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MURA-KRS-7

FFAG WITH SPIRAL POLES - SMOOTH APPROXIMATION*

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1. Equations of motion.

The equation of motion of a particle in a magnetic field is

$$m\frac{d\vec{v}}{dt} + \vec{v}\,\frac{dm}{dt} = \frac{e}{c}\,\vec{v} \times \vec{H} \tag{1}$$

We will omit the second term on the left, which gives rise to adiabatic damping. We assume that the orbit lies near a reference circle of radius r, and we use polar coordinator x, y, θ as shown in Fig. 1. We will write:

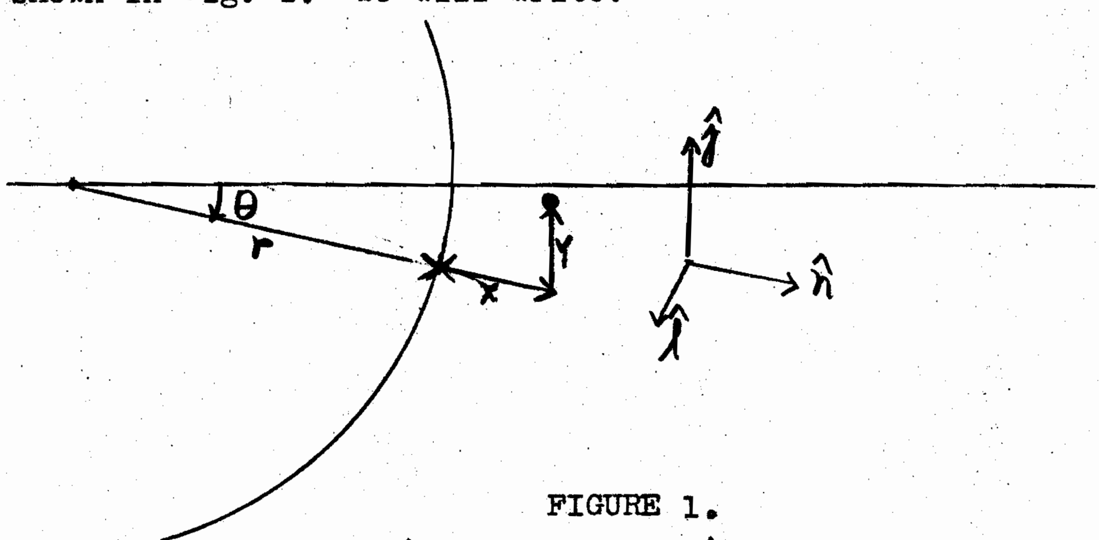

FIGURE 1.

$$\vec{H} = H_y\hat{j} + H_x\hat{n} + H_\theta\hat{l}, \tag{2}$$

$$\vec{V} = v\hat{l} + \dot{x}\hat{n} + \dot{y}\hat{j}. \tag{3}$$

Substituting Equations (2) and (3) in Equation (1), we obtain

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three component equations:

$$\ddot{x} - \frac{v^2}{r+x} = -\frac{ev}{mc} H_Y + \frac{e\dot{y}}{mc} H_\theta - \frac{\dot{x}}{m}\frac{dm}{dt}, \qquad (4)$$

$$\ddot{y} = \frac{ev}{mc} H_x - \frac{e\dot{x}}{mc} H_\theta - \frac{\dot{y}}{m}\frac{dm}{dt}, \qquad (5)$$

$$\dot{v} + \frac{v\dot{x}}{r+x} = \frac{e\dot{x}}{mc} H_Y - \frac{e\dot{y}}{mc} H_x . \qquad (6)$$

