MURA - 47
Pt. 2

Stability Theory

Part II

Some special canonical transformations in two-dimensional phase space

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Part II Some Special Canonical Transformations in Two-dimensional Phase Space

(1) General characterization

In this part we shall study some special classes of canonical transformations which are most useful for an approximation technique to be discussed later. They are of the form

$$T: \quad \begin{aligned} p_1 &= p + A(p,q) \\ q_1 &= q + B(p,q) \end{aligned} \qquad (1\text{-}1)$$

where A and B are a sum of homogeneous polynomials beginning with the second order. Every canonical transformation which is analytical in the neighborhood of $p = q = 0$ and has this point for a fixed point can be brought into this form by a suitable linear transformation of the variables (cf. Section 4 part I; we shall ignore here the exceptional special case III[11] for which this is not true). When we express the condition for a canonical transformation in terms of A and B we obtain

$$\begin{vmatrix} \rho + A_p, & A_q \\ B_p, & \rho^{-1} + B_q \end{vmatrix} = 1 + \rho^{-1} A_p + \rho B_q + A_p B_q - A_q B_p = 1 \qquad (1\text{-}2)$$

or $$\rho^{-1} A_p + \rho B_q = A_q B_p - A_p B_q \,, \tag{1-3}$$

where $A_p \equiv \dfrac{\partial A}{\partial p}$, etc.

Thus any pair of functions A and B of the specified form which satisfy Equ. (1-3) give a canonical transformation (1-1).

(2) <u>A Special Class of Transformation</u>

We satisfy Equ. (1-3) by assuming

$$\rho^{-1} A_p + \rho B_q = 0 \tag{2-1}$$

$$A_p B_q - A_q B_p = 0 \tag{2-2}$$

These two equations are a consequence of Equ. (1-3) for instance in the case that A and B are even functions of p and q. For in this case the two sides of Equ. (1-3) are of opposite parity under reflection on the origin and therefore must vanish separately.

The Equ. (2-1) implies existence of a function ϕ (p,q) such that

$$\left.\begin{aligned} A &= -\rho\, \phi_q \\ B &= \rho^{-1} \phi_p \end{aligned}\right\} \tag{2-3}$$

The Equ. (2-2) when expressed in terms of ϕ leads to the relation

$$\left(\phi_{qp}\right)^2 = \phi_{qq}\,\phi_{pp} \tag{2-4}$$

This partial differential equation for ϕ is called the equation of Ampère and Monge. Its general solution is known and can be represented in different ways. Geometrically the equation $z = \phi(x,y)$ represents a surface with Gaussian curvature = 0. An analytical representation can be given as follows: Let $\varphi(c)$ and $\psi(c)$ be two arbitrary (analytical) functions of a parameter C. The equation

$$Z = cx + \varphi(c)y + \psi(c) \tag{2-5}$$

represents a one-parameter family of planes. The envelope of these planes represents a surface $Z = \phi(x,y)$ with the desired property. That is $\phi(x,y)$ is a solution of the Equ. (2-4) (with $p = x$, $q = y$). An analytical expression for $\phi(xy)$ is obtained by eliminating the parameter c from Equ. (2-5) and from

$$0 = x + \varphi'(c)y + \psi'(c) \tag{2-5'}$$

(3) <u>The P-Transformations</u>

The special subclass of transformations which is obtained from the functions

$$\begin{aligned} \varphi(c) &= \lambda c \\ \psi(c) &= \text{arbitrary} \end{aligned} \tag{3-1}$$

shall be called polynomial transformations (P-transformations). The elimination of the parameter c from the two equations (2-5) and (2-5)' results in a transformation

$$\phi\,(pq) = \phi\,(w) \tag{3-2}$$

where $w = \alpha p + \beta q$

and $\phi\,(w)$ is an arbitrary function of w. Powell considered in particular the transformation which is associated with

$$\phi(w) = \frac{A}{4} w^4 \tag{3-3}$$

We obtain in this case

$$T_p : \left.\begin{aligned} p_1 &= \rho p - A\rho\beta(\alpha p + \beta q)^3 \\ q_1 &= \rho^{-1} q + A\rho^{-1}\alpha(\alpha p + \beta q)^3 \end{aligned}\right\} \tag{3-4}$$

