves is unnecessary and therefore, only some of the curves for the pure cubic force and for the N values 200, 400, 800, and 1440 are included in this report.

In addition, a series of photographs of curves in the phase plane have been included with the parameters, ranges of axes, and initial conditions listed under each. The general features of the phase curves are clearly shown but the comparison of different motions or different N values is difficult due to the large differences in the scales of the axes. Photos 1 - 15 show the entire phase plane. Photos 16 - 23 are blow-ups of local regions.

Phase Curves:

The curves for N = 200, 400, and 800 are shown in Figs. 1 - 6 and some rather general remarks can be made about them. In the neighborhood of the origin, where the forces are nearly linear, the curves are very elliptical in shape. As one goes to larger displacements the curves depart more and more from their elliptical shape but appear to remain smooth and stable. Finally a stability limit, beyond which the motion is unstable, is reached.

The η -motion is more like a linear motion in that there seem to be no points in the phase plane, except the origin, which are fixed under a transformation through one focussing-defocussing pair. On the other hand, four such fixed points, aside from the origin, appear in the ξ -motions.

Two of the fixed points in the ξ -motion are obvious from the differential equation since, at the points $\xi = \pm\sqrt{3}$, $\dot{\xi} = 0$, the force is always zero. An expansion about the point $\xi = \sqrt{3}$, $\dot{\xi} = 0$ gives the differential equation

$$\ddot{\xi}' = -N\left[2\xi' + \sqrt{3}\,\xi'^2 + \frac{1}{3}\xi'^3\right] U(at),$$

and it is seen that the force about this point now includes a quadratic term and that the linear part of the force corresponds to an N value twice as large as the N value about the origin. Since linear motions are unstable for N values larger than N = 1406, one would expect the motions about this point to be unstable for original N values larger than N = 703 and this seems to be the case. The regions near such a fixed point are shown in Fig. 1 and Fig. 3.

The other two fixed points are not at all obvious from the differential equation. Actually these points are fixed only under a transformation through a focussing-defocussing pair and move as one passes continuously through the pair. It has been possible by approximating to the solution of the d.e. to obtain approximations for the locations of these points. These approximations, which appear as expansions in powers of the parameter $\alpha = \sqrt{\frac{N}{6}}\,\frac{\pi}{4}$, are given below. They give the fixed point which falls in the positive coordinate half of the phase plane and the corresponding fixed point is obtained by reflection through the origin.

Middle of focussing sector:

$$\xi = 1 + 2\alpha^2 - \frac{16}{35}\alpha^4 - \frac{44}{5}\alpha^6 + O(\alpha^8)$$

$$\dot{\xi} = 0$$

Beginning of defocussing sector:

$$\xi = 1 - \frac{16}{35}\alpha^4 + O(\alpha^8)$$

$$\dot{\xi} = -\frac{16}{\pi}\alpha^2\left[1 - \frac{16}{5}\alpha^4 + O(\alpha^8)\right]$$

Middle of defocussing sector:

$$\xi = 1 - 2\alpha^2 - \frac{16}{35}\alpha^4 + \frac{44}{5}\alpha^6 + O(\alpha^8)$$

$$\dot{\xi} = 0$$

Beginning of focussing sector:

$$\xi = 1 - \frac{16}{35}\alpha^4 + O(\alpha^8)$$

$$\dot{\xi} = \frac{16}{\pi}\alpha^2\left[1 - \frac{16}{5}\alpha^4 + O(\alpha^8)\right]$$

These approximations seem to work quite well for all values of N for which the linear motions are stable. It turns out that these fixed points are unstable which means that the phase curves in the immediate neighborhood of these points are hyperbolas.

It is quite convenient to discuss the ξ -motion with reference to the five fixed points. For values of N up to some value between N = 200 and N = 300 the stable region encloses all five fixed points as shown in Fig. 1. As one goes to larger values of N an unstable region separates the stable regions about the origin and the $\xi = \sqrt{3}$ point. Such a case is shown in Fig. 3. It is possible that the stable region about the origin extends out to the unstable fixed point for N values up to about N = 400 but this is difficult to determine since there is a tendency for large errors to enter the calculations of motions near such points. The stable region about the origin definitely does pull away from this fixed point for larger N values and this is the only stable region for values larger than N = 703. This last case is illustrated by Fig. 5.

Wavelength:

One of the more important features of the non-linear focussing equations is the variation of the wavelength of betatron oscillations with the amplitude of the motion. This variation is shown in Figs. 7 and 8 for the ξ – motions about the origin and in Figs. 9, 10, and 11 for the η -motions. The wavelength is expressed as the number of focussing-defocussing pairs per betatron oscillation.

In the case of ξ -motions, as one might expect, the wavelength increases with amplitude since the cubic term opposes the linear term and results in a weaker force. At the origin, the wave-

length has the value given by the linear theory and it approaches infinity in the neighborhood of the unstable fixed point provided the N values are such that the motion is stable in this neighborhood.

On the other hand, η -motions result in a decrease in wavelength with amplitude. In this case the cubic term aids the linear term and this results in a stronger force. One might expect η -motions to be stable for all amplitudes with wavelengths greater than two focussing-defocussing pairs since this wavelength separates the stable and unstable linear motions. It turns out, however, that instability sets in long before this wavelength is reached as the figures clearly show.

Fig. 12 shows the wavelength of ξ -motions as a function of amplitude when the stable region encloses all five fixed points. The N value in this case is 200.

Stability Limits:

For $N < 1406$, a linear force gives rise to motions which are stable for all amplitudes. The cubic term then has the effect of making these motions unstable for amplitudes larger than some value which is referred to as the stability limit. As one goes to larger amplitudes the phase curves become very irregular and become definitely unstable at still larger amplitudes. Because of the irregularity and detailed structure near the stability limit a certain amount of guessing is involved in its determination. It is always possible to obtain the limiting amplitudes more accurately by making more and longer runs with the computer for initial amplitudes near the limits.

Tables I and II list the stability limits which have been obtained as a function of N for each type of motion. The value of the coordinate listed is that value for which the velocity is zero.

STABILITY LIMITS

Table I: ξ-motion

N	Region around origin	Region around $\xi = \sqrt{3}$	Region around five points
36	1.01	1.01 - 2.01	4.58
200	1.04	1.04 - 1.96	2.20
300	1.06	1.25 - 1.90	
400	1.07	1.50 - 1.85	
600	0.99	1.72 - 1.74	
800	0.83		
1000	0.66		
1200	0.64		

Table II: η-motion

N	Region around origin
36	5.22
200	2.07
300	1.69
400	1.36
600	1.00
800	0.76
1000	0.64
1200	0.61

N = 1440:

N = 1440 is a special case in which the unstable linear motion has been stabilized by a cubic term which opposes the linear term. Representative phase curves have been plotted in Fig. 13 which shows the positive coordinate half of the phase plane. The cubic term has the effect of connecting the asymptotes of the hyperbolas of the linear motion, thus giving a stable region resembling a figure eight. In the immediate neighborhood of the origin the curves have a hyperbolic character as they should since the linear term is predominant in this region. For the smaller amplitudes there are two unconnected curves and successive points alternate between these two curves. At the centers of these curves are points which are fixed under a transformation through two focussing-defocussing pairs and are found at $\xi = \pm .277$ for this N value. The stability limit is about $\xi = .55$. No stable region was found for the case where the cubic term aids the linear term.

Pure Cubic:

The special case of a pure cubic force was investigated and the resulting phase curves are shown in Fig. 14. The parameters used in the differential equation were n = 0, e = 1000, and h = .03 which fixes the period at .6. With this choice of parameters the stability limit is around .162. As in the other figures, the initial conditions were applied at the middle of a focussing sector. Fig. 15 shows the variation of wavelength with amplitude for the larger amplitudes.

Loops:

A fine survey of the phase plane reveals the existence of sets of points which are fixed under a transformation through several sectors and that a pair of initial conditions near such a point leads to a curve about each of the points in the corresponding set. In other words, certain initial conditions may lead to a series of "loops" instead of to a single smooth curve. A brief study has indicated that the presence of such sets of loops is not merely a result of using the Runge-Kutla method for solving the differential equation but is due to a resonance effect which is inherent in the equations themselves. Phase plots displaying such loops are shown in Fig. 5 and in Photos 16 thru 23.

There is a definite wavelength associated with each set of these fixed points and the wavelength of a corresponding set of loops fluctuates but averages to this same value. This wavelength is always a rational number and when it is expressed in lowest terms the numerator gives the number of loops which arise from a pair of initial conditions and the denominator gives the number of traversals about the origin before the phase point again falls on the original loop. Due to the symmetry involved, there must always be an even number of loops in any given set and it is possible that only half of them are generated by a given pair of initial conditions. This occurs whenever the numerator of the rational number in lowest terms is odd . This is the case in Fig. 5 and in Photos 17, 20, and 21.

Table III lists the wavelengths of those sets of loops which have been found. It is by no means a complete list since the necessary fine survey of the phase plane was not made for all N values.

Table III

Type of Motion	N	λ
ξ	400	
	600	26/2, 14/1 $\mathcal{U}$
	800	6/1, 14/2, 22/3 $\mathcal{U}$
	1000	4/1, 14/3, 24/5, 34/7 $\mathcal{U}$
	1200	
η	400	8/1, 22/3, 14/2, 34/5, 20/3 46/7, 72/11, 26/4 $\mathcal{U}$
	600	10/2 $\mathcal{U}$
	800	26/6, 34/8, 38/9, 42/10 $\mathcal{U}$
	1000	10/3 $\mathcal{U}$, 26/8 $\mathcal{U}$
	1200	6/2, 8/3

It was, at one time, thought that the wavelengths of the sets of loops for a given N value could be represented by an ordered sequence of rational numbers but many exceptions have been found. For example, the sequence for N = 400, which was most extensively studied, might be written as $\lambda_m = \frac{6m+4}{5}, m = 1, 2, \cdots$. Evidence has been found for the wavelengths corresponding to $m = 2$ through $m = 8$ but on the other hand the wavelength 72/11 appears and this doesn't fit into the scheme. Similar sequences for the other N values are also suggested but more work would be necessary to prove or disprove this idea.

Another point which is suggested by the available results is that instability sets in on a set of loops and photos 20, 21, and 22 lend support to this idea. It seems that the larger loops around

the fixed points have amplitudes which are large enough to make the motion unstable even though the smaller loops appear to be quite stable. In Table III, those sets of loops which seem to lead to instability are marked with a $\mathcal{U}$.

* Supported by the National Science Foundation.

January 12, 1955.

