MURA 40

MURA NOTES

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CONCERNING THE ATTAINMENT OF STABLE ORBITS WITH NEGATIVE MOMENTUM COMPACTION

1. Motivation:

At recent meetings of the MURA technical group it has been pointed out that it would be desirable to have a machine with stable orbits for which $\alpha \equiv \frac{d\langle X\rangle}{dp/p_0}$ (where $X = r - r_0$) is negative. If such a machine were possible, it would possess the definite advantage of removing the transition-energy problem.

Although an ordinary alternate-gradient machine with $\alpha < 0$ may be contrived readily [f. ex. (Fig. 1), $n_2 = 4n_1$, $\theta_2 = (1/2)\theta_1$, and $\psi_1 = \psi_2 = 1.3$, with indices 1 and 2 designating respectively the radially focusing and defocusing sectors, for which $n_1\alpha = -3.053$], there can appear instability in such cases. This situation in regard to radial stability or instability is perhaps not surprising, as the following argument demonstrates that α passes through infinity to attain negative values (cf. Fig. 2) at just those values for the parameters of an ordinary A-G machine for which $\cos\sigma_r = 1$:

With $\xi = n_2/n_1$, $\tau = \psi_2/\psi_1$,

$$n_1\alpha = \frac{\sqrt{\xi} - \tau/\sqrt{\xi}}{\sqrt{\xi} + \tau} + \frac{1}{(\psi_1/2)}\,\frac{(1+\xi)^2}{\xi(\sqrt{\xi}+\tau)}\,\frac{1}{\mathrm{Ctnh}(\psi_2/2) - \sqrt{\xi}\,\mathrm{ctn}(\psi_1/2)}$$

and $\alpha = \infty$ when

$$\sqrt{\xi}\,\mathrm{Tanh}\frac{\psi_2}{2} = \tan\frac{\psi_1}{2} \qquad \text{or} \qquad \sqrt{\xi} = \frac{\tan\frac{\psi_1}{2}}{\mathrm{Tanh}\frac{\psi_2}{2}}.$$

Now

$$\cos\sigma_r = \frac{\xi - 1}{2\sqrt{\xi}}\,\mathrm{Sinh}\,\psi_2\,\sin\psi_1 + \mathrm{Cosh}\,\psi_2\cos\psi_1$$

and, in the present instance ($\alpha = \infty$)

$$\frac{\xi - 1}{2\sqrt{\xi}} = \frac{\frac{\tan^2(\psi_1/2)}{\mathrm{Tanh}^2(\psi_2/2)} - 1}{2\,\frac{\tan(\psi_1/2)}{\mathrm{Tanh}(\psi_2/2)}} = \frac{\tan^2(\psi_1/2) - \mathrm{Tanh}^2(\psi_2/2)}{2\tan(\psi_1/2)\,\mathrm{Tanh}(\psi_2/2)}$$

$$= \frac{\frac{(1-\cos\psi_1)^2}{\sin^2\psi_1} - \frac{(\mathrm{Cosh}\,\psi_2 - 1)^2}{\mathrm{Sinh}^2\psi_2}}{2\,\frac{1-\cos\psi_1}{\sin\psi_1}\,\frac{\mathrm{Cosh}\,\psi_2 - 1}{\mathrm{Sinh}\,\psi_2}};$$

hence

$$\cos\sigma_r = \frac{(1-\cos\psi_1)^2\,\mathrm{Sinh}^2\psi_2 - (\mathrm{Cosh}\,\psi_2 - 1)^2\sin^2\psi_1}{2\,(1-\cos\psi_1)(\mathrm{Cosh}\,\psi_2 - 1)} + \mathrm{Cosh}\,\psi_2\cos\psi_1$$

$$= \frac{(1-\cos\psi_1)^2(\cosh^2\psi_2 - 1) - (\cosh\psi_2 - 1)^2(1-\cos^2\psi_1)}{2\,(1-\cos\psi_1)\,(\cosh\psi_2 - 1)} + \cosh\psi_2\cos\psi_1$$

$$= \frac{1}{2}\left[(1-\cos\psi_1)(\cosh\psi_2 + 1) - (\cosh\psi_2 - 1)(1+\cos\psi_1)\right] + \cosh\psi_2\cos\psi_1$$

$$= 1 .$$

It appears of interest, therefore, to examine whether the existence of stable orbits always requires α to be positive or, if not, to present some specific example of stable motion with negative α. We undertake to examine this question, for radial motion, in the following section.

2. General Expression for α:

A recent BNL report by Courant and Snyder [EDC-15] indicates that stable solutions of the differential equation

$$\frac{d^2X}{d\tau^2} + F(\tau)\,X = 0 ,$$

where $F(\tau)$ is periodic with period T, may be written as

$$x(\tau) = w(\tau)\,e^{\pm i\,\phi(\tau)} , \quad \text{or} \quad w(\tau)\begin{cases}\sin \\ \text{OR} \\ \cos\end{cases}\phi(\tau) ,$$

where $\phi(\tau) = \frac{\sigma}{T}\tau + \psi(\tau)$,

$w(\tau)$ and $\psi(\tau)$ are real functions, each periodic with period T, and σ is a real constant.

The constancy of the Wronskin insures that $[w(\tau)]^2\frac{d\phi}{d\tau} = K^2$, a constant.

A periodic solution (period T) of the inhomogeneous equation

$$\frac{d^2X}{d\tau^2} + F(\tau)\,X = a , \quad \text{where } a \text{ is a constant},$$

may be obtained by inductive reasoning [see Appendix -- the writer is indebted to Dr. B. C. Carlson for discussions on this point] or confirmed by substitution:

$$X(\tau) = \frac{a}{2K^6\sin\frac{\sigma}{2}}\int_{\tau-T}^{\tau}[w(\tau)]^3[w(\tau^x)]^3\,\phi'(\tau)\phi'(\tau^x)\cos\left[(\phi-\phi^x) - \frac{\sigma}{2}\right]d\tau^x .$$

Hence

$$\langle X\rangle = \frac{a}{2K^6 T\sin\frac{\sigma}{2}}\int_0^{\sigma}d\phi\int_{\phi-\sigma}^{\phi}d\phi^x\,w^3\,w^{x3}\cos(\phi-\phi^x-\frac{\sigma}{2})$$

$$= \frac{a}{K^6 T\sin\frac{\sigma}{2}}\int_0^{\sigma}d\phi\int_{\phi-\sigma}^{0}d\phi^x\,w^3\,w^{x3}\cos(\phi-\phi^x-\frac{\sigma}{2}) \qquad (1)$$

$$= \frac{a}{K^2\,T\sin\frac{\sigma}{2}}\int_0^{\sigma}d\phi\int_0^{\phi}d\phi^x\,w^3\,w^{x3}\cos(\phi^x-\phi+\frac{\sigma}{2}) .$$

If, regarding w as a periodic function of ϕ with period σ, we write

$$w^3(\phi) = A_o + \sum_{k=1}^{\infty} \left[A_k \cos \frac{2\pi k}{\sigma}\phi + B_k \sin \frac{2\pi k}{\sigma}\phi \right]. \qquad (2)$$

we find, from (1),

$$\langle X \rangle = a \frac{\sigma}{K^6 T} \left[A_o^{\,2} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{A_k^{\,2} + B_k^{\,2}}{\left(\frac{2\pi k}{\sigma}\right)^2 - 1} \right],$$

or

$$\alpha = \frac{\langle X \rangle}{a} = \frac{\sigma}{K^6 T} \left[A_o^{\,2} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{A_k^{\,2} + B_k^{\,2}}{\left(\frac{2\pi k}{\sigma}\right)^2 - 1} \right]. \qquad (3)$$

This result may be checked for an ordinary (non A-G) synchrotron, for which $A_o = C^3$, $\sigma = (1-n)^{1/2} T$, $K^2 = C^2 (1-n)^{1/2}$, $K^6 = C^6 (1-n)^{3/2}$:

$$\alpha = \frac{\langle X \rangle}{a} = \frac{(1-n)^{1/2}\, T\, C^6}{C^6 (1-n)^{3/2}\, T} = \frac{1}{1-n},$$ in agreement with the well-known result.