with the inverse

$$\left.\begin{aligned} p &= \rho^{-1} p_1 + A\beta(\alpha\rho^{-1} p_1 + \beta\rho q_1)^3 \\ q &= \rho q_1 - A\alpha(\alpha\rho^{-1} p_1 + \beta\rho q_1)^3 \end{aligned}\right\} \tag{3-5}$$

since

$$\alpha p + \beta q = \alpha \rho^{-1} p_1 + \beta \rho q_1 = w$$

The reality condition in case I (see part I Equ. (4-15) ff.) are $p_1^* = q_1$ while in case II p and q are real. It follows from this that in case I $\phi(w)$ must be purely imaginary

$$\phi^*(w) = -\phi(w) \qquad \text{(Case I)}$$

while in case II it is real

$$\phi^*(w) = \phi(w) \qquad \text{(Case II)}$$

Thus the constants A , α and β in the special case (3-3) must be chosen as to satisfy these conditions vz.

$$\left.\begin{array}{l} \alpha^* = \beta \\ A^* = -A \end{array}\right\} \qquad \text{Case I)}$$

$$\left.\begin{array}{l} \alpha^* = \alpha, \quad \beta^* = \beta \\ A^* = A \end{array}\right\} \qquad \text{(Case II)}$$

(4) <u>The Generating Function for the P-Transformation</u>

Since the P-transformation is canonical there exists a generating function $F(p_1,q)$ such that the corresponding values $p_1 q_1$, p and q satisfy

$$\frac{\partial F}{\partial p_1} = q_1 \qquad \frac{\partial F}{\partial q} = p \qquad \text{4-1}$$

We shall determine this function first for the case

$$\phi(w) = \frac{1}{n+1} w^{n+1} \qquad \text{(4-2)}$$

so that

$$\left.\begin{aligned} p_1 &= \rho p - \rho\beta w^n \\ q_1 &= \rho^{-1} q + \rho^{-1}\alpha w^n \end{aligned}\right\} \tag{4-3}$$

$$\left.\begin{aligned} p &= \rho^{-1} p_1 + \beta w^n \\ q &= \rho q_1 - \alpha w^n \end{aligned}\right\} \tag{4-4}$$

It follows from (4-1), (4-3) and (4-4) that

$$\frac{\partial F}{\partial p_1} = \rho^{-1} q + \rho^{-1}\alpha w^n \tag{4-5}$$

$$\frac{\partial F}{\partial q} = \rho^{-1} p_1 + \beta w^n \tag{4-6}$$

In the following we need the expression for

$$\left(\frac{\partial w}{\partial q}\right)_{p_1} = \beta\left(1 + \frac{\alpha}{\beta}\left(\frac{\partial p}{\partial q}\right)_{p_1}\right) \tag{4-7}$$

and $$\left(\frac{\partial w}{\partial p_1}\right)_q = \alpha\rho^{-1}\left(1 + \frac{\beta}{\alpha}\rho^2\left(\frac{\partial q_1}{\partial p_1}\right)_q\right) \tag{4-8}$$

Differentiating the first of Equ. (4-3) with the condition $dp_1 = 0$ gives

$$0 = \rho dp - \rho\beta n w^{n-1}(\alpha dp + \beta dq), \tag{4-9}$$

from which we obtain

$$\left(\frac{\partial p}{\partial q}\right)_{p_1} = \frac{\beta^2 w^{n-1}}{1 - n\alpha\beta w^{n-1}} \tag{4-10}$$

Similarly the last of Equ. (4-4) gives with dq = 0

$$0 = \rho\, dq_1 - n\alpha w^{n-1}\left(\alpha\rho^{-1} dp_1 + \beta\rho\, dq_1\right) \tag{4-11}$$

or

$$\left(\frac{\partial q_1}{\partial p_1}\right)_p = \frac{1}{\rho^2}\,\frac{n\alpha^2 w^{n-1}}{1 - n\alpha\beta\, w^{n-1}} \tag{4-12}$$

It follows

$$1 + \frac{\alpha}{\beta}\left(\frac{\partial p}{\partial q}\right)_{p_1} = \frac{1}{1 - n\alpha\beta\, w^{n-1}} = 1 + \frac{\beta}{\alpha}\rho^2\left(\frac{\partial q_1}{\partial p_1}\right)_q \tag{4-13}$$

The equs. (4-5) and (4-6) can now be integrated with the result

$$F = \rho^{-1} q p_1 + \frac{1}{n+1} w^{n+1} - \frac{\alpha\beta}{2} w^{2n} \tag{4-14}$$

This is the desired generating function in the special case of Equ. (4-2). In this expression w must be considered as a function of p_1 and q. We can generalize this result without difficulty for the case of a convergent power series

$$\phi(w) = \sum_{\nu=0}^{\infty} a_\nu w^\nu \tag{4-15}$$

with arbitrary coefficients a_ν.