We will neglect the last two terms on the right in Eqs. (4) and (5), and will neglect $\dot{x}$, $\dot{y}$ in comparison with v, so that v becomes the constant speed of the particle. We now introduce θ as independent variable:

$$vdt = (r + x)\, d\theta, \qquad (7)$$

so that Eqs. (4) and (5) become:

$$\frac{v}{r+x}\frac{d}{d\theta}\frac{v}{r+x}\frac{dx}{d\theta} - \frac{v^2}{r+x} = -\frac{ev}{mc} H_Y \qquad (8)$$

$$\frac{v}{r+x}\frac{d}{d\theta}\frac{v}{r+x}\frac{dY}{d\theta} = \frac{ev}{mc} H_x \qquad (9)$$

which reduce to

$$x'' = -\frac{H_Y}{H\rho}(r+x)^2 + (r+x)\left[1 + \left(\frac{x'}{r+x}\right)^2\right] \qquad (10)$$

$$Y'' = \frac{H_x}{H\rho}(r+x)^2 + (r+x)\left(\frac{Y'}{r+x}\right)^2 . \qquad (11)$$

We will neglect the terms involving $\left(\frac{x'}{r}\right)^2$, $\left(\frac{Y'}{r}\right)^2$. We now assume that the mean field $\bar{H}$ on the reference circle is such as to maintain the particle on that circle if it alone were acting:

$$\frac{\bar{H}r^2}{H\rho} = r. \qquad (12)$$

Equations (10) and (11) can then be written:

$$x'' = \frac{(\bar{H} - H_Y)r^2 - 2rxH_Y - H_Y x^2}{H\rho} + x \qquad (13)$$

$$Y'' = \frac{H_x(r+x)^2}{H\rho} \qquad (14)$$

We now assume that x,y are small, and expand the field in powers of x, y, keeping only first order terms, and assuming $H_x = 0$ in the horizontal plane:

$$H_Y = H(r,\theta) + H_r(r,\theta)x \tag{15}$$

$$H_x = H_r(r,\theta)Y \tag{16}$$

Where the subscript "r" denotes partial differentiation. To first order in x, y, the equations of motion are now:

$$\frac{d^2x}{d\theta^2} + \left[\frac{r^2 H_r(r,\theta)}{H\rho} + \frac{2r}{\rho} - 1\right]x = \frac{r^2[\bar{H}(r) - H(r,\theta)]}{H\rho} \tag{17}$$

$$\frac{d^2Y}{d\theta^2} - \frac{r^2 H_r(r,\theta)}{H\rho}Y = 0 \tag{18}$$

The solution of Eq. (17) can be written as a steady state solution, which represents the deviation of the equilibrium orbit from the reference circle (scallops) plus a solution of the homogeneous equation which represents the betatron oscillations. Hence in a linear approximation, scalloping of the equilibrium orbit has no effect on the betatron oscillations. We will neglect the last two terms in brackets on the left in Eq. (17), which is valid if $m \gg 1$, and write the betatron focussing equations in the form:

$$x'' = -\frac{r^2 H_r}{H\rho}x, \tag{19}$$

$$Y'' = \frac{r^2 H_r}{H\rho}Y \tag{20}$$

2. The smooth approximation for force functions of arbitrary shape. Let us consider an equation of the form:

$$x'' = f(\theta)x, \tag{21}$$

where $f(\theta)$ is a force function with N sectors in the period 2π, so that we can write the Fourier series:

$$f(\theta) = \bar{f} + \sum_{g=1}^{\infty} A_g \sin(gN\theta - \delta_g)] \tag{22}$$

According to KRS(MURA)-4, an approximate solution of Eq. (22) is:

$$x = X + \xi \tag{23}$$

where

$$\xi(\theta) = X \iint \sum_g A_g \sin(gN\theta - \delta_g)\, d\theta\, d\theta \tag{24}$$

and X is a solution of

$$X'' = \bar{f} X + \overline{f(\theta)\xi} \tag{25}$$

The integration constants in ξ are to be chosen so that ξ is periodic with zero mean:

$$\xi = -X \sum_\gamma \frac{A_\gamma}{\gamma N^2} \sin(\gamma N\theta - \delta_\gamma) \tag{26}$$

The cross terms in $f\xi$ drop out when we average over θ, so that we obtain:

$$X'' = \left[\bar{f} - \sum_\gamma \frac{A_\gamma^2}{2\gamma^2 N^2}\right] X \tag{27}$$

3. Application to spirally ridged fields.

Let us assume that the magnetic field in the median plane of a spiral ridge machine with N sectors around the circumference is given by:

$$H(r,\theta) = H_0 F(r)\left\{1 + \sum_\gamma f_\gamma \sin \gamma N[\theta - \phi(r)]\right\} \tag{28}$$