3. Selection of Orbits for which $\alpha < 0$:

The last result suggests a means for determining certain situations in which α is negative for radially stable orbits, the coefficients of the Fourier expansion for w^3 [eqn. (2)] being selected to achieve this condition.

With the functional dependence of w on ϕ thus selected, the relation $w^2 \frac{d\phi}{d\tau} = K^2$ should, if the integration can be performed, give the form of the relation connecting ϕ and τ. w can then be expressed also as a function of τ.

Finally, the required form for the function $F(\tau)$ appearing in the original differential equation characterizing the machine, is determined by

$$w \phi'^2 - w'' = wF,$$

$$\text{or} \quad F = \phi'^2 - \frac{w''}{w},$$ primes denoting differentiation with respect to τ.

4. Example:

A simple, albeit strange, example of radially stable motion with negative α may be constructed by the methods outlined in section 3. We take

$$w(\phi) = 1 + E\cos\frac{\phi}{\epsilon}, \quad \text{with } E = 0.8 \text{ and } \epsilon = 0.8 \ [\text{or } \sigma = 0.8\,(2\pi)];$$

$$w(\phi) = 1 + 0.8\cos 1.25\phi$$

$$w^3(\phi) = 1 + 2.4\cos 1.25\phi + 1.92\cos^2 1.25\phi + 0.512\cos^3 1.25\phi$$

$$= 1.96 + 2.784\cos 1.25\phi + 0.96\cos 2.5\phi + 0.128\cos 3.75\phi \; .$$

For this function we would expect, from eq. (3), that

$$\alpha = \frac{\sigma}{K^6 T}\left\{(1.96)^2 - \frac{1}{2}\left[\frac{(2.784)^2}{0.5625} + \frac{(0.96)^2}{5.25} + \frac{(0.128)^2}{13.0625}\right]\right\}$$

$$= -3.136\frac{\sigma}{K^6 T} = -3.136\frac{0.8(2\pi)}{K^6 T} \; .$$

The relation between ϕ and τ is determined by the relation

$$w^2\frac{d\phi}{d\tau} = K^2 :$$

$$\tau = \frac{1}{K^2}\int^{\phi} w^2\, d\phi$$

$$= \frac{1}{K^2}\int^{\phi}(1 + 0.8\cos 1.25\phi)^2\, d\phi$$

$$= \frac{1}{K^2}\left[\text{const.} + 1.32\phi + 1.28\sin 1.25\phi + 0.128\sin 2.5\phi\right],$$

$$\text{with} \quad T = \frac{1.32 \times 0.8 \times 2\pi}{K^2} = 1.056\,\frac{2\pi}{K^2} \qquad \text{or}$$

$$K^2 = 1.056\frac{2\pi}{T} \; .$$

The function F is given parametrically, in terms of ϕ, by

$$F = \left(\frac{d\phi}{d\tau}\right)^2 - \frac{1}{w}\frac{d^2w}{d\tau^2}$$

$$= \frac{\frac{d\tau}{d\phi}\left(w - \frac{d^2w}{d\phi^2}\right) + \frac{dw}{d\phi}\frac{d^2\tau}{d\phi^2}}{w\,(d\tau/d\phi)^3}$$

$$= \frac{K^4}{w^6}\left[w\left(w - \frac{d^2w}{d\phi^2}\right) + 2\left(\frac{dw}{d\phi}\right)^2\right]$$

$$= (1.056)^2\left(\frac{2\pi}{T}\right)^2\frac{3 + 2.85\cos 1.25\phi - 0.36\cos^2 1.25\phi}{(1 + 0.8\cos 1.25\phi)^6}$$

$$= (1.056)^2\left(\frac{2\pi}{T}\right)^2\frac{2.82 + 2.85\cos 1.25\phi - 0.18\cos 2.5\phi}{(1 + 0.8\cos 1.25\phi)^6} \; .$$

We thus have the differential equation

$$\frac{d^2X}{d\tau^2} + F\cdot X = 0$$

with F expressed as

$$F = \left(\frac{2.112\pi}{T}\right)^2 \frac{3 + 2.85\cos 1.25\phi - 0.36\cos^2 1.25\phi}{(1 + 0.8\cos 1.25\phi)^6},$$

$$\tau = \frac{T}{2.112\pi}\left[\text{const.} + 1.32\phi + 1.28\sin 1.25\phi + 0.128\sin 2.5\phi\right].$$

The solution to this homogeneous equation is

$$X = (1 + 0.8\cos 1.25\phi)\, e^{\pm i\phi}.$$

This result may be checked directly by confirming that

$$\frac{\frac{d\tau}{d\phi}\frac{d^2X}{d\phi^2} - \frac{dX}{d\phi}\frac{d^2\tau}{d\phi^2}}{\left(\frac{d\tau}{d\phi}\right)^3} + F\cdot X = 0,$$

it being noted that

$$\frac{d\tau}{d\phi} = \left(\frac{T}{2.112\pi}\right)(1 + 1.6\cos 1.25\phi + 0.64\cos^2 1.25\phi)$$

$$= \frac{T}{2.112\pi}(1 + 0.8\cos 1.25\phi)^2 = \frac{T}{2.112\pi} w^2.$$

The solution to the inhomogeneous equation

$$\frac{d^2X}{d\tau^2} + F\cdot X = a$$ [the right hand side being taken as a constant, representing $r_o\,(\Delta p/p_o)$]

is obtained from eq. (A-2) of the Appendix as

$$X = \left(\frac{T}{2.112\pi}\right)^3 a\left(\frac{d\phi}{d\tau}\right)(1 + 0.8\cos 1.25\phi)^3(1.96 - 4.9493\cos 1.25\phi - 0.1829\cos 2.5\phi - 0.0098\cos 3.75\phi)$$

$$= \left(\frac{T}{2.112\pi}\right)^3 a\left(\frac{d\phi}{d\tau}\right)(1 + 0.8\cos 1.25\phi)^3(2.1428 - 4.9199\cos 1.25\phi - 0.3656\cos^2 1.25\phi - 0.0392\cos^3 1.25\phi)$$

$$= \left(\frac{T}{2.112\pi}\right)^2 a\,(1 + 0.8\cos 1.25\phi)(1.96 - 4.9493\cos 1.25\phi - 0.1828\cos 2.5\phi - 0.0098\cos 3.75\phi)$$

$$= \left(\frac{T}{2.112\pi}\right)^2 a\,(1 + 0.8\cos 1.25\phi)(2.1428 - 4.9199\cos 1.25\phi - 0.3656\cos^2 1.25\phi - 0.0392\cos^3 1.25\phi). \quad (5)$$