In this case

$$\left(\frac{\partial p}{\partial q}\right)_{p_1} = \frac{\beta^2\phi'}{1-\alpha\beta\,\phi'} \tag{4-16}$$

$$\left(\frac{\partial q_1}{\partial p_1}\right)_{q} = \frac{1}{\rho^2}\,\frac{\alpha^2\phi'}{1-\alpha\beta\,\phi'} \tag{4-17}$$

The integration of the equations

$$\frac{\partial F}{\partial p_1} = \rho^{-1}q + \rho^{-1}\alpha\,\phi' \tag{4-18}$$

$$\frac{\partial F}{\partial q} = \rho^{-1}p_1 + \beta\,\phi' \tag{4-19}$$

gives

$$F = \rho^{-1}p_1 q + \phi - \frac{\alpha\beta}{2}(\phi')^2 \tag{4-20}$$

(5) <u>The Fixed Points of the P-transformation</u>

We consider the P-transformation of the form

$$T: \quad \begin{aligned} p_1 &= \rho\,(p-\beta\theta) \\ q_1 &= \rho^{-1}(q+\alpha\theta) \\ w &= \alpha p + \beta q \end{aligned} \qquad \theta = A\,w^n \tag{5-1}$$

and shall try to find under what condition there exists a fixed point (other than the origin) p = u, q = v with the property

$$\begin{aligned} p_1 &= p = u \\ q_1 &= q = v \end{aligned} \tag{5-2}$$

We discuss first Case II where ρ is real. Defining the real polar coordinates r and φ by putting

$$p = r\cos\varphi \qquad p_1 = r_1\cos\varphi_1$$
$$q = r\sin\varphi \qquad q_1 = r_1\sin\varphi_1 \tag{5-3}$$

we find

$$\theta = r^n\theta_0$$
$$\theta_0 = (\alpha\cos\varphi + \beta\sin\varphi)^n \equiv w_0^{\,n}$$

The equations (5-1) become then

$$\left.\begin{aligned} r_1\cos\varphi_1 &= \rho(r\cos\varphi - \beta r^n\theta_0) \\ r_1\sin\varphi_1 &= \rho^{-1}(r\sin\varphi + \alpha r^n\theta_0) \end{aligned}\right\} \tag{5-4}$$

Eliminating θ_0 we find

$$r(\alpha\cos\varphi + \beta\sin\varphi) = r_1(\alpha\rho^{-1}\cos\varphi_1 + \beta\rho\sin\varphi_1)$$

or

$$\frac{\alpha\cos\varphi + \beta\sin\varphi}{\alpha\rho^{-1}\cos\varphi_1 + \beta\rho\sin\varphi_1} = \frac{r_1}{r} \tag{5-5}$$

For a fixed point $r_1 = r$ and $\varphi_1 = \varphi$, consequently the angle φ must satisfy the condition

$$\frac{\alpha + \beta\,\mathrm{tg}\,\varphi}{\alpha\rho^{-1} + \beta\rho\,\mathrm{tg}\,\varphi} = 1$$

or

$$\mathrm{tg}\,\varphi = \frac{\alpha}{\beta\rho} \tag{5-6}$$

This is a necessary but in general not sufficient condition for a fixed point. We may state this result as a

<u>Theorem</u>: In Case II the fixed points of the P-transformation must all be situated on the ray

$$u = r \cos \varphi$$
$$v = r \sin \varphi \qquad (5\text{-}7)$$

with $\quad tg\, \varphi = \dfrac{\alpha}{\beta \rho}$

The angle φ is determined by (5-6) up to an additive multiple of π.