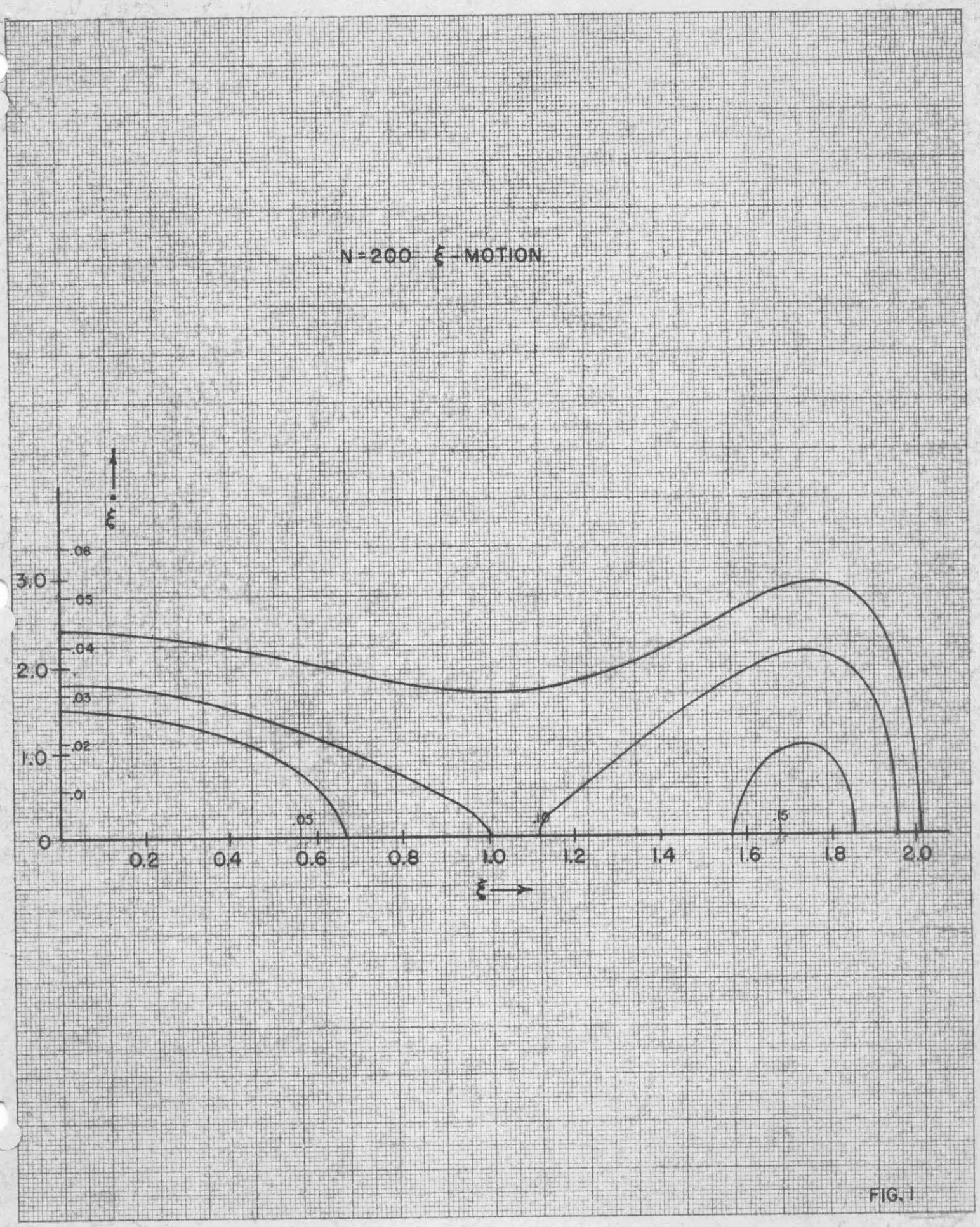


FIG. 1

NO. 318A. MILLIMETERS, 200 BY 250 DIVISIONS. CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS. PRINTED IN U.S.A.

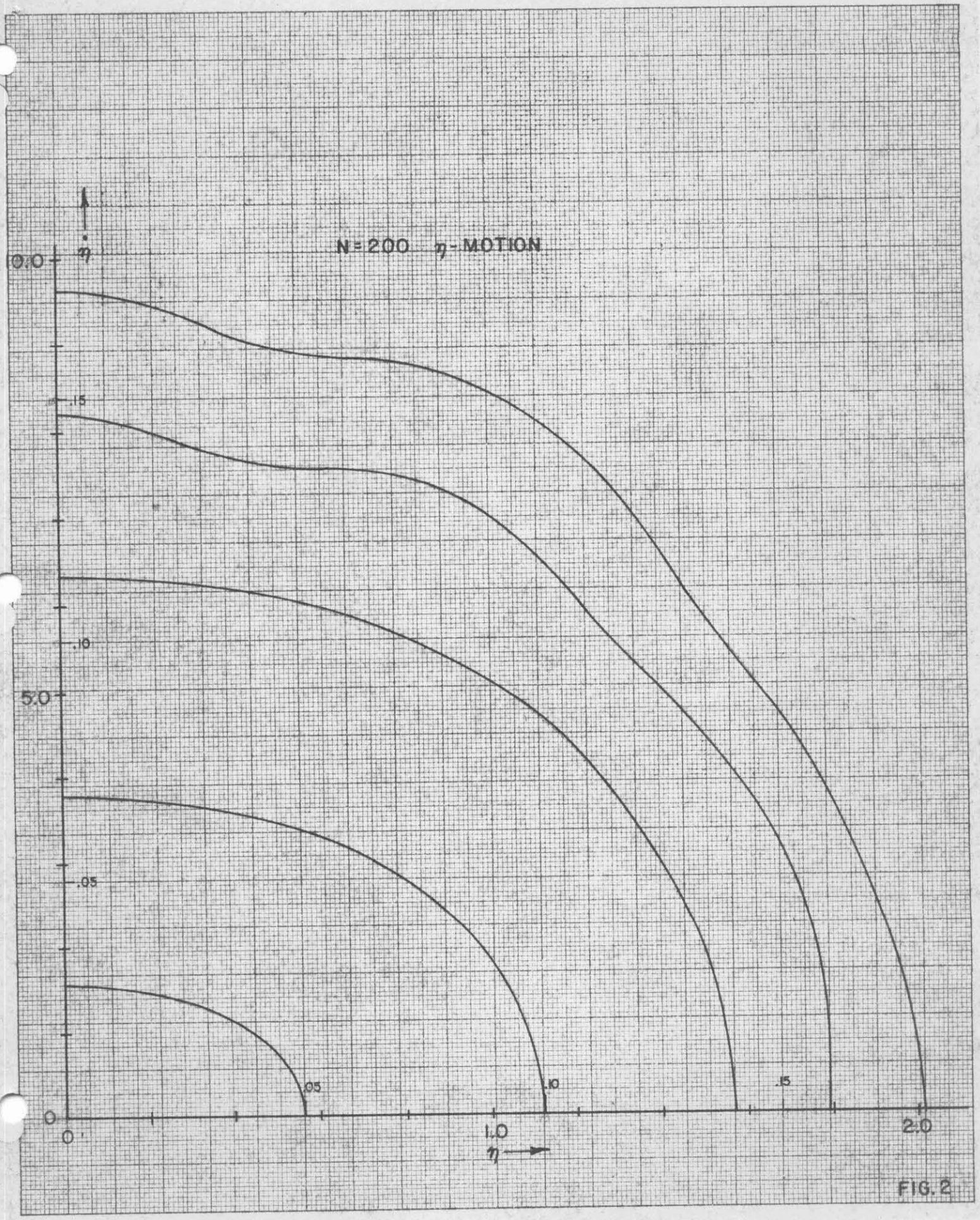


FIG. 2

NO. 319A. MILLIMETER. 200 DIVISIONS PRINTED IN U.S.A.

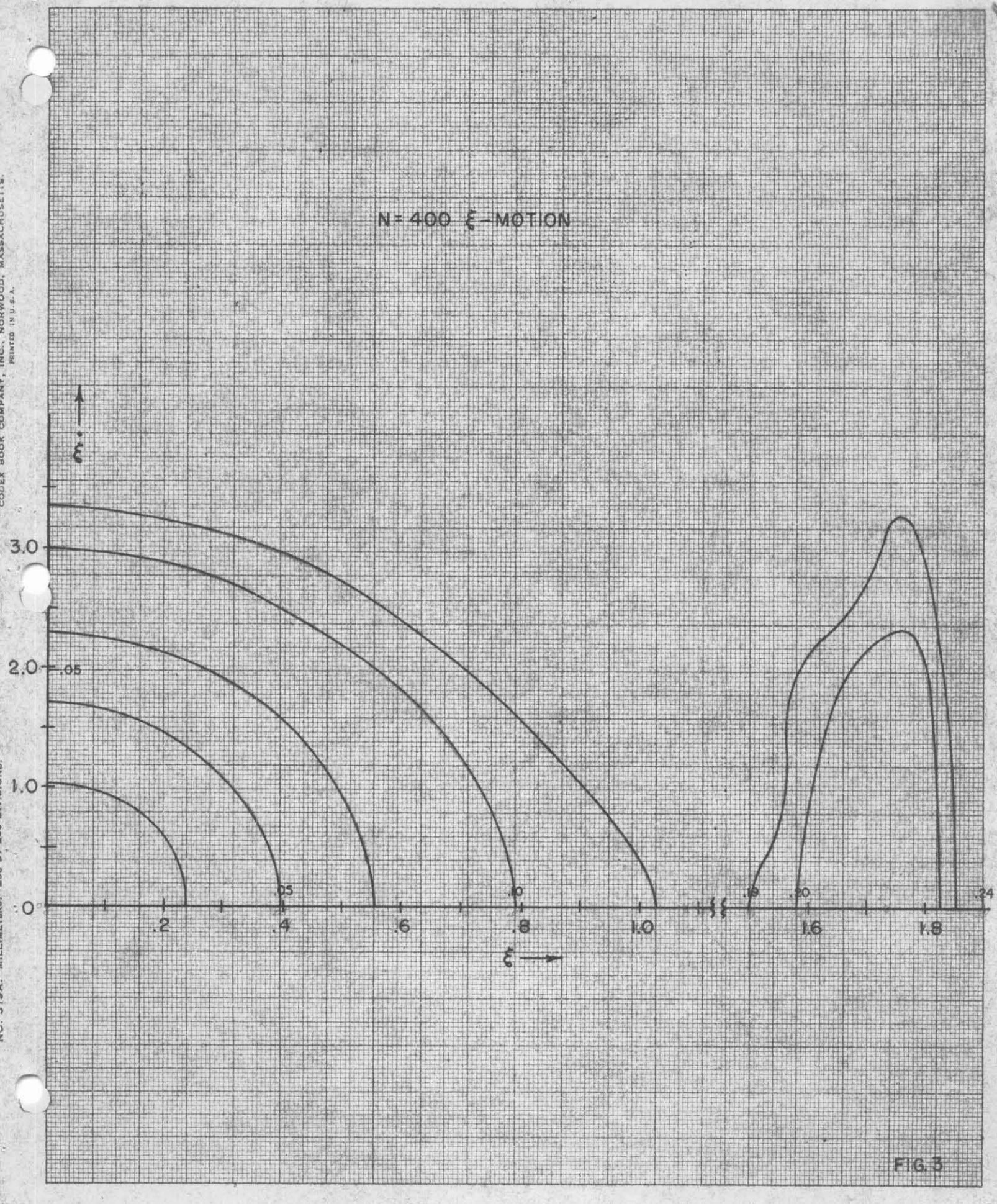


FIG. 3

CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS.
PRINTED IN U.S.A.