The average value of X is obtainable in a direct manner from eq. (4), to yield

$$\alpha = \frac{\langle X \rangle}{a} = \frac{1}{T} \int_{\phi = 0}^{\phi = \sigma = 0.8(2\pi)} \left(\frac{T}{2.112\pi}\right)^3 (1 + 0.8 \cos 1.25\phi)^3 (2.1428 - 4.9199 \cos 1.25\phi - 0.3656 \cos^2 1.25\phi - 0.0392 \cos^3 1.25\phi)\, d\phi$$

$$= \frac{0.8(2\pi)}{T} \left(\frac{T}{2.112\pi}\right)^3 (-3.136)$$

$$= -3.136 \frac{0.8(2\pi)}{K^6 T} , \quad \text{as expected.}$$

The validity of the expression (5) as a solution of the inhomogeneous equation may be checked directly by confirming that

$$\frac{\frac{d\tau}{d\phi} \frac{d^2X}{d\phi^2} - \frac{dX}{d\phi} \frac{d^2\tau}{d\phi^2}}{\left(\frac{d\tau}{d\phi}\right)^3} + F \cdot X = a .$$

A graph of the phase parameter ϕ, vs. τ, is shown in Fig. 3. The function F is plotted in Figs. 4 and 5. The function w, the solutions $w \cos \phi$, $w \sin \phi$, and the periodic solution of the inhomogeneous equation are shown in Figs. 6 and 7.

5. Conclusion:

It appears from the foregoing development and example that it is not impossible to have radially stable motion associated with negative momentum compaction. Whether axial stability can also be achieved (in a Maxwell field) and practicable conditions achieved with negative α remains to be investigated.

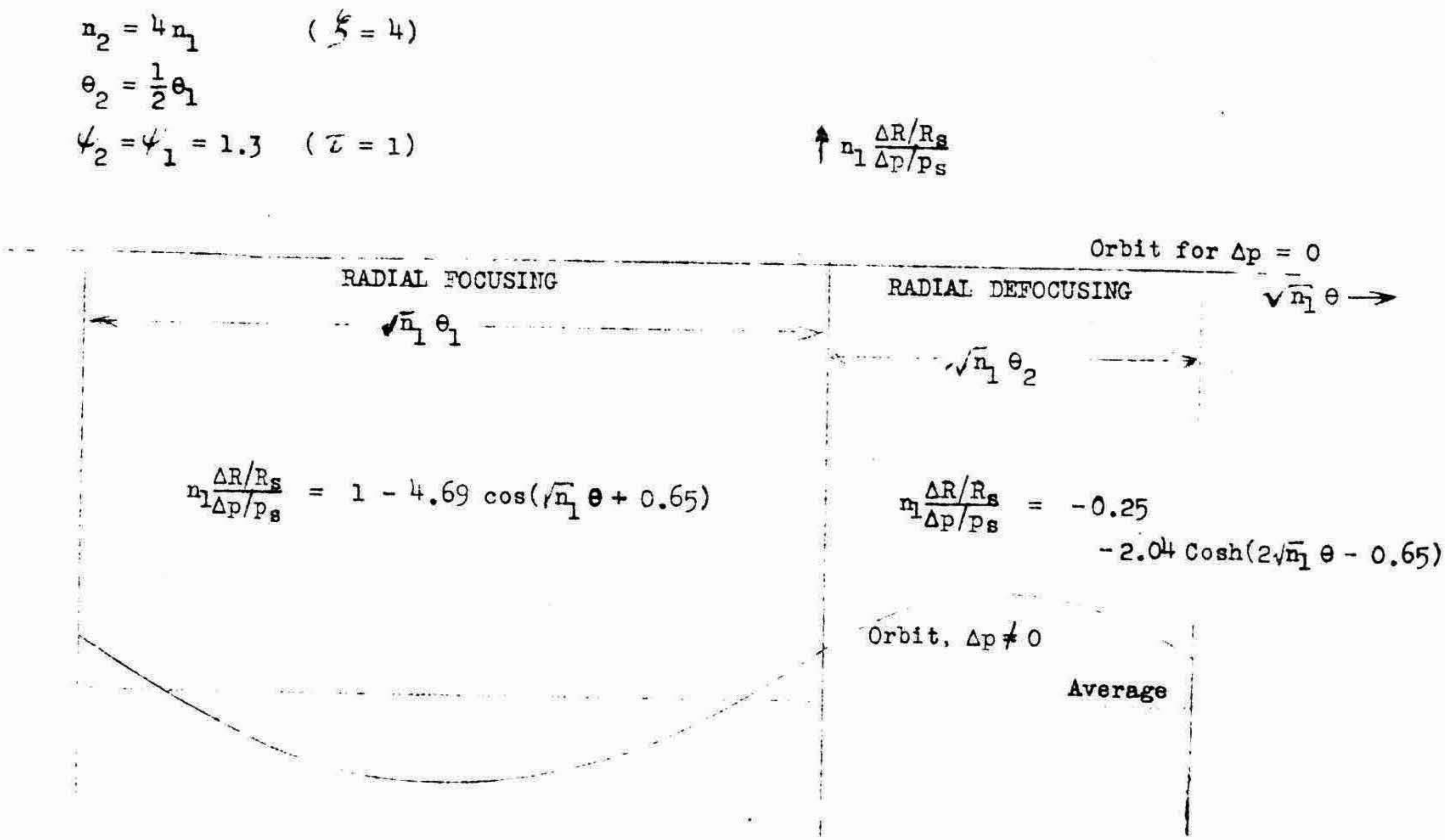


FIG. 1. Unstable equilibrium orbit, with $\alpha = -3.05/n_1$; $\cos \sigma_r = 1.75$.