Where the f_γ are constant coefficients which determine the shape of the field as a function of θ, $\phi(r)$ is a phase function which determines the spiral shape of the ridges, and F(r) is a function determining the radial dependence of the field. We will require, at the maximum energy orbit,:

$$F(r_0) = 1, \quad \phi(r_0) = 0 \tag{29}$$

We now calculate:

$$H_r = H_0 F'(r)\{\ \} - H_0 F(r)\phi'(r) \sum_\gamma \gamma N f_\gamma \cos\gamma N(\theta - \phi) \tag{30}$$

In order that $H_r(r,\theta)$ shall have the same functional dependence on θ at all radii, we require:

$$F\phi' = -KF' \tag{31}$$

where K is constant, so that:

$$\phi = -K \ln F \,. \tag{32}$$

Equation (30) then becomes:

$$H_r = H_0 F'(r)\left\{1 + \sum_\gamma f_\gamma \sqrt{1+\gamma^2 K^2 N^2}\, \sin(\gamma N\theta - \gamma N\phi + \delta_\gamma)\right\} \tag{33}$$

where:

$$\tan \delta_j = j K N \tag{34}$$

We substitute in Eqs. (19) and (20), compare with Eq. (21), and use Eq. (27). The equations for the smooth betatron oscillations are then:

$$X'' = -\omega_x^2 X \quad , \quad Y'' = -\omega_y^2 Y, \tag{35}$$

with:

$$\omega_x^2 = k + \frac{k^2 K^2}{2} \sum_j f_j^2 \left(1 + \frac{1}{j^2 K^2 N^2}\right) \tag{36}$$

$$\omega_y^2 = -k + \frac{k^2 K^2}{2} \sum_j f_j^2 \left(1 + \frac{1}{j^2 K^2 N^2}\right) \tag{37}$$

where we have set:

$$k = \frac{r^2 H_0 F'}{H\rho} = \frac{r H_0 F'}{H} = \frac{r F'}{F} \tag{38}$$

The phase shifts per sector are given by:

$$\sigma_x = \frac{2\pi}{N} \omega_x \quad , \quad \sigma_y = \frac{2\pi}{N} \omega_y \ . \tag{39}$$

If we require that σ_x, σ_y be independent of energy, then k must be constant, and the function F is determined by Eq. (38):

$$F = \left(\frac{r}{r_0}\right)^k . \tag{40}$$

The function ϕ is then, by Eq. (32):

$$\phi = -K k \ln r/r_0 \tag{41}$$

We will show later that KN is of the order of the number of ridges counting radially outward across the donut, say $\gtrsim$ 10. If we assume $j^2K^2N^2 \gg 1$, we have the very simple result:

$$\omega_x^2 = k + \overline{f^2} k^2 K^2 , \tag{42}$$

$$\omega_y^2 = -k + \overline{f^2} k^2 K^2 , \tag{43}$$

where $f(\theta)$ is the periodic part of the azimuthal field dependence:

$$f(\theta) = \sum_j f_j \sin jN\theta \tag{44}$$

If we substitute Eqs. (42), (43) in (39), and solve for $\overline{f^2}$, k, we obtain, for a given N, K, σ_x, σ_y:

$$k = \frac{N^2}{8\pi^2}\left(\sigma_x^2 - \sigma_y^2\right) \tag{45}$$

$$\overline{f^2} = \frac{8\pi^2}{N^2 k^2}\frac{\sigma_x^2 + \sigma_y^2}{(\sigma_x^2 - \sigma_y^2)^2} \tag{46}$$

If we set:

$$\Delta = \sigma_y/\sigma_x \,, \tag{47}$$

$$\Lambda = 2\pi/\sigma_x \,, \tag{48}$$

where Λ is the number sectors per radial betatron wavelength, these can be written:

$$k = \frac{1}{2}\left(\frac{N}{\Lambda}\right)^2 (1-\Delta^2) \tag{49}$$

$$\overline{f^2} = \frac{2}{K^2}\left(\frac{\Lambda}{N}\right)^2 \frac{1+\Delta^2}{(1-\Delta^2)^2} \tag{50}$$