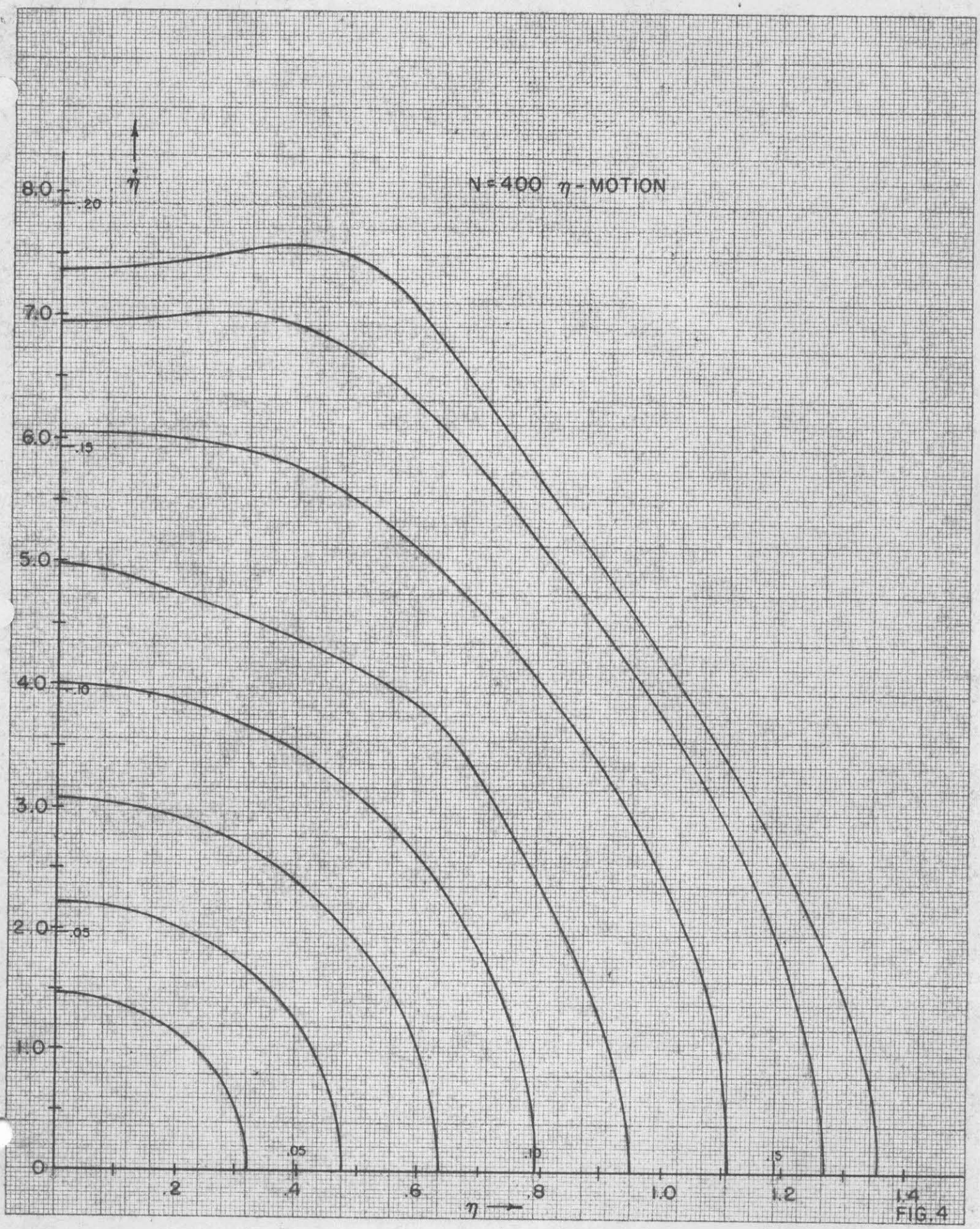


FIG. 4

NO. 319A. MILLIMETERS. 200 BY 250 DIVISIONS. CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS. PRINTED IN U.S.A.

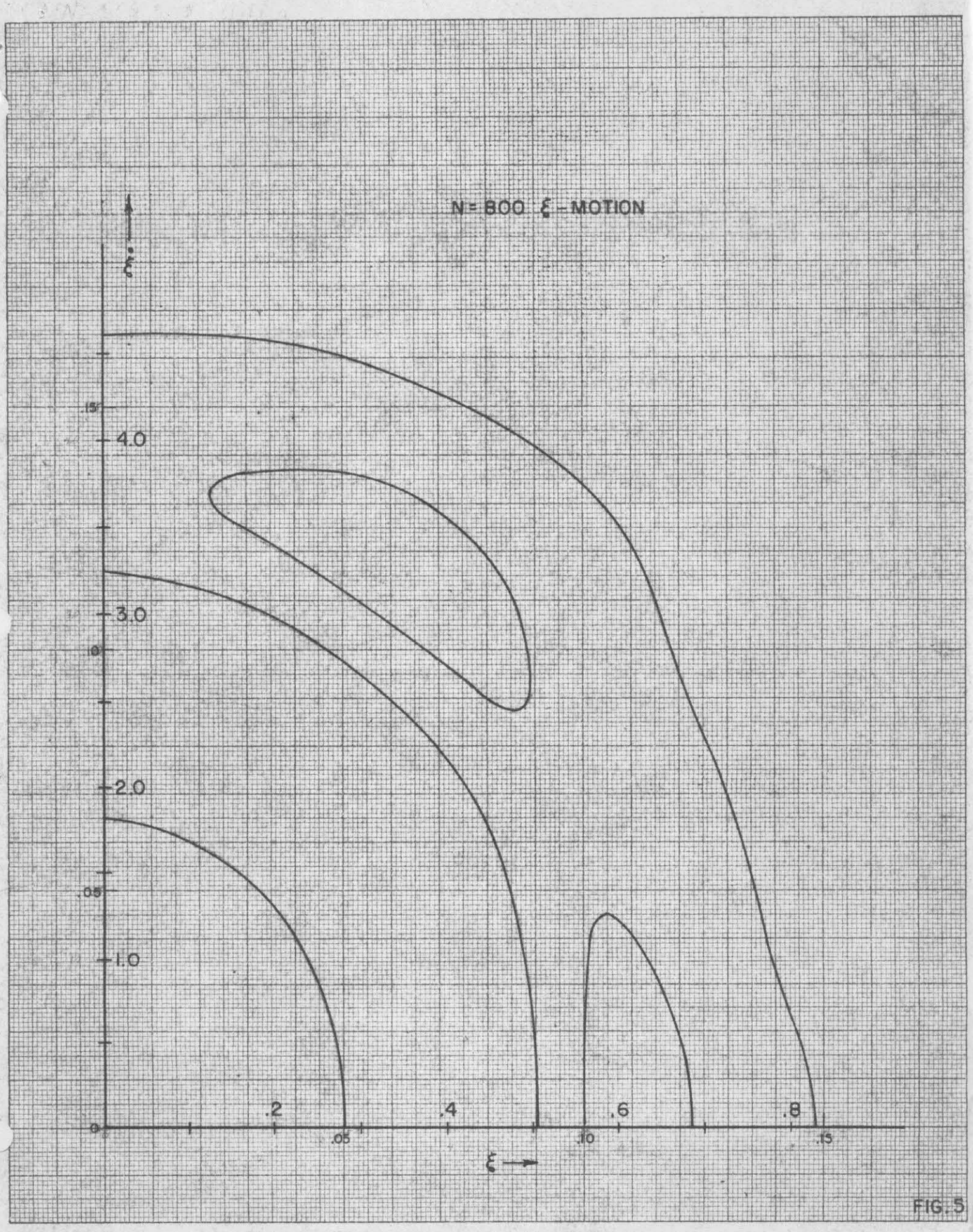


FIG. 5

NO. 319A. MILLIMETERS. 200 BY 250 DIVISIONS. CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS. PRINTED IN U.S.A.

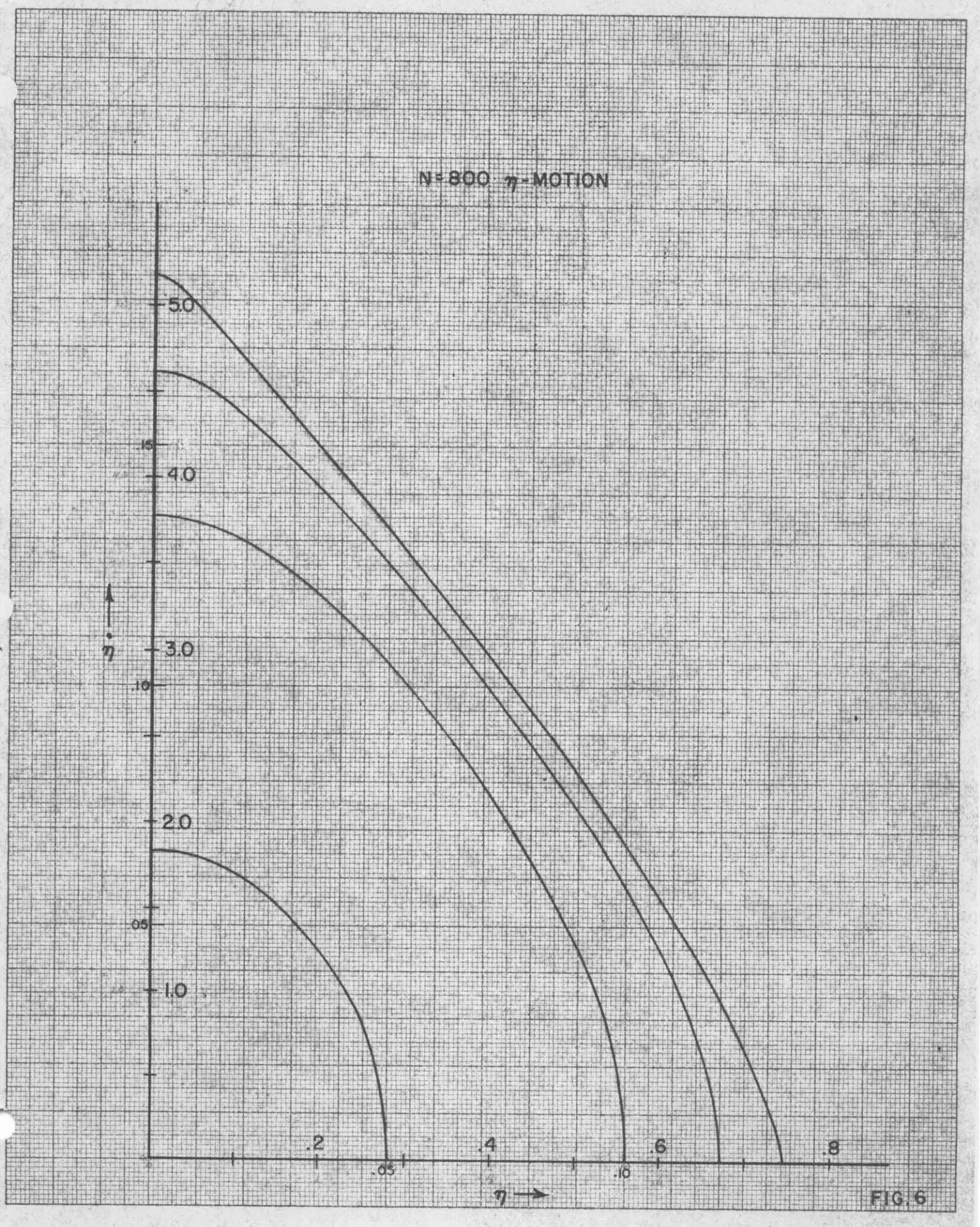


FIG. 6

CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS. PRINTED IN U.S.A.
NO. 319A. MILLIMETERS. 200 BY 250 DIVISIONS.