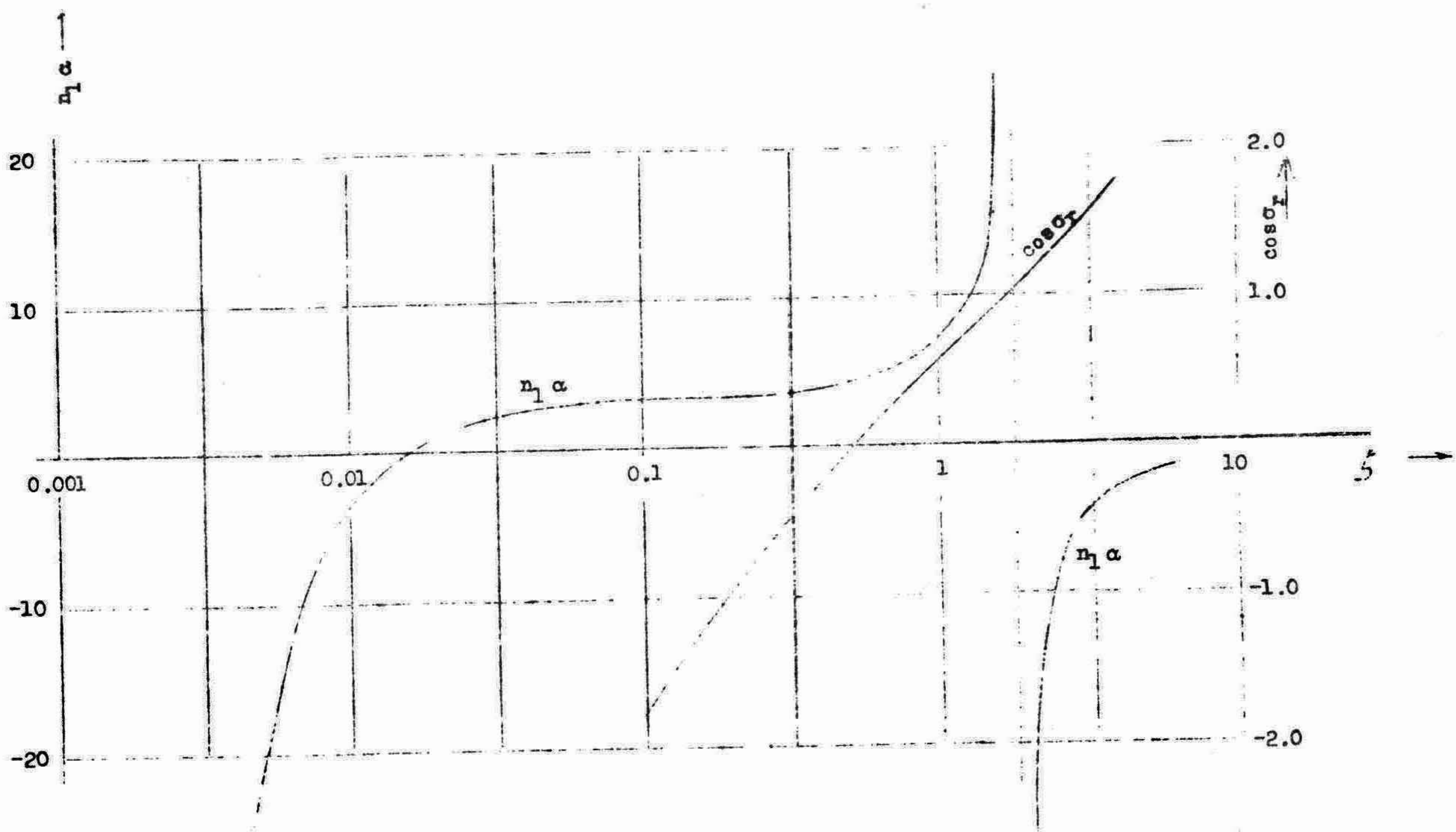


FIG. 2. $n_1\alpha$ and $\cos\sigma_r$ as functions of $\zeta \equiv n_2/n_1$.
plotted for $\psi_1 = \psi_2 = 1.3$.

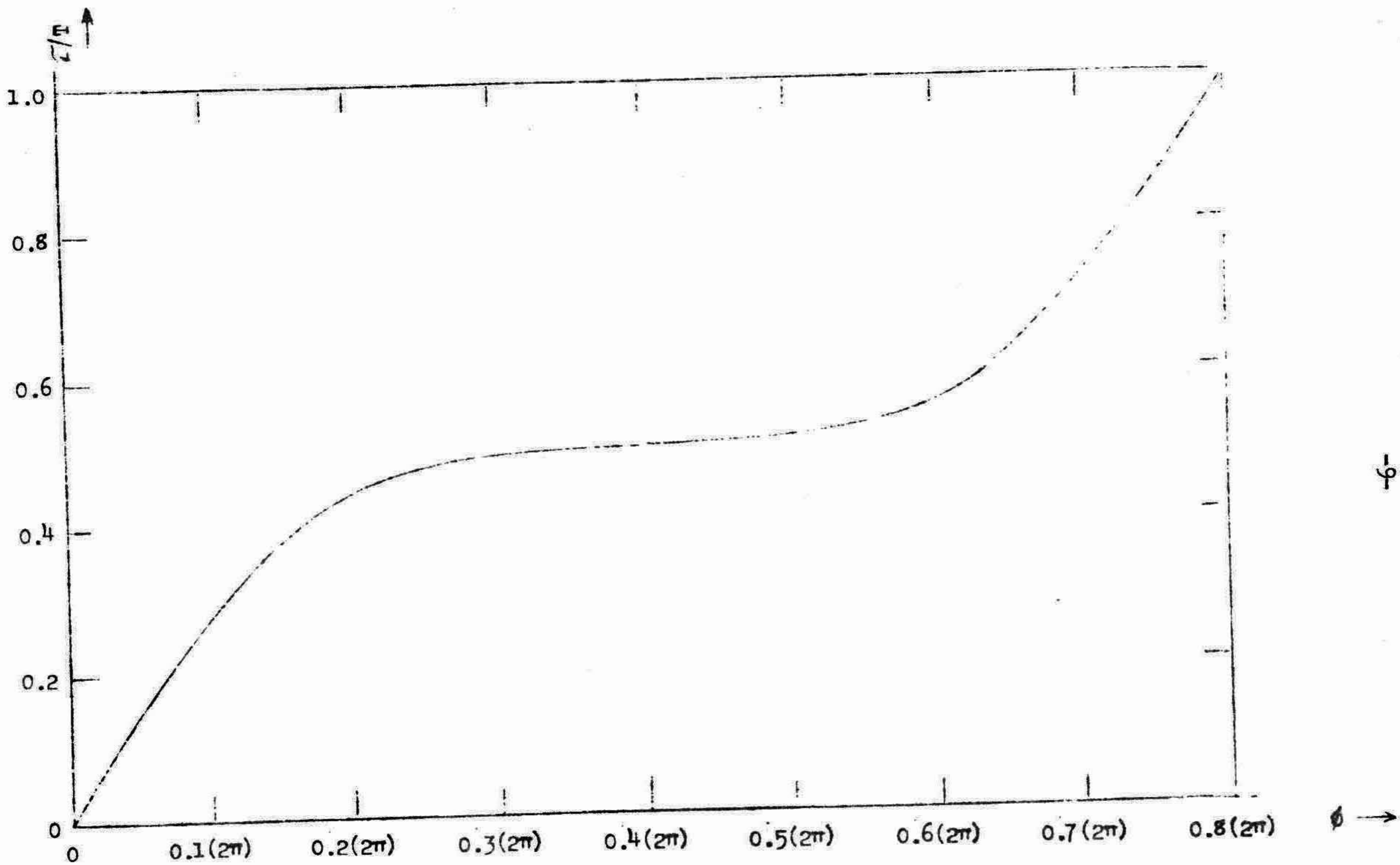


FIG. 3. Relation between the independent variable τ, in units of the period T, and the phase function ϕ — for $w = 1 + 0.8 \cos(\phi/0.8)$.