In order to find the distance of the fixed point from the origin on the ray (5-6) we return to Equ. (5-4) and determine $r = r_1$ from either

$$\cos \varphi = \rho \cos \varphi - \rho \beta r^{n-1} \theta_0$$

or $\quad \sin \varphi = \rho^{-1} \sin \varphi + \rho^{-1} \alpha r^{n-1} \theta_0 .$

$$r^{n-1} = \frac{(\rho - 1) \sin \varphi}{\alpha \theta_0} = \frac{(\rho - 1) \cos \varphi}{\rho \beta \theta_0} \qquad (5\text{-}8)$$

We must have a positive root r for a real fixed point. When ρ is real the condition for this is easily found. We discuss the case n odd ($\neq 1$). In this case r^{n-1} must be positive or

$\dfrac{\sin \varphi}{\alpha \theta_0}$ of the same sign as $\rho - 1$.

When this is expressed in terms of α and β we find

$$\frac{\sin\varphi}{\alpha\theta_0} = \frac{\cos\varphi}{\beta\rho\,\theta_0} = \frac{(\alpha^2+\beta^2\rho^2)^{\frac{n-1}{2}}}{(\alpha\beta)^n}\;\frac{1}{(\rho+1)^n} \tag{5-9}$$

The condition for real and positive r may then be reduced to

$$\frac{\rho-1}{\alpha\beta(\rho+1)} > 0 \tag{5-10}$$

This result is summarized with the

<u>Theorem</u>: The P-transformation with n odd ($\neq 1$) and ρ real has exactly two fixed points outside the origin if and only if either

$$\rho > 1 \;,\quad \alpha\beta > 0$$

or $\rho < -1 \;,\quad \alpha\beta > 0$

or $-1 < \rho < 1 \;,\quad \alpha\beta < 0 \;.$

The coordinates of the fixed points are then

$$u = \pm\beta\rho\left[\frac{\rho-1}{(\alpha\beta)^n(\rho+1)^n}\right]^{\frac{1}{n-1}} ,\quad v = \pm\alpha\left[\frac{\rho-1}{(\alpha\beta)^n(\rho+1)^n}\right]^{\frac{1}{n-1}} \tag{5-11}$$

In Case I we define the real quantities r and φ through

$$p = re^{-i\varphi} \qquad p_1 = r_1 e^{-i\varphi_1} \tag{5-3'}$$
$$q = re^{i\varphi} \qquad q_1 = r_1 e^{i\varphi_1}$$

This leads to

$$\theta = r^n \theta_0$$

$$\theta_0 = i(\alpha e^{-i\varphi} + \beta e^{i\varphi})^n = i w_0^{\,n}$$

The Equ. (5-1) becomes then

$$r_1 e^{-i\varphi_1} = \rho(r e^{-i\varphi} - \beta r^n \theta_0)$$

$$r_1 e^{i\varphi_1} = \rho^{-1}(r e^{i\varphi} + \alpha r^n \theta_0) \,. \qquad (5\text{-}4)'$$

Corresponding to (5-5) we find the necessary condition for a fixed point:

$$e^{2i\varphi} = \frac{\alpha}{\beta\rho} \qquad (5\text{-}6)'$$

The equations which correspond to (5-8) are then

$$r^{n-1} = \frac{\rho^{-1}}{\alpha\theta_0} e^{i\varphi} = \frac{\rho^{-1}}{\rho\beta\theta_0} e^{-i\varphi} \qquad (5\text{-}8)'$$

Because θ_0 is purely imaginary, the last two expressions are conjugate to each other, hence their common value is real. Closer examination shows that for n odd it is always positive, while for n even it is of opposite signs for the two angles φ which satisfy Equ. (5-6)'. Hence we find the

<u>Theorem</u>: The P-transformation with $n \neq 1$ and ρ complex has exactly two fixed points for n odd and exactly one fixed point for n even outside the origin.

We shall now investigate the nature of these fixed points in the Case I for odd n. To this end let

$$u = re^{-i\varphi}$$
$$v = re^{i\varphi} \tag{5-12}$$

be the coordinates of the fixed point ω_0 with r and φ given by (5-6)' and (5-8)'. For a point ω in the neighborhood of ω_0 we write

$$p = u + x$$
$$q = v + y \tag{5-13}$$

where x and y are considered small quantities.