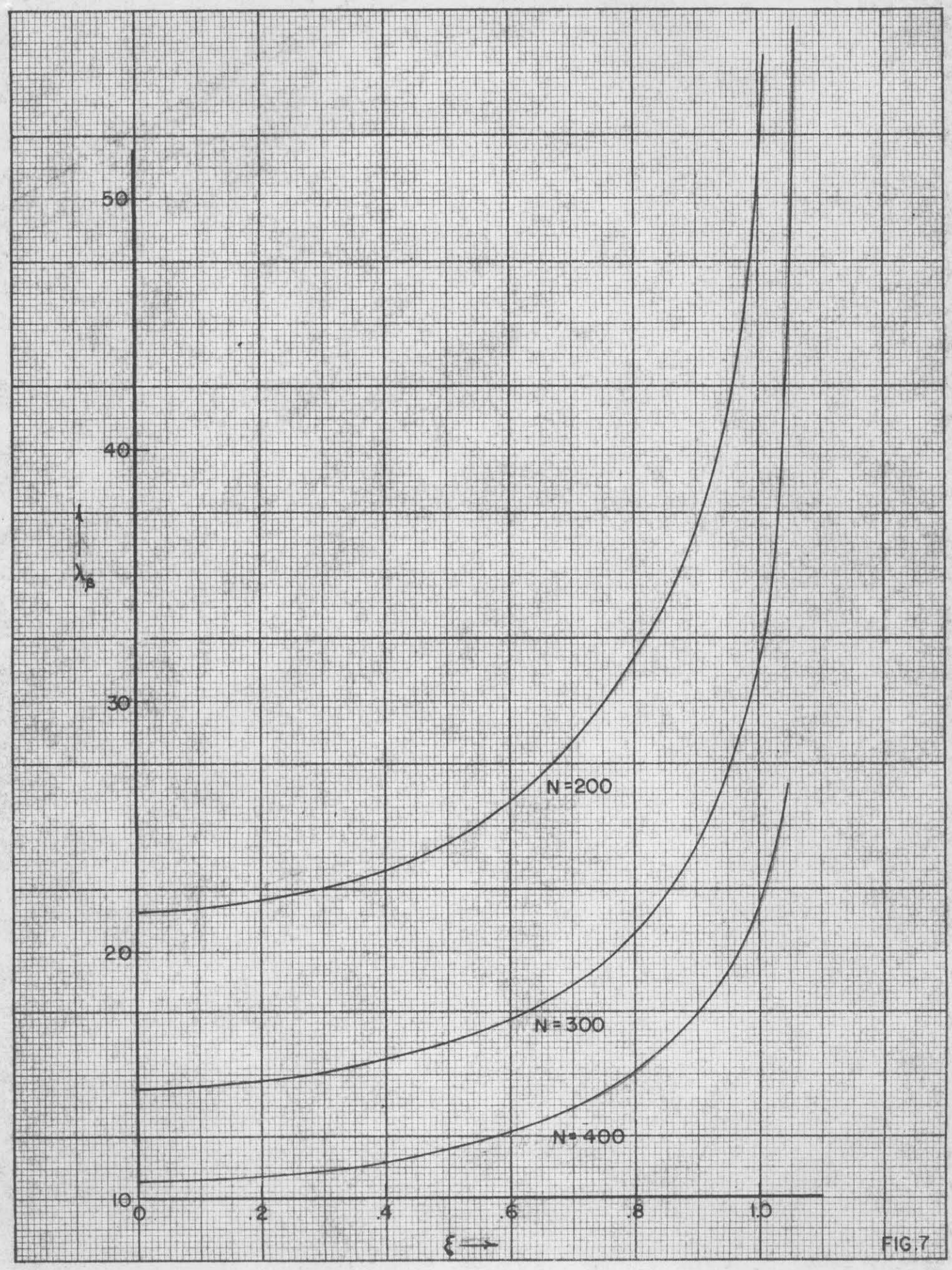


FIG. 7

NO. 318A. 20 DIVISIONS PER INCH BOTH WAYS. 150 BY 200 DIVISIONS.
CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS.
PRINTED IN U.S.A.

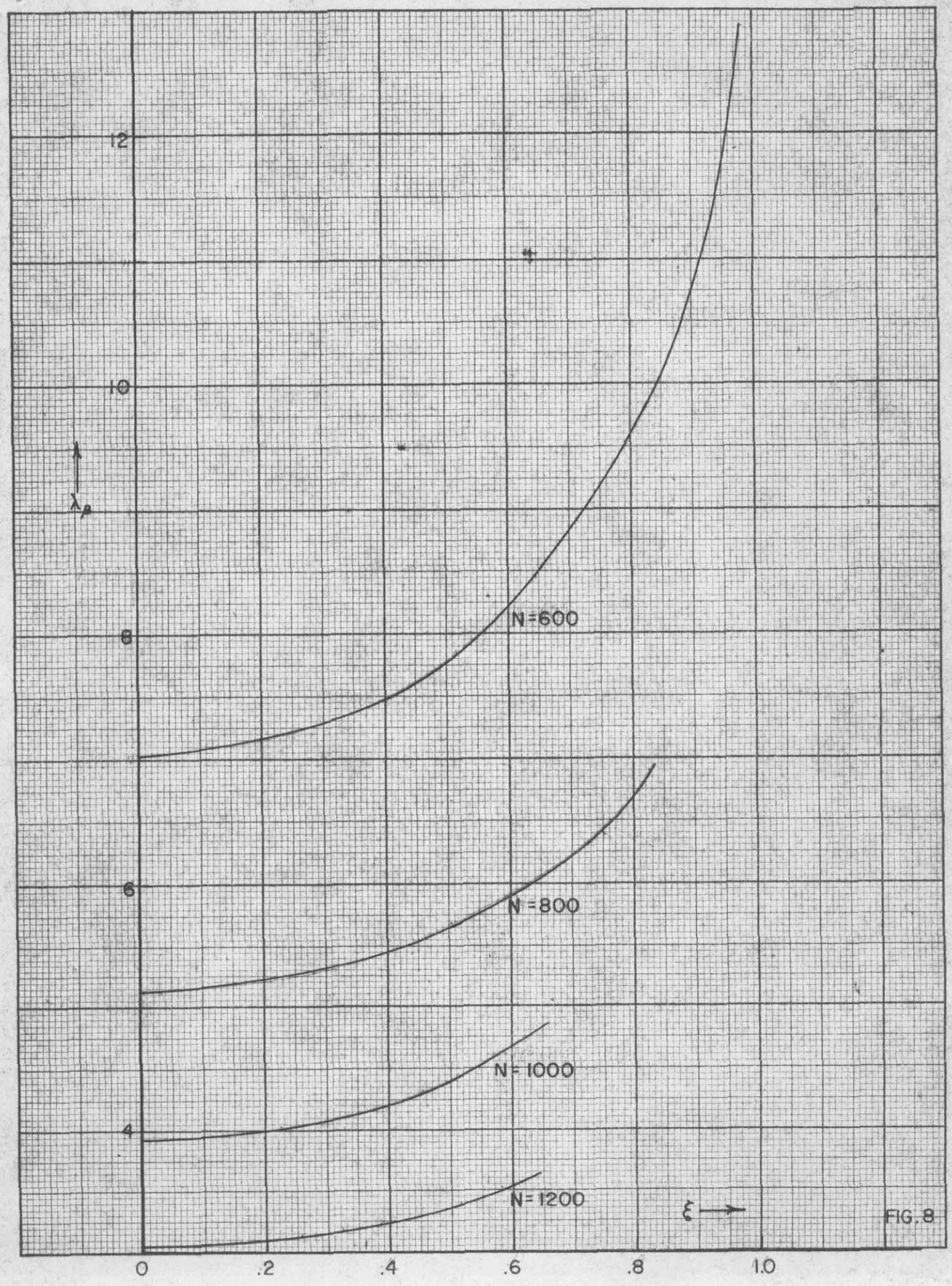
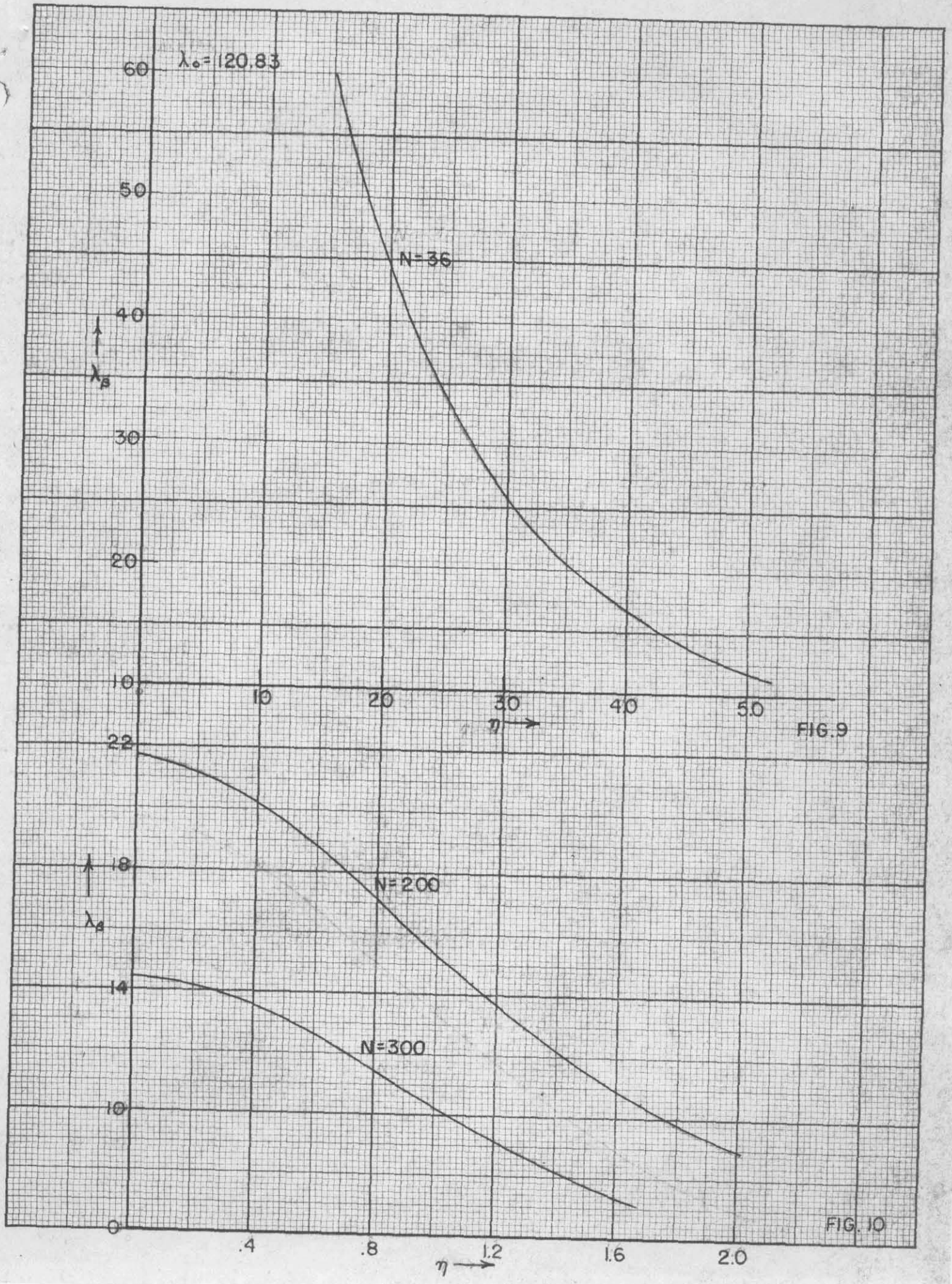


FIG. 8

NO. 318A. 20 DIVISIONS PER INCH BOTH WAYS. 150 BY 200 DIVISIONS.
PRINTED IN U.S.A.



CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS. PRINTED IN U.S.A.
NO. 318A. 20 DIVISIONS PER INCH BOTH WAYS. 150 BY 200 DIVISIONS.

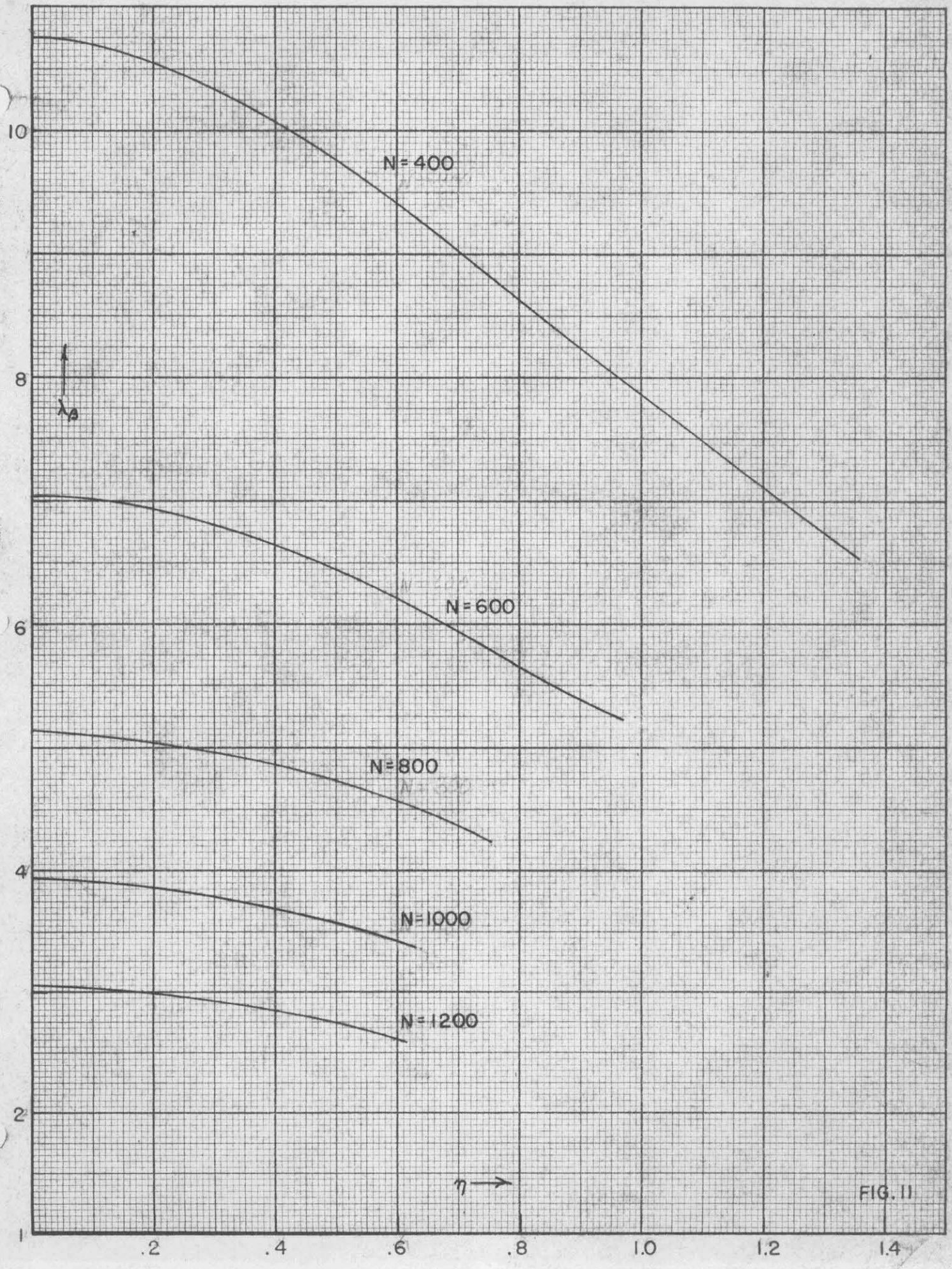


FIG. 11

NO. 318A. 20 DIVISIONS PER INCH BOTH WAYS. 150 BY 200 DIVISIONS.
CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS.
PRINTED IN U.S.A.

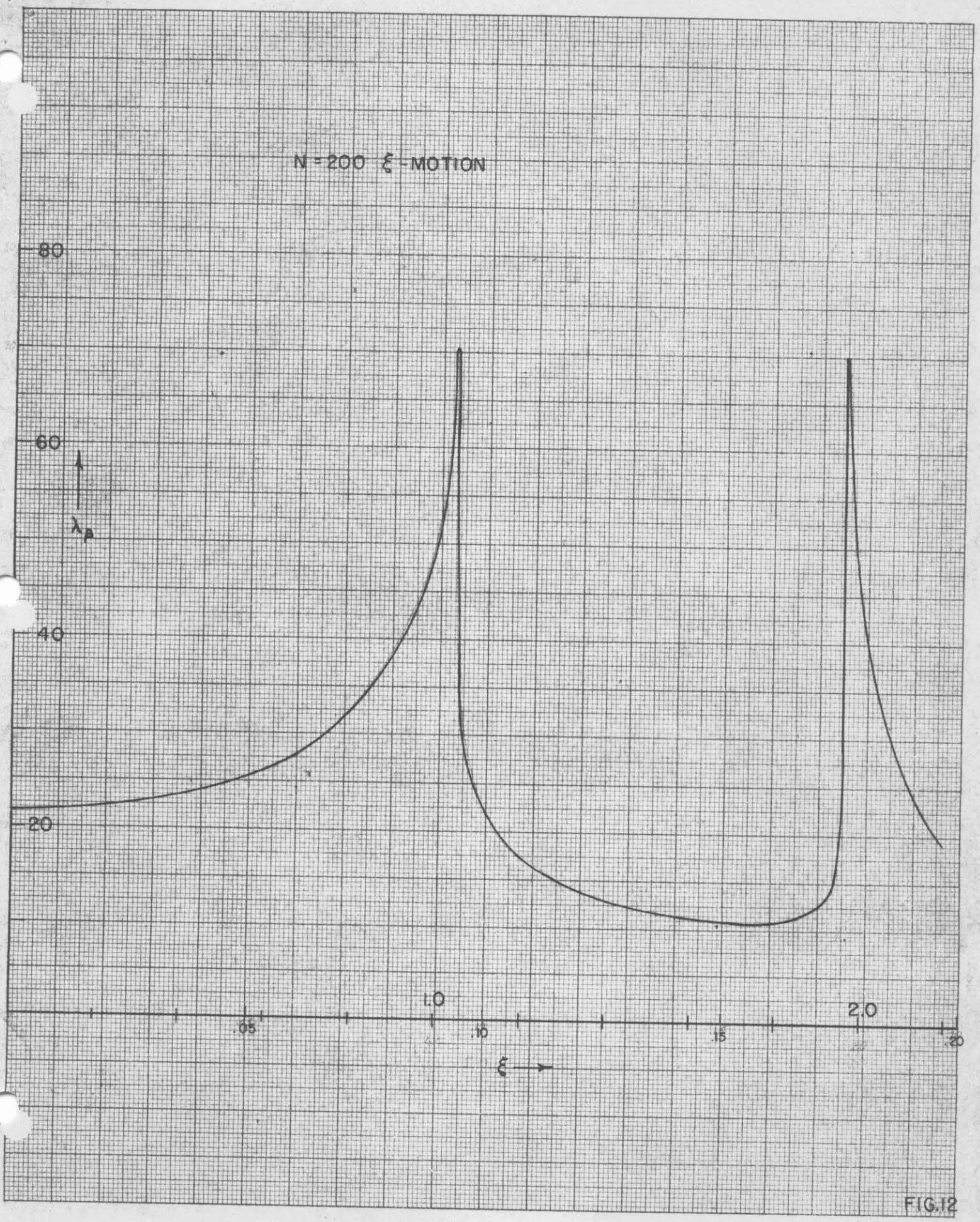


FIG. 12

CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS. PRINTED IN U.S.A.
NO. 319A. MILLIMETERS. 200 BY 250 DIVISIONS.

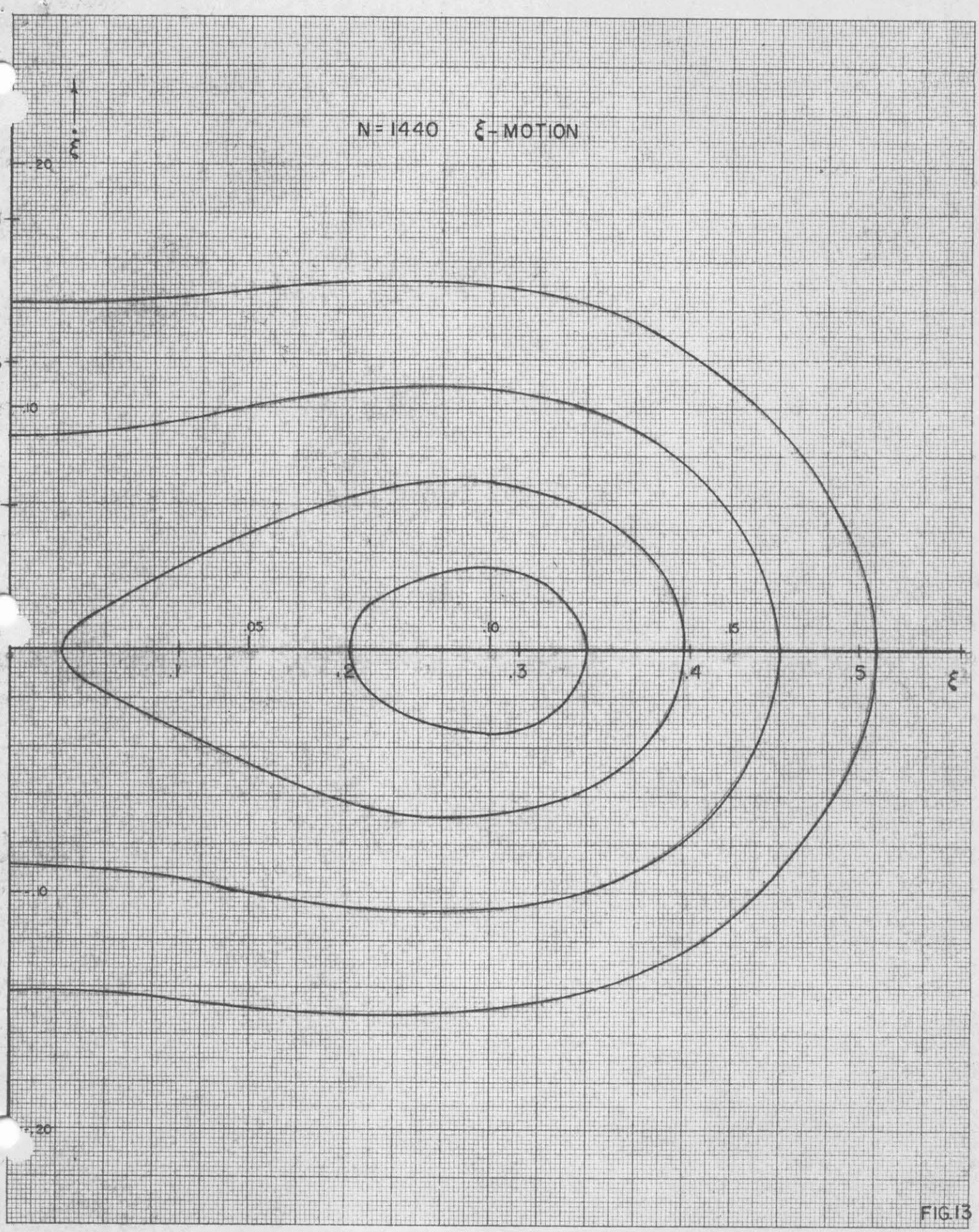


FIG. 13

NO. 319A. MILLIMETERS. 200 BY 250 DIVISIONS. CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS. PRINTED IN U.S.A.