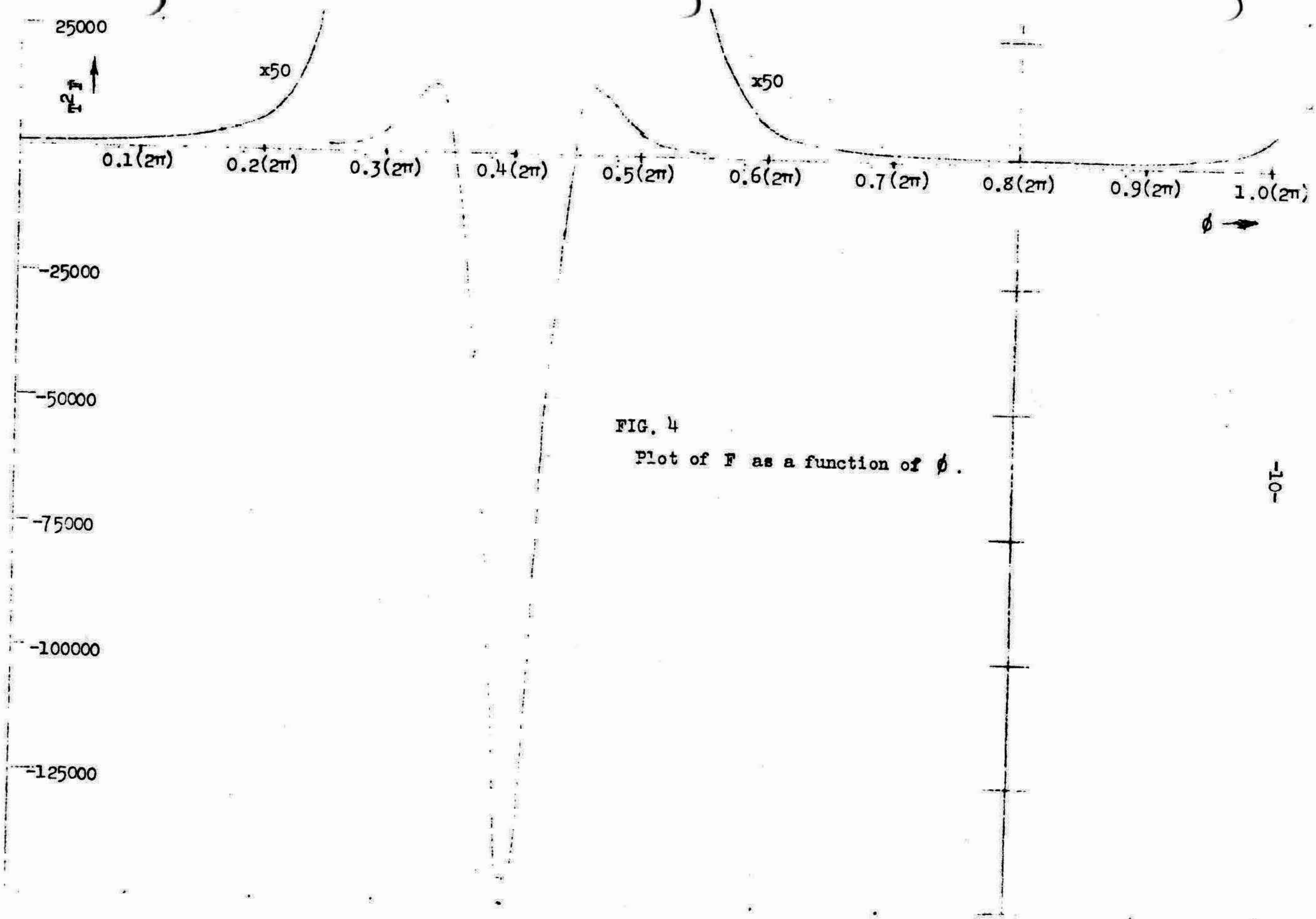


FIG. 4

Plot of F as a function of ϕ.

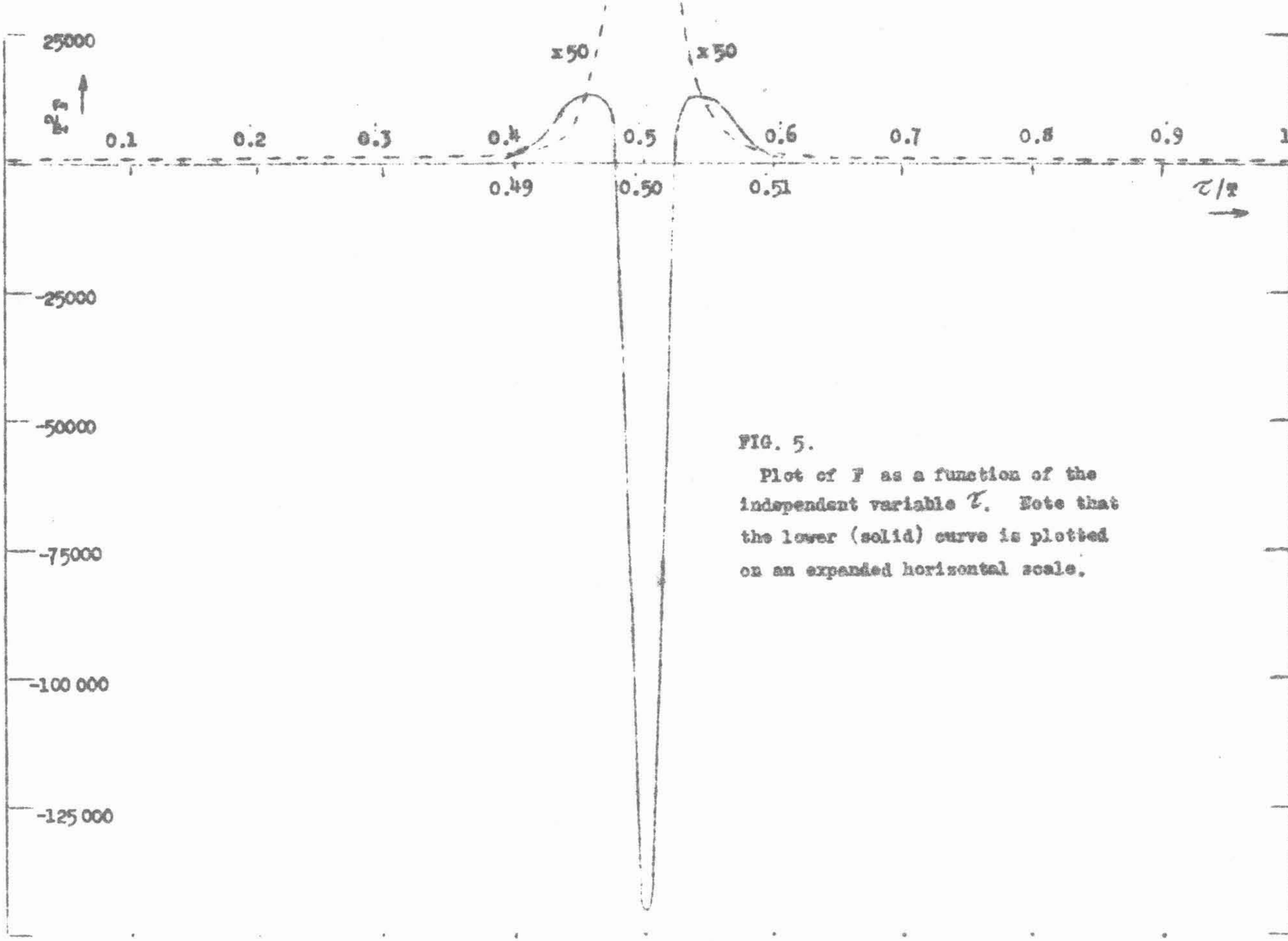


FIG. 5.