The number of ridges counted radially across the donut is:

$$m = \frac{N\phi_i}{2\pi} = \frac{Nk\,\ln F_i}{2\pi} = \epsilon N k \,, \tag{51}$$

where the subscript i refers to the values at injection, and:

$$\epsilon = \frac{-\ln F_i}{2\pi} \doteq \frac{\ln p_0/p_i}{2\pi} \sim 1 \,, \tag{52}$$

if $p_o \sim 400 p_i$, and $|r_o - r_i| \ll r_o$. We can then write Eq. (50) in terms of m:

$$\overline{f^2} = \frac{2\epsilon^2\Lambda^2}{m^2}\frac{1+\Delta^2}{(1-\Delta^2)^2} \tag{53}$$

The circumference factor for the spiral pole machine is:

$$C = \frac{H_{max}}{\bar{H}} = 1 + f_{max} \tag{54}$$

where f_{max} is the maximum value of the function defined by Eq. (44). The function $f(\theta)$ which yields a minimum f_{max} for given $\overline{f^2}$ is the square wave:

$$f(\theta) = \begin{cases} +f \,, & 0 < gN\theta < \pi/2 \,, \\ -f \,, & \pi/2 < gN\theta < \pi \,, \end{cases} \tag{55}$$

with: $$f = \sqrt{\overline{f^2}} = \frac{\sqrt{2}\,\epsilon\,\Lambda}{m}\,\frac{\sqrt{1+\Delta^2}}{1-\Delta^2} \qquad (56)$$

The circumference factor is then:

$$C = 1 + \frac{\sqrt{2}\,\epsilon\,\Lambda}{m}\,\frac{\sqrt{1+\Delta^2}}{1-\Delta^2} \qquad (57)$$

We note that, happily, it does not help much to use very small values of Δ. If we take $\Delta = \frac{1}{2}$, $\Lambda = 4$ (in order that the smooth approximation be at least roughly correct), we obtain:

$$C = 1 + \frac{8.5\,\epsilon}{m} \qquad (58)$$

Thus with $m \sim 15$, we get $C \sim 1.5$, and hence an overall radius no larger than in conventional AG machines.

It should be pointed out that non-linear effects may be much more important in the spiral pole machine, so that the above conclusions based on a linear smooth approximation must be regarded as provisional.

For a 20 Bev FFAG synchrotron

Take $\Lambda = 4$ so that the smooth approximation is valid

and $\Delta = 1/2$

Then $kN = m/\epsilon$ BY (51)

and by (56) $\overline{f^2} = 71\,\epsilon^2/m^2$

or $\sqrt{\overline{f^2}} = 8.5\,/m$ with $\epsilon \sim 1$ FOR $p_0/p_i \sim 400$

For $f(\theta) = \pm f$, a step function, we have

$f \cong 8.5/m$. So if $m = 25$ ridges <u>across</u> pole,

$f \cong .34$ or a 34% step up and the same step down must be built into the field, and $C = 1.35$.

Although the smooth approximation doesn't hold for Λ small, we can see the trend if we take $\Lambda = 12/5$, $\Delta = \frac{\pi}{6} / \frac{5\pi}{6}$

Then $f \cong 3.4/m = .14$ for 25 ridges <u>across</u>
$C = 1.14$

If we take $k = 100$ and $p_o/p_i = 400$, then $(r_o/r_i)^{k+1} = p_o/p_i$ and $r_o/r_i = 1.061$. At $r_o = 10^4$ cm this means a 6. meter radial aperture with N becoming 35 ridges <u>around</u> the machine; but if we take $k = 500$, then $r_o/r_i = 1.012$ or a 1.2 meter radial aperture with 77 ridges around the machine results.

By (42), (43) and (39) it is clear that the sign of k can be reversed and σ_x and σ_y just exchange numerical values. If k is negative, we have negative momentum compaction-that is the high energy orbit has a smaller radius than the low energy orbit. The advantage of this is that a frequency modulated synchrotron using such a magnet would have no transition energy because the particles would always increase their angular velocity about the machines as their energy increased.