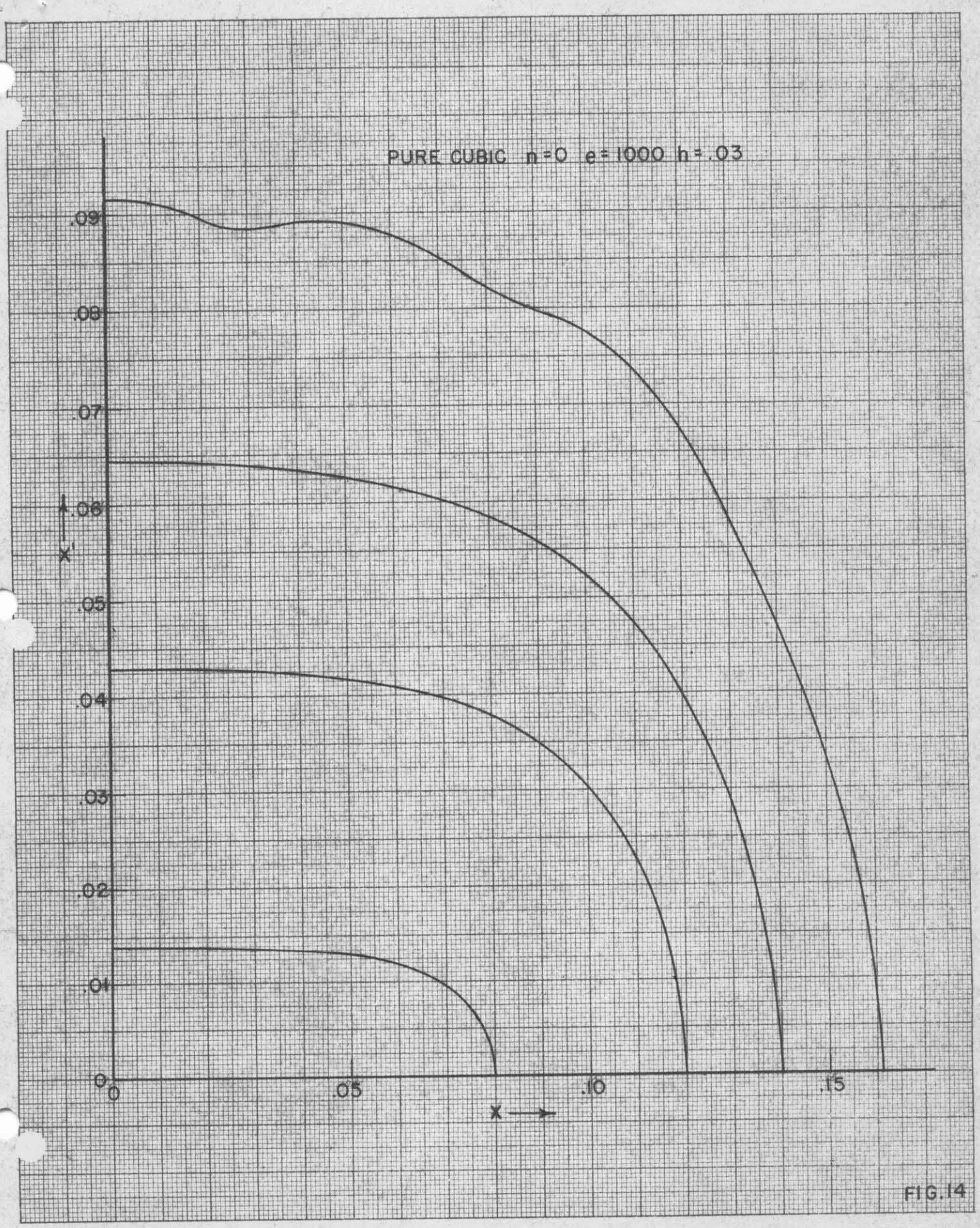


FIG. 14

CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS. PRINTED IN U.S.A.
NO. 319A. MILLIMETERS. 200 BY 250 DIVISIONS.

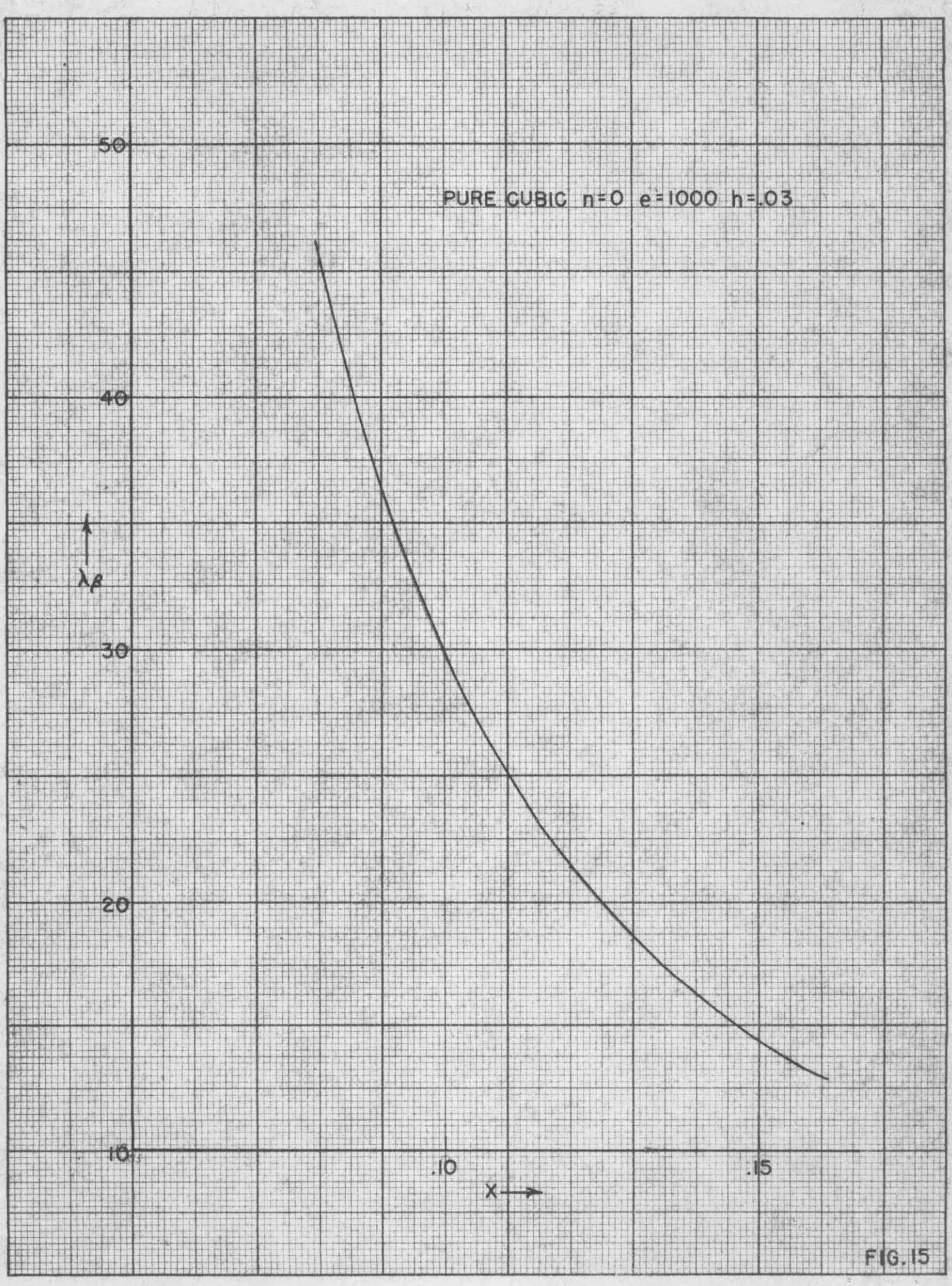
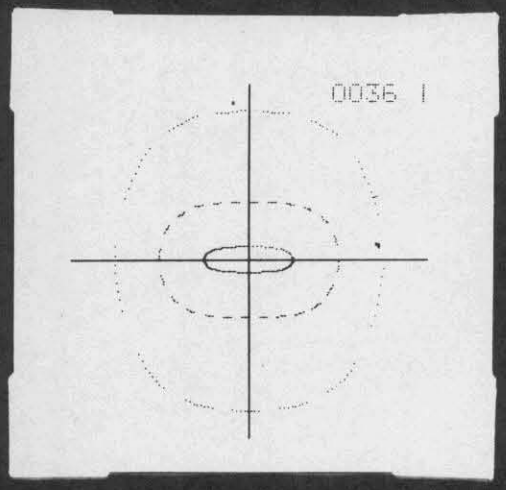


FIG. 15

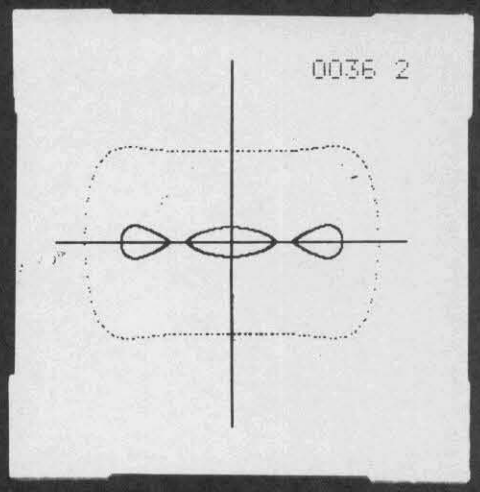
NO. 318A. 20 DIVISIONS PER INCH BOTH WAYS. 150 BY 200 DIVISIONS.
CODEX BOOK COMPANY, INC., NORWOOD, MASSACHUSETTS.
PRINTED IN U.S.A.