Plot of F as a function of the independent variable τ. Note that the lower (solid) curve is plotted on an expanded horizontal scale.

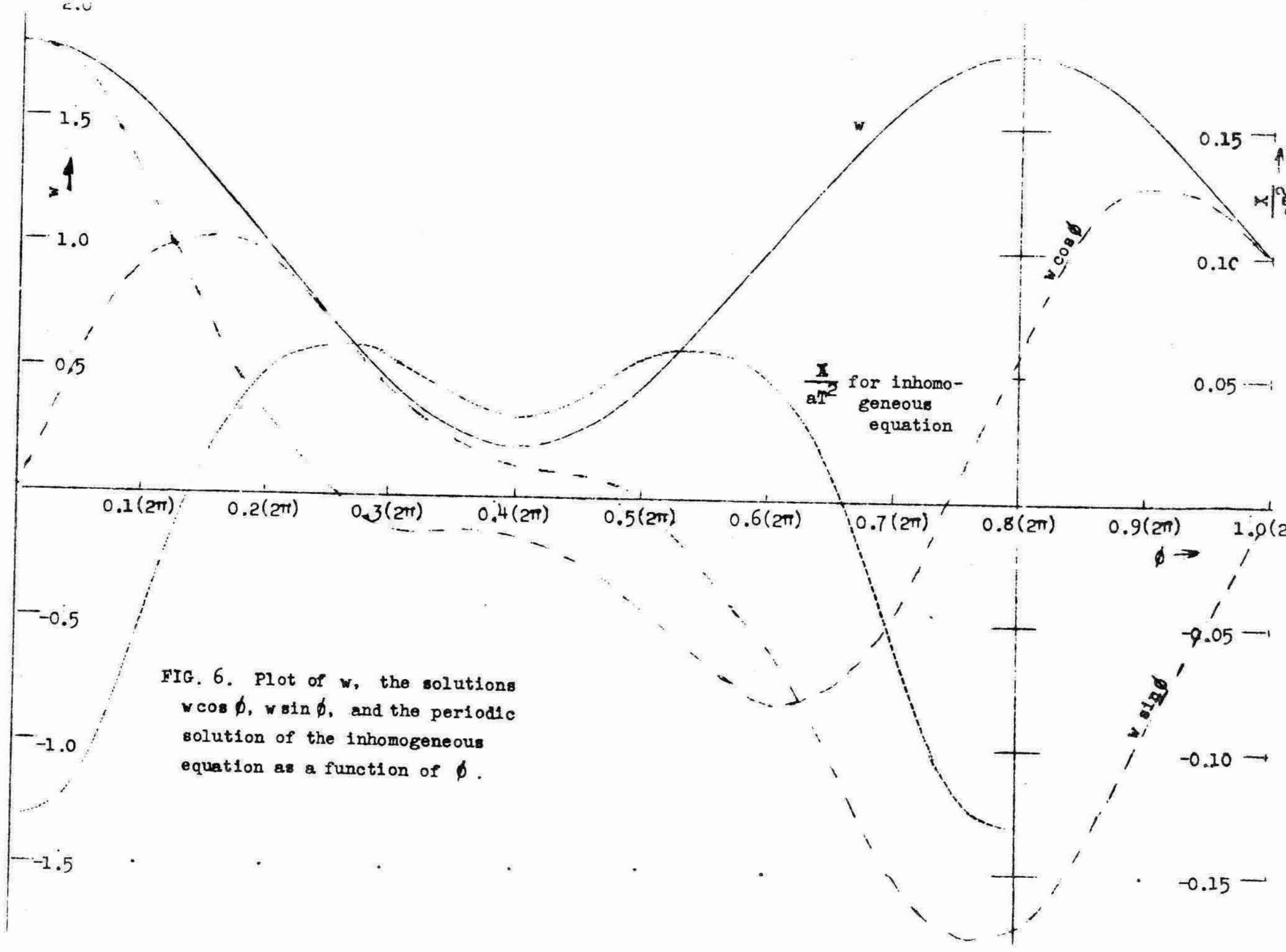


FIG. 6. Plot of w, the solutions w cos ϕ, w sin ϕ, and the periodic solution of the inhomogeneous equation as a function of ϕ.

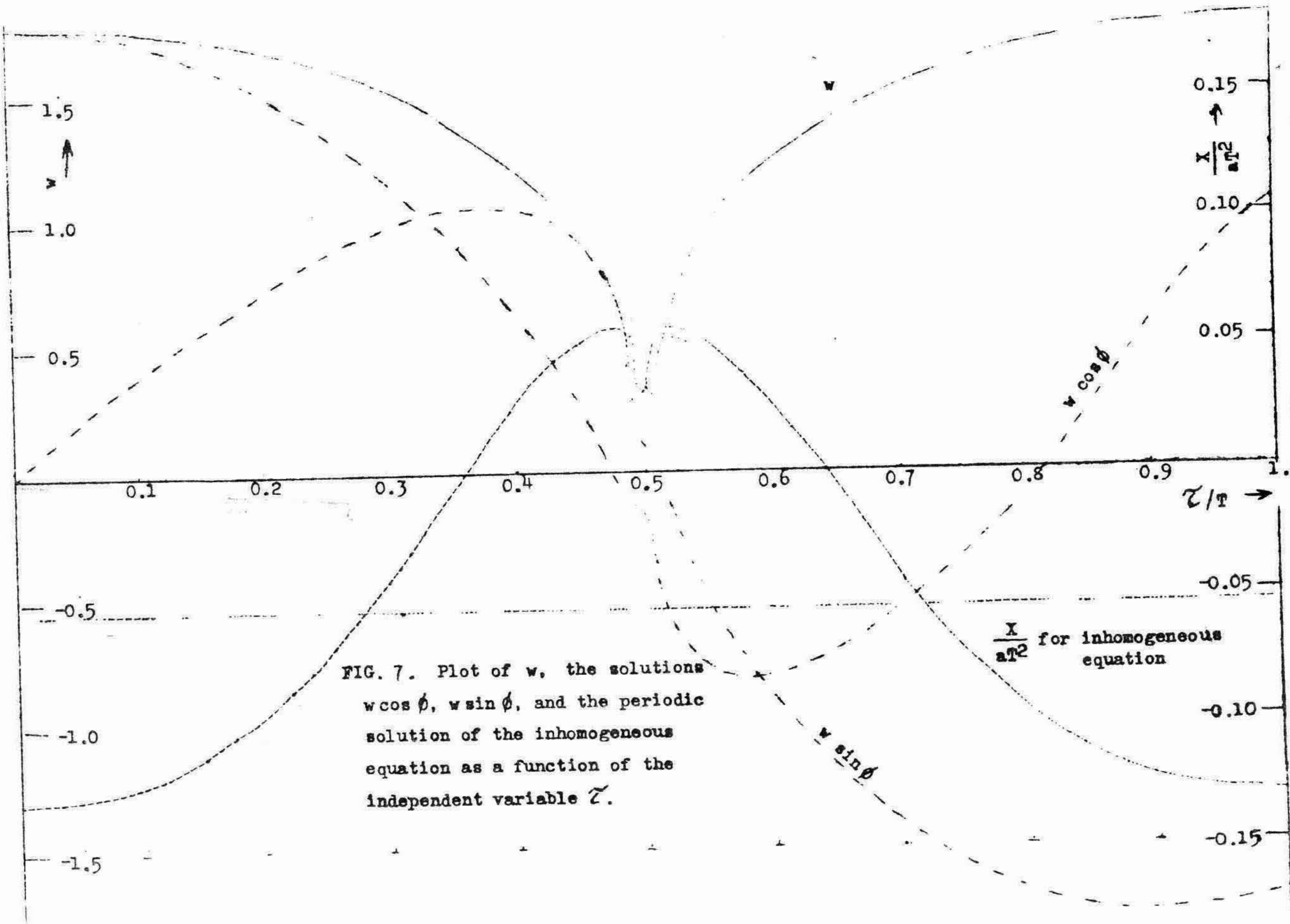


FIG. 7. Plot of w, the solutions w cos ϕ, w sin ϕ, and the periodic solution of the inhomogeneous equation as a function of the independent variable τ.