Since

$$\Theta = i(\alpha p + \beta q)^n = i[\alpha(u+x) + \beta(v+q)]^n$$

we find to first order in x and y

$$\Theta = \Theta_0 + in(\alpha x + \beta y) + \dots$$
$$\text{with} \quad \Theta_0 = i(\alpha u + \beta v)^n \tag{5-14}$$

The transformed variables

$$p_1 = u + x_1$$
$$q_1 = v + y_1$$

are then related to x and y by the linear transformation

$$\left.\begin{aligned} x_1 &= (1 - in\alpha\beta)\rho x - in\beta^2\rho y \\ y_1 &= in\alpha^2\rho^{-1}x + (1 + in\alpha\beta)\rho^{-1}y \end{aligned}\right\} \tag{5-15}$$

This fixed point is stable when the characteristic roots of this transformation matrix are complex. The condition for this is

$$\rho+\rho^{-1}+i n\alpha\beta\,(\rho^{-1}-\rho) < 2 \tag{5-16}$$

Since ρ is complex (Case I) we may write

$$\rho = e^{i\psi}$$

Expressed in ψ, condition (5-16) becomes

$$\cos\psi + n\alpha\beta\sin\psi < 1 \tag{5-17}$$

This can be further simplified by introducing the angle δ

$$\sin\delta = \frac{n\alpha\beta}{\sqrt{1+(n\alpha\beta)^2}}$$

We obtain thus for the stability condition

$$\cos(\psi-\delta) < \cos\delta \tag{5-18}$$

The result may be stated with the

Theorem: In Case I for odd n the fixed point ω_0 is <u>unstable</u> if

$$0 \leq \frac{1}{i}\ln\rho \leq \delta \qquad \sin\delta = \frac{n\alpha\beta}{\sqrt{1+(n\alpha\beta)^2}}$$

Otherwise it is stable.

The same is true for the second fixed point $-\omega_0$. The stability of the two fixed points ω_0 and $-\omega_0$ will determine to a considerable extent the topological structure of

the lines $\Gamma(pq)$ = const. This property is therefore of **considerable importance in determining the stability** limits of the canonical transformation T.

The topographic system Γ = const. is expected to be of the general structure indicated in Figs. 1 and 2 for the two cases. The shaded area indicates the stable region surrounding the origin

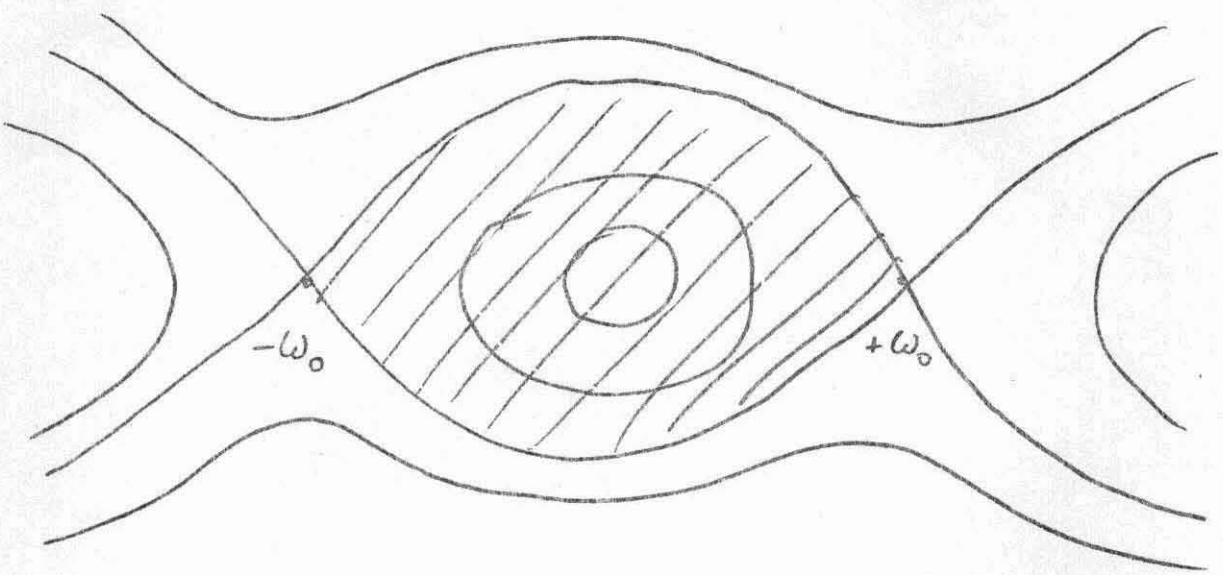


Fig. 1 The system Γ = const. for an <u>unstable</u> fixed point ω_0.