Photo 1



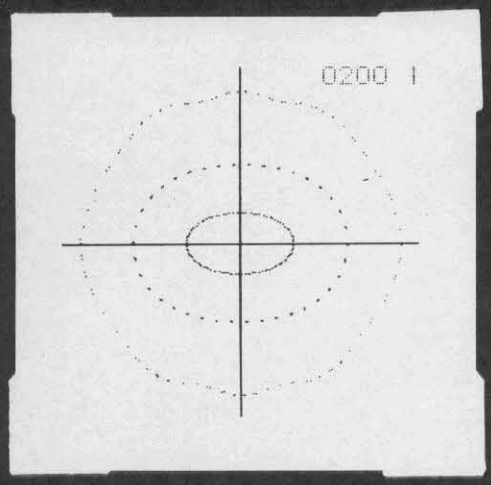
$n = 1$ $e = 1000$ $h = .030$
$N = 36$ $x = x' = 0$
$-.20 < y < .20$ $-.10 < y' < .10$
$y_0 = .050$ $y_0' = 0$
$.100$ 0
$.150$ 0

Photo 2



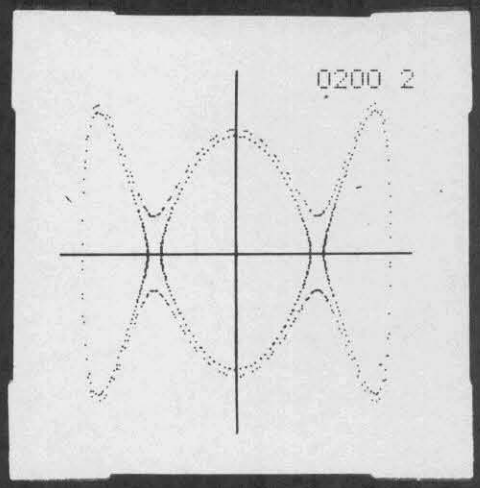
$n = 1$ $e = 1000$ $h = .030$
$N = 36$ $y = y' = 0$
$-.10 < x < .10$ $-.02 < x' < .02$
$x_0 = .025$ $x_0' = 0$
$.035$ 0
$-.035$ 0
$.030$ $.020$

Photo 3



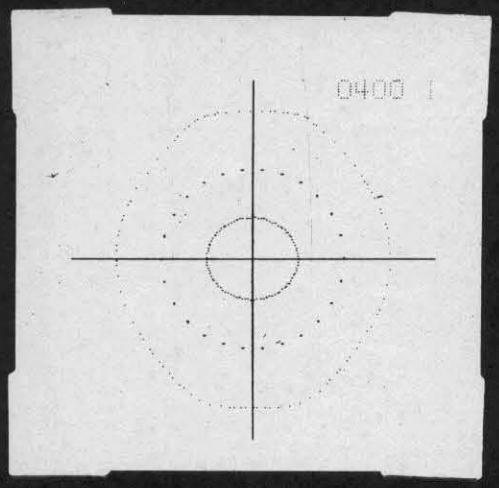
$n = 8$ $e = 1000$ $h = .025$
$N = 200$ $x = x' = 0$
$-.20 < y < .20$ $-.20 < y' < .20$
$y_0 = .060$ $y_0' = 0$
$.120$ 0
$.180$ 0

Photo 4



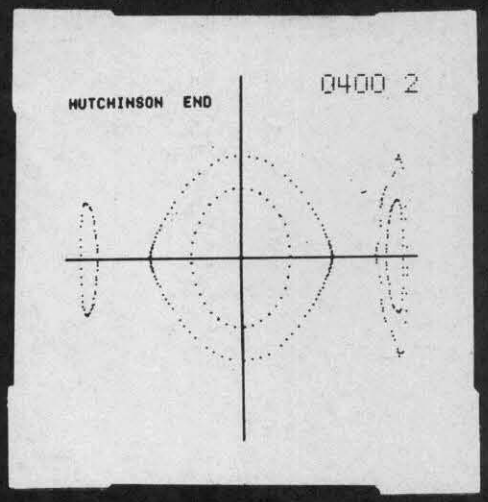
$n = 8$ $e = 1000$ $h = .025$
$N = 200$ $y = y' = 0$
$-.20 < x < .20$ $-.05 < x' < .05$
$x_0 = .085$ $x_0' = 0$
$.100$ 0
$-.100$ 0
$.092$ $.010$

Photo 5



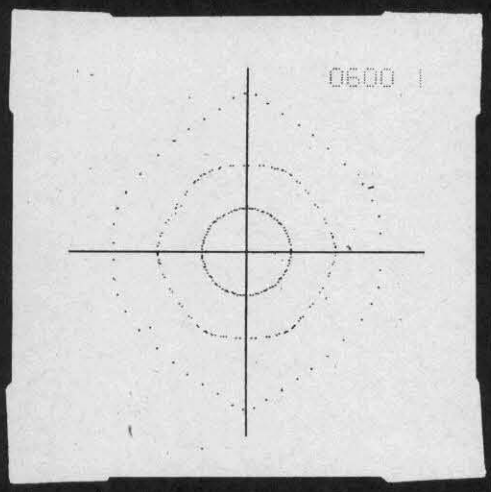
$n = 16$ $e = 1000$ $h = .025$
$N = 400$ $x = x' = 0$
$-.20 < y < .20$ $-.20 < y' < .20$
$y_0 = .050$ $y_0' = 0$
$.100$ 0
$.150$ 0

Photo 6



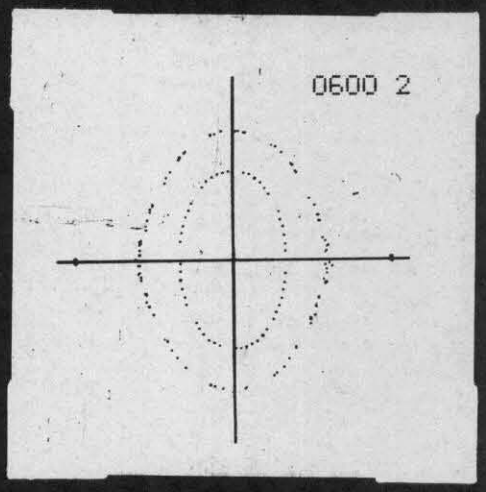
$n = 16$ $e = 1000$ $h = .025$
$N = 400$ $y = y' = 0$
$-.25 < x < .25$ $-.15 < x' < .15$
$x_0 = .070$ $x_0' = 0$
$.130$ 0
$.190$ 0
$.205$ 0
$-.205$ 0

Photo 7



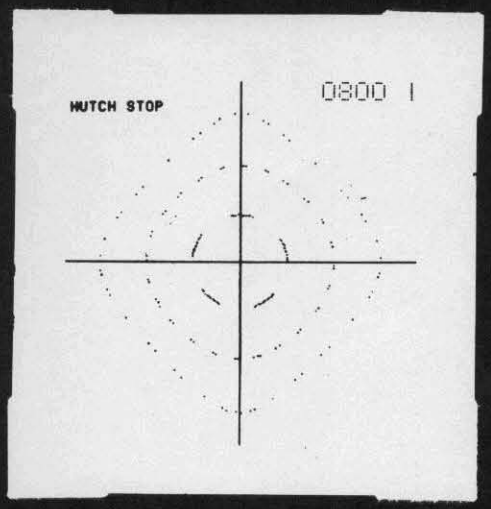
$n = 24$ $e = 1000$ $h = .025$
$N = 600$ $x = x' = 0$
$-.20 < y < .20$ $-.25 < y'' < .25$
$y_0 = .050$ $y_0' = 0$
$.100$ 0
$.150$ 0

Photo 8



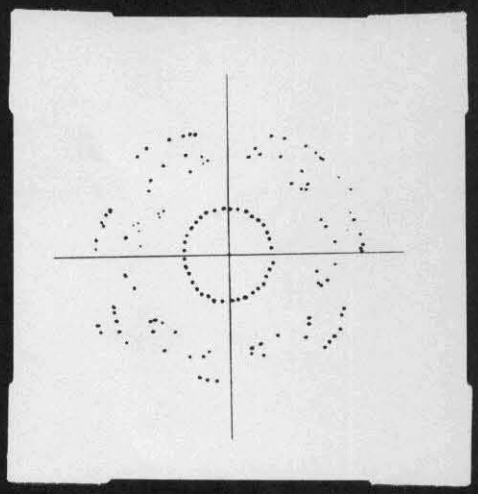
$n = 24$ $e = 1000$ $h = .025$
$N = 600$ $y = y'' = 0$
$-.30 < x < .30$ $-.20 < x' < .20$
$x_0 = .090$ $x_0' = 0$
$.170$ 0
$.268$ 0
$-.268$ 0

Photo 9



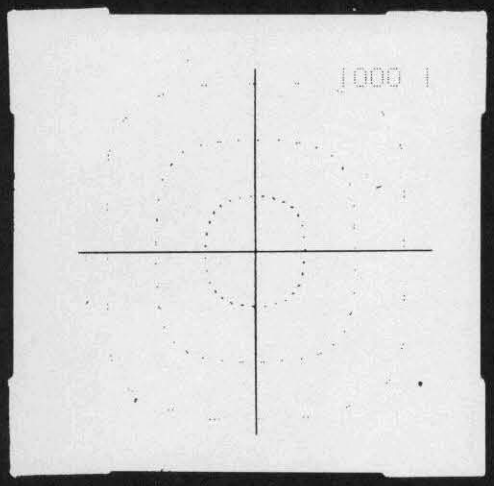
$n = 32$ $e = 1000$ $h = .025$
$N = 800$ $x = x' = 0$
$-.15 < y < .15$ $-.20 < y' < .20$
$y_0 = .040$ $y_0' = 0$
$.080$ 0
$.120$ 0

Photo 10



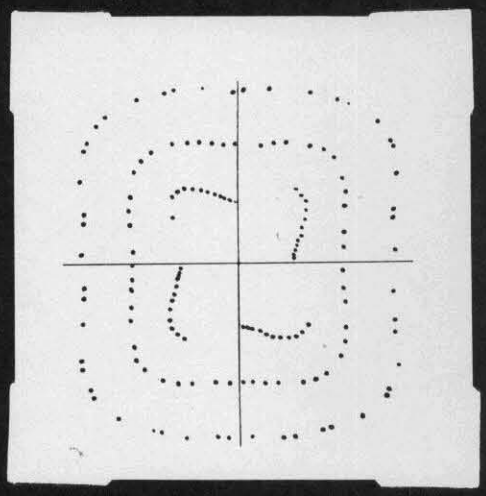
$n = 32$ $e = 1000$ $h = .025$
$N = 800$ $y = y' = 0$
$-.20 < x < .20$ $-.25 < x' < .25$
$x_0 = .050$ $x_0' = 0$
$.100$ 0
$.150$ 0

Photo 11



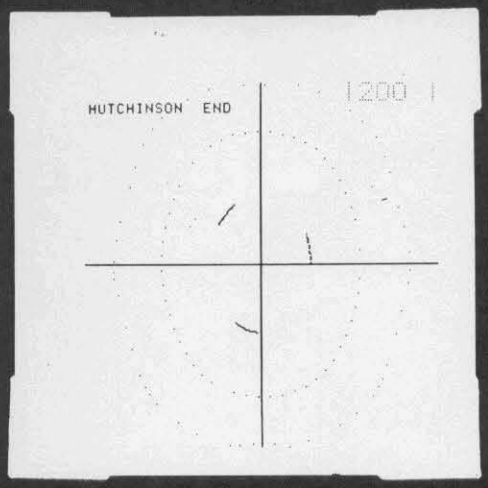
$n = 250$ $e = 2000$ $h = .010$
$N = 1000$ $x = x' = 0$
$-.25 < y < .25$ $-.75 < y' < .75$
$y_0 = .070$ $y_0' = 0$
$.140$ 0
$.210$ 0