APPENDIX

DERIVATION OF THE PERIODIC SOLUTION OF THE INHOMOGENEOUS EQUATION

1. The differential equation

$$\frac{d^2X}{d\tau^2} + F(\tau)\, X = a,$$

$F(\tau)$ being periodic in τ with period T, may be written in vector form with $\bar{X} = \binom{X}{dX/d\tau}$:

$$\frac{d\bar{X}}{d\tau} + A(\tau)\,\bar{X} = \bar{b},$$

where $A = \begin{pmatrix} 0 & -1 \\ F & 0 \end{pmatrix}$, $\bar{b} = \binom{0}{a}$.

The solution of the <u>homogeneous</u> equation relates $\bar{X}(\tau)$ to $\bar{X}(\tau_0)$ by a matrix $U(\tau, \tau_0)$ such that

$$\bar{X}(\tau) = U(\tau, \tau_0)\,\bar{X}(\tau_0).$$

In the inhomogeneous case we obtain, in addition, a response from an impulse $\binom{0}{a\,d\tau^x}$ which immediately augments X' by $a\,dt^x$ -- hence, for the inhomogeneous problem,

$$\bar{X}(\tau) = U(\tau, \tau_0)\,\bar{X}(\tau_0) + \int_{\tau_0}^{\tau} U(\tau, \tau^x)\binom{0}{a}\,d\tau^x .$$

For a solution of the inhomogeneous equation which is to be periodic with the period T, we require

$$\bar{X}(\tau) = \bar{X}(\tau - T)$$

$$[1 - U(\tau, \tau - T)]\,\bar{X}(\tau) = \int_{\tau - T}^{\tau} U(\tau, \tau^x)\binom{0}{a}\,d\tau^x, \quad \text{or,}$$

defining

$$P(\tau) = U(\tau, \tau - T),$$

$$\bar{X}(\tau) = [1 - P(\tau)]^{-1}\int_{\tau - T}^{\tau} U(\tau, \tau^x)\binom{0}{a}\,d\tau^x .$$

2. In terms of the solutions $X_1 = w\,e^{i\phi}$, $X_2 = w\,e^{-i\phi}$, the transformation matrix may be written

$$U(\tau, \tau_0) = \begin{pmatrix} \dfrac{w_0 w \phi_0' \cos(\phi - \phi_0) - w w_0' \sin(\phi - \phi_0)}{w_0^2\,\phi_0'} & \dfrac{w_0\, w \sin(\phi - \phi_0)}{w_0^2\,\phi_0'} \\[2ex] \dfrac{\begin{bmatrix}(w_0 w' \phi_0' - w\, w_0' \phi')\cos(\phi - \phi_0) \\ -(w_0' w' + w_0 w \phi_0' \phi')\sin(\phi - \phi_0)\end{bmatrix}}{w_0^2\,\phi_0'} & \dfrac{w_0 w \phi' \cos(\phi - \phi_0) + w_0 w' \sin(\phi - \phi_0)}{w_0^2\,\phi_0'} \end{pmatrix}$$

for which the determanant is unity by virtue of the relation $w^2\phi' = w_0^2\phi_0'$, as follows from the constancy of the Wronskin.

Thus

$$U(\tau, \tau - T) = P(\tau) = \begin{pmatrix} \cos\sigma - \frac{w'}{w\phi'}\sin\sigma & \frac{\sin\sigma}{\phi'} \\ -(\phi' + \frac{w'^2}{w^2\phi'})\sin\sigma & \cos\sigma + \frac{w'}{w\phi'}\sin\sigma \end{pmatrix},$$

$$1 - P(\tau) = \begin{pmatrix} 1 - \cos\sigma + \frac{w'}{w\phi'}\sin\sigma & -\frac{\sin\sigma}{\phi'} \\ (\phi' + \frac{w'^2}{w^2\phi'})\sin\sigma & 1 - \cos\sigma - \frac{w'}{w\phi'}\sin\sigma \end{pmatrix}$$

$$[1 - P(\tau)]^{-1} = \begin{pmatrix} \frac{1}{2}\left[1 - \frac{w'}{w\phi'}\frac{\sin\sigma}{1-\cos\sigma}\right] & \frac{1}{2\phi'}\frac{\sin\sigma}{1-\cos\sigma} \\ -\frac{1}{2}\left[\phi' + \frac{w'^2}{w^2\phi'}\right]\frac{\sin\sigma}{1-\cos\sigma} & \frac{1}{2}\left[1 + \frac{w'}{w\phi'}\frac{\sin\sigma}{1-\cos\sigma}\right] \end{pmatrix}$$

and, with $a(\tau)$ taken as constant,

$$U(\tau,\tau^x)\begin{pmatrix}0\\a\end{pmatrix} = a\,U(\tau,\tau^x)\begin{pmatrix}0\\1\end{pmatrix} = a\begin{pmatrix} \frac{w}{w^x\phi'^x}\sin(\phi-\phi^x) \\ \frac{w\phi'}{w^x\phi'^x}\cos(\phi-\phi^x) + \frac{w'}{w^x\phi'^x}\sin(\phi-\phi^x) \end{pmatrix}.$$

Our periodic solution (period T) to the inhomogeneous equation accordingly then becomes given explicitly by use of these forms in the expression

$$\overline{X}(\tau) = [1 - P(\tau)]^{-1}\int_{\tau-T}^{\tau} U(\tau,\tau^x)\begin{pmatrix}0\\a\end{pmatrix} d\tau^x \ ;$$

the coördinate function is

$$X(\tau) = \frac{a}{2}\left\{\left[1 - \frac{w'}{w\phi'}\frac{\sin\sigma}{1-\cos\sigma}\right]\int_{\tau-T}^{\tau}\frac{w}{w^x\phi'^x}\sin(\phi-\phi^x)\,d\tau^x \right.$$

$$\left. + \frac{1}{\phi'}\frac{\sin\sigma}{1-\cos\sigma}\int_{\tau-T}^{\tau}\left[\frac{w\phi'}{w^x\phi'^x}\cos(\phi-\phi^x) + \frac{w'}{w^x\phi'^x}\sin(\phi-\phi^x)\right]d\tau^x\right\}$$

$$= \frac{a}{2}w\left\{\int_{\tau-T}^{\tau}\frac{1}{w^x\phi'^x}\sin(\phi-\phi^x)\,d\tau^x + \frac{\sin\sigma}{1-\cos\sigma}\int_{\tau-T}^{\tau}\frac{1}{w^x\phi'^x}\cos(\phi-\phi^x)\,d\tau^x\right\}$$

$$= \frac{a}{2(1-\cos\sigma)}w\int_{\tau-T}^{\tau}\frac{1}{w^x\phi'^x}\left\{\sin(\phi-\phi^x) + \sin[\sigma - (\phi-\phi^x)]\right\}d\tau^x$$

$$= \frac{a}{2\sin(\sigma/2)}w\int_{\tau-T}^{\tau}\frac{1}{w^x\phi'^x}\cos[(\phi-\phi^x) - \frac{\sigma}{2}]\,d\tau^x .$$