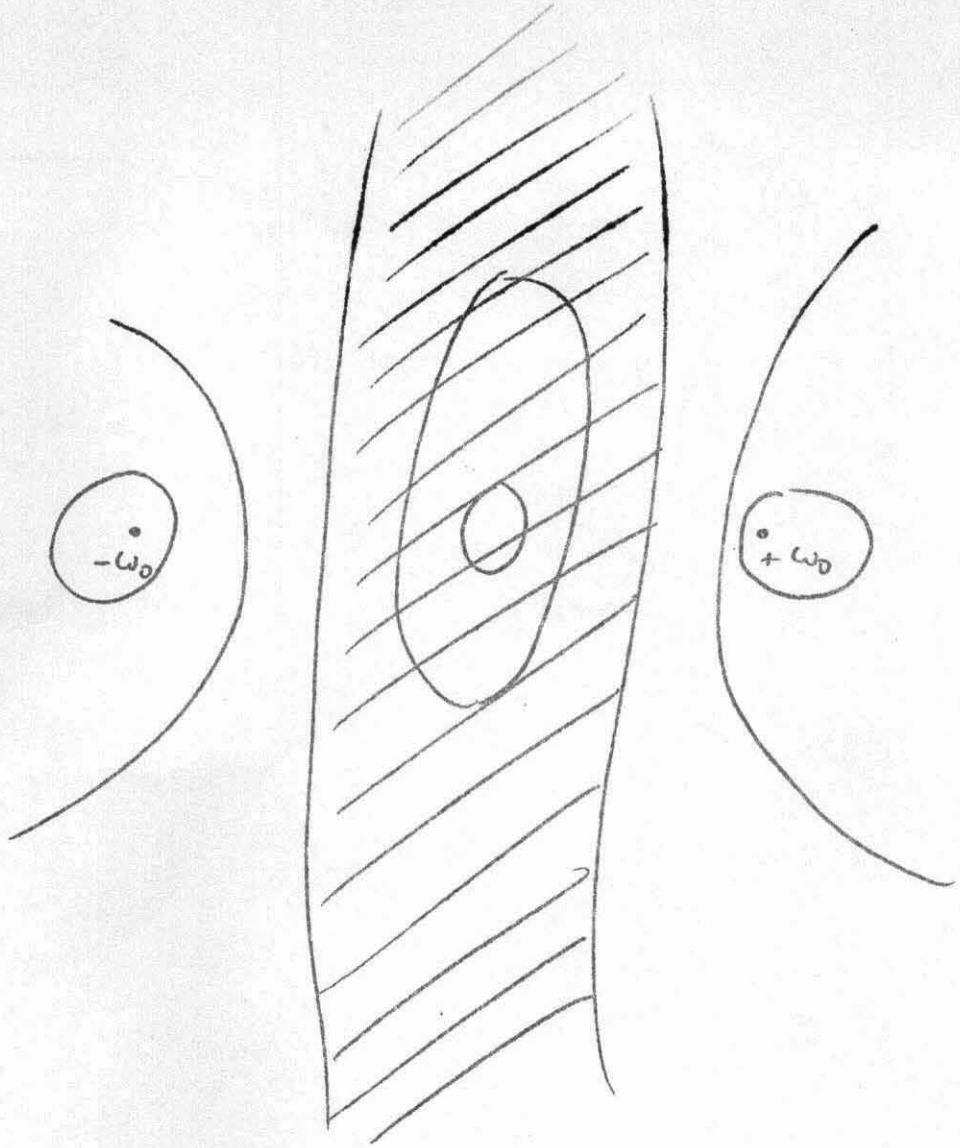


Fig. 2 The system Γ = const. for a <u>stable</u> fixed point ω_0

(6) Fixed Points of the Iterated Transformations

In this section we shall give a rigorous proof of a conjecture by Powell[1] that for a linear transformation

1. J. L. Powell, MAC-JLP-3, March 30, 1954.

T the fixed points of T^n are everywhere dense in a suitably restricted neighborhood of 0.