Photo 12



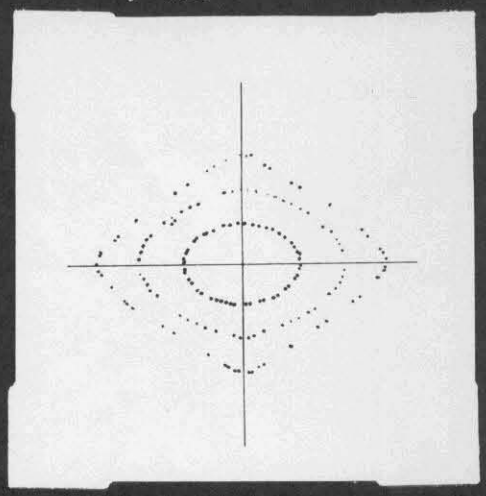
$n = 250$ $e = 2000$ $h = .010$
$N = 1000$ $y = y' = 0$
$-.25 < x < .25$ $-.75 < x' < .75$
$x_0 = .080$ $x_0' = 0$
$.150$ 0
$.221$ 0

Photo 13



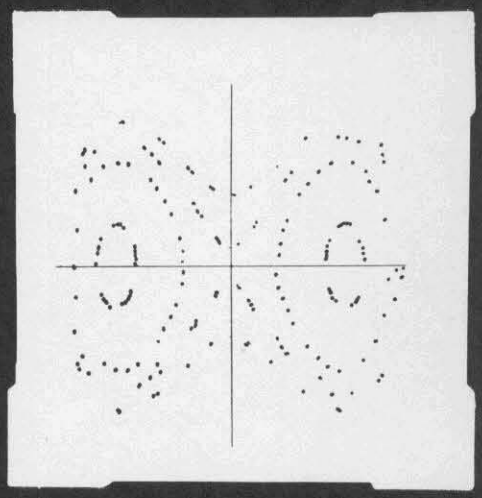
$n = 300$ $e = 2000$ $h = .010$
$N = 1200$ $x = x' = 0$
$-.25 < y < .25$ $-.50 < y' < .50$
$y_0 = .070$ $y_0' = 0$
$.140$ 0
$.210$ 0

Photo 14



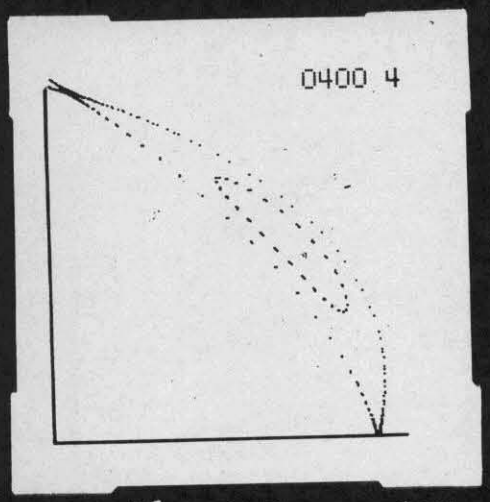
$n = 48$ $e = 2000$ $h = .025$
$N = 1200$ $y = y' = 0$
$-.12 < x < .12$ $-.20 < x' < .20$
$x_0 = .040$ $x_0' = 0$
$.070$ 0
$.100$ 0

Photo 15



$n = 250$ $e = 2000$ $h = .012$
$N = 1440$ $y = y' = 0$
$-.15 < x < .15$ $-.10 < x' < .10$
$x_0 = .040$ $x_0' = 0$
$.080$ 0
0 $.010$
0 $.040$

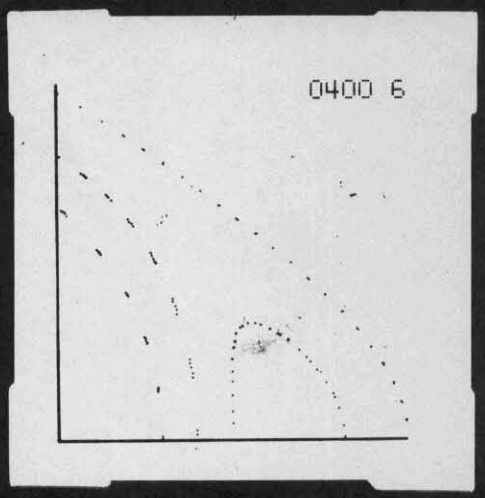
Photo 16



$n = 16$ $e=1000$ $h=.025$
$N = 400$ $x = x' = 0$
$.075 < y < .125$ $0 < y' < .100$
$y_0 = .1205$ $y_0' = 0$
$.1210$ 0
$.1150$ $.035$

$\lambda_\beta = 8/1$

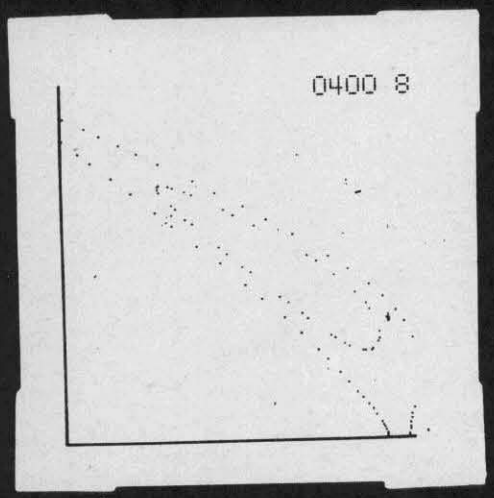
Photo 17



$n = 16$ $e = 1000$ $h=.025$
$N = 400$ $x=x' = 0$
$.148 < y < .158$ $0 < y' < .050$
$y_0 = .151$ $y_0' = 0$
$.152$ 0
$.153$ 0
$.158$ 0

$\lambda_\beta = 14/2$

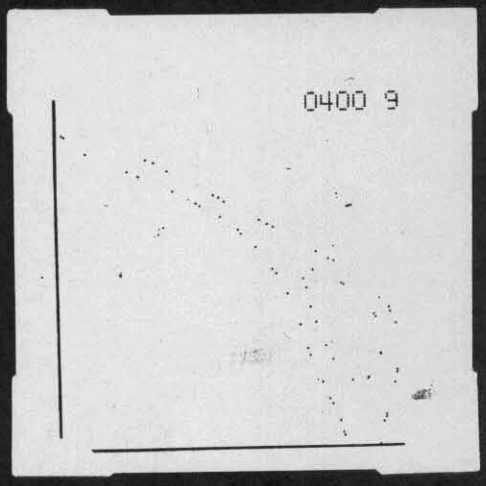
Photo 18



$n=16$ $e=1000$ $h=.025$
$N=400$ $x=x'=0$
$.158 < y < .168$ $0<y' < .050$
$y_0 = .1672$ $y_0' = 0$
$.1678$ 0
$.1670$ $.015$

$\lambda_\beta = 20/3$

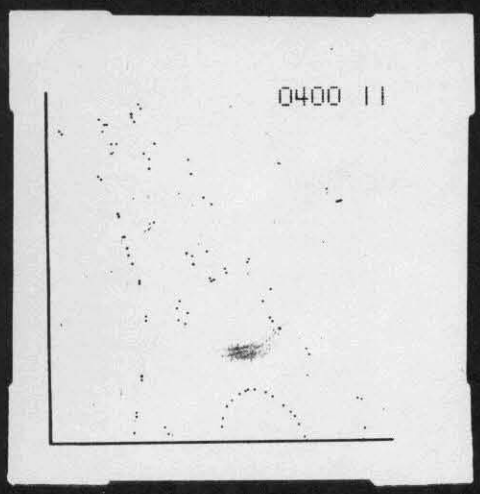
Photo 19



$n=16$ $e=1000$ $h=.025$
$N=400$ $x = x' = 0$
$.168 < y < .171$ $0 < y' < .025$
$y_0 = .1705$ $y_0 = 0$
$.1708$ 0
$.1705$ $.005$

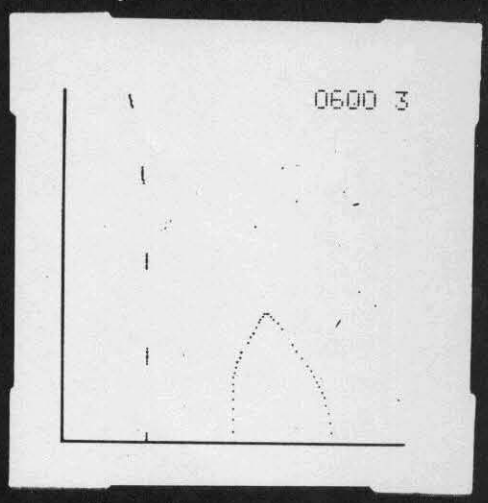
$\lambda_\beta = 46/7$

Photo 20



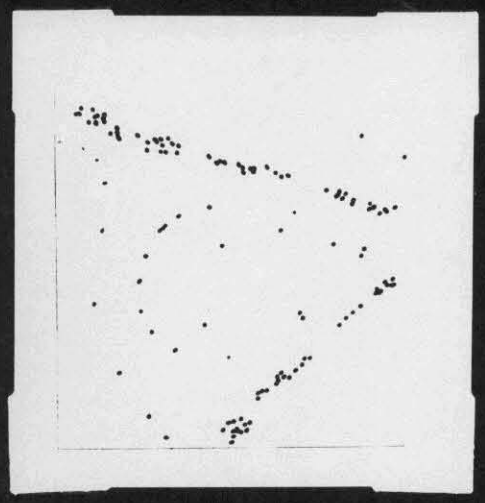
n=16 e= 1000 h=.025
N=400 x= x' = 0
.170<y<.176 0<y'<.020
y_0=.1715 y_0' = 0
.1720 0
.1730 0
.1758 0
$\lambda\beta$=26/4

Photo 21



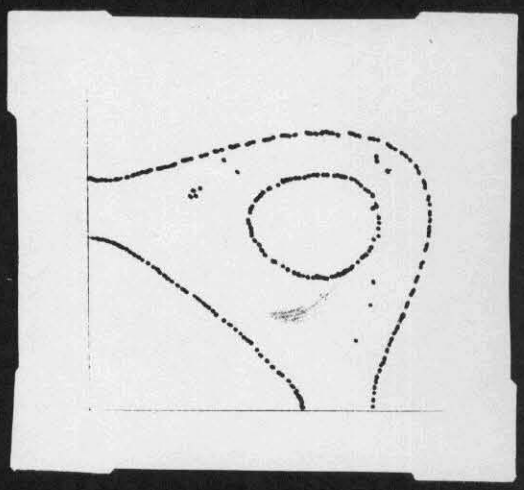
n=24 e=1000 h=.025
N=600 x= x' = 0
.140<y<.180 0<y'<.050
y_0 =.150 y_0' = 0
.160 0
.178 0
$\lambda\beta$=5/1

Photo 22



n = 250 e=2000 h=.01
N =1000 x=x' = 0
.216 < y < .246 0<y'<.600
y_0=.226 y_0' = 0
.230 0
.228 .150
$\lambda\beta$=10/3

Photo 23



n = 250 e =2000 h=.01
N = 1000 y= y' = 0
0 < x < .10 0 < x' < .40
x_0 = .06 x_0' = 0
.08 0
.07 .15
$\lambda\beta$ = 4/1