Introducing the constant $K^2 = w^2\phi'$, this result may be conveniently expressed

$$X(\tau) = \frac{a}{2K^2\sin(\sigma/2)}\int_{\tau-T}^{\tau} w\,w^x\cos[(\phi-\phi^x) - \frac{\sigma}{2}]\,d\tau^x \qquad \text{(A-1)}$$

or, alternatively,

$$X(\tau) = \frac{a}{2K^6 \sin(\sigma/2)} \int_{\tau-T}^{\tau} w^3 w^{x3} \phi' \phi^{x'} \cos[(\phi - \phi^x) - \frac{\sigma}{2}] d\tau^x . \quad (A-2)$$

Hence

$$\langle X \rangle = \frac{a}{2K^6 T \sin(\sigma/2)} \int_0^T d\tau \int_{\tau-T}^{\tau} w^3 w^{x3} \phi' \phi^{x'} \cos[(\phi - \phi^x) - \frac{\sigma}{2}] d\tau^x$$

$$= \frac{a}{2K^6 T \sin(\sigma/2)} \int_0^{\sigma} d\phi \int_{\phi-\sigma}^{\phi} w^3 w^{x3} \cos[(\phi - \phi^x) - \frac{\sigma}{2}] d\phi^x ,$$

w now being expressed in terms of ϕ as a periodic function with period σ.

A rewriting of this last result, employing the periodicity of w, leads to the forms cited in Section 2:

$$\langle X \rangle = \frac{a}{K^6 T \sin(\sigma/2)} \int_0^{\sigma} d\phi \int_{\phi-\sigma}^{0} w^3 w^{x3} \cos(\phi - \phi^x - \frac{\sigma}{2}) d\phi^x$$

$$= \frac{a}{K^6 T \sin(\sigma/2)} \int_0^{\sigma} d\phi \int_{\phi}^{\sigma} w^3 w^{x3} \cos(\phi^x - \phi - \frac{\sigma}{2}) d\phi^x$$

$$= \frac{a}{K^6 T \sin(\sigma/2)} \int_0^{\sigma} d\phi \int_0^{\phi} w^3 w^{x3} \cos(\phi^x - \phi + \frac{\sigma}{2}) d\phi^x .$$

3. The result (A-1) may be checked directly, as follows:

$$X' = \frac{a}{2K^2 \sin(\sigma/2)} \int_{\tau-T}^{\tau} \{ w' w^x \cos(\phi - \phi^x - \frac{\sigma}{2}) - w w^x \phi' \sin(\phi - \phi^x - \frac{\sigma}{2}) \} d\tau^x ,$$

the terms introduced from the upper and lower limits of (A-1) cancelling.

$$X'' = \frac{a}{2K^2 \sin(\sigma/2)} \Big[2w^2 \phi' \sin\frac{\sigma}{2} + \int_{\tau-T}^{\tau} \{ [w'' - w\phi'^2] w^x \cos(\phi - \phi^x - \frac{\sigma}{2}) - (2w'\phi' + w\phi'') w^x \sin(\phi - \phi^x - \frac{\sigma}{2}) \} d\tau^x \Big] .$$

which, by use of $2w'\phi' + w\phi'' = 0$, $w^2\phi' = K^2$, $w\phi'^2 - w'' = wF$ becomes

$$X'' = \frac{a}{2K^2 \sin(\sigma/2)} \Big[2w^2 \phi' \sin\frac{\sigma}{2} - F(\tau) \int_{\tau-T}^{\tau} w w^x \cos(\phi - \phi^x - \frac{\sigma}{2}) d\tau^x$$

$$= a - F(\tau) X ;$$

hence

$$X'' + F(\tau) X = a , \quad \text{as required.}$$

The form (A-1) is, in addition, clearly periodic in τ with the period T.

ADDENDUM

22 November 1954

A situation involving step-wise constant values for $F(\tau)$ can be synthesized to represent closely the example given in Section 4. With fairly minor, but somewhat critical adjustments, it appears that the situation can be adapted to give both radial and vertical stability. The constants suggested are listed below:

$$\frac{d^2X}{d\tau^2} + F(\tau)X = 0$$

Interval for τ in units of the period T	$T^2 F(\tau)$	
0 to 0.200	$(2.81)^2 = 7.8961$	
0.200 to 0.300	$(3.53)^2 = 12.4609$	
0.300 to 0.400	$(5.25)^2 = 27.5625$	
0.400 to 0.430	$(8.17)^2 = 66.7489$	
0.430 to 0.460	$(12.33)^2 = 152.0289$	
0.460 to 0.475	$(20.00)^2 = 400.0000$	
0.475 to 0.490	$(38.70)^2 = 1497.6900$	
0.490 to 0.495	$(79.60)^2 = 6336.1600$	
0.495 to 0.498	Straight Section (F = 0)	
0.498 to 0.500	$-(147.07)^2 = -21629.5849$	radially defocusing
0.500 to 0.502	$-(147.07)^2 = -21629.5849$	
0.502 to 0.505	Straight Section (F = 0)	
0.505 to 0.510	$(79.60)^2 = 6336.1600$	
etc.		

It is believed that the radial motion is characterized by $\cos\sigma_r \doteq 0.718$. The periodic solution of the inhomogeneous equation

$$\frac{d^2X}{d\tau^2} + F(\tau)X = a$$

appears to start off at $\tau = 0$ with an ordinate $-0.3834\,aT^2$ and with zero slope, leading to the rather pronounced compaction factor

$$\alpha \equiv \langle X\rangle/a \doteq -0.22\,T^2 .$$

It is believed that the constants listed above are such as to result in stability of the vertical motion, represented by the equation

$$\frac{d^2Y}{d\tau^2} - F(\tau)Y = 0 ,$$

the value of $\cos\sigma_v$ being estimated as -0.2.

Characteristic solutions of the radial equation, and the periodic solution of the inhomogeneous equation, are illustrated in Fig. 8.

The nature of the solutions to the problem presented indicates the possibility of attaining stable solutions characterized by $\alpha < 0$. One might thereby be encouraged to seek systematically other, more practical, situations in which alpha is negative (or possesses a sufficiently small positive value) and the transition-energy problem is eliminated. L. W. Jones has recently made mention of FFAG accelerators in which negative alpha appears to be realized [see: Minutes of the Michigan Working Group for 11-12 November 1954, MURA-LWJ/DWK/LJL/KRS/KMT-2 (12 November 1954), MURA-LWJ-6 (12 November 1954), Abstract for the (January, 1955) New York meeting of the American Physical Society].