We begin with a few definitions and preliminaries. The region of points ω in the neighborhood of a stable point 0 such that the curve

$$\Gamma(pq) = \Gamma(\omega)$$

is bounded is called the stable neighborhood U_s of the stable point 0.

A curve C defined by

C: $\Gamma(\omega) = A$

where A is a constant is called an integral curve of T. It is said to be in U_s if all its points are in U_s.

$C \in U_s$ if $\omega \in U_s$ for all ω such that $\Gamma(\omega) = A$. An integral curve in U_s is also called a stable integral curve. Let us now consider such a stable integral curve. To every such curve we can assign a real number κ between 0 and 1 which we shall call its character. This number

is defined in the following way. According to the theorem of part I there exists a canonical transformation such that in the neighborhood of 0 the transformation T is in the form

$$P_1 = \rho(P \cdot Q)\, P$$

$$Q_1 = \rho^{-1}(P \cdot Q)\, Q$$

and $\rho(P \cdot Q)$ is a function of theproduct PQ only. In the stable case P and Q are complex conjugate and hence

$$\rho^*(PQ) = \rho^{-1}(P \cdot Q)$$

Thus there exists a real phase angle $\psi(PQ)$ such that

$$\rho(PQ) = e^{i\psi(PQ)} \qquad (0 \leq \psi < 2\pi)$$

We define

$$\kappa = \frac{\psi(PQ)}{2\pi}$$

Since PQ is the invariant function κ is a constant on each curve Γ = const. We find immediately: the character of T^n is κ' where

$$\kappa' \equiv n\kappa \pmod 1$$

The curve C contains a periodic point of order s if and only if κ is rational and s is the smallest denominator:

$$\kappa = \frac{r}{s}$$

(r,s integers without common divisor)

Indeed for a fixed point of order s,

$$s\kappa = \text{integer} = r$$

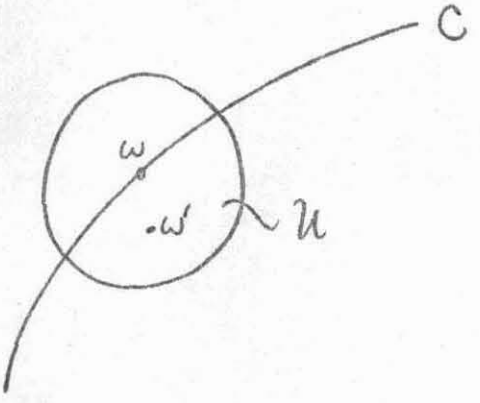
therefore
$$\kappa = \frac{r}{s}$$

If C contains one periodic point then all its points are periodic of the same order. This is so because κ if a function of the curve only and its value is independent of the point on the curve.

We are now prepared to formulate the

<u>Theorem</u>: For every non-linear canonical transformation, which is analytical in the neighborhood of the stable fixed point 0 there exists a neighborhood U of 0 such that in U the periodic points are everywhere dense.

<u>Proof</u>: Let ω be any point in U_s, let C be the closed integral curve through C and κ its character. We wish to prove that every neighborhood U of ω contains at least one periodic point. Now the function $\kappa(\omega)$ is analytical and it is not identically a constant, because the transformation T is analytical and non-linear. Therefore there exists a point ω' in U such that $\kappa' \equiv \kappa(\omega') \neq \kappa(\omega) = \kappa$. Since $\kappa(\omega)$ is analytical it is also continuous. Therefore there exist



points ω' such that $\kappa(\omega')$ assumes any value between and κ' and κ. In particular there exists at least one ω' in U such that $\kappa(\omega')$ is rational.

According to our preceding remarks this point is periodic, q.e.d.

Remarks:

(a) It seems fairly certain that the assumption of analyticity is much too strong for the validity of the theorem. We shall however not try to prove the theorem in its most general form.

(b) The assumption of non-linearity is essential. Indeed for a linear transformation ψ is a constant and therefore if $\kappa = \frac{\psi}{2\pi}$ is irrational there exists a neighborhood of 0 which is entirely free of periodic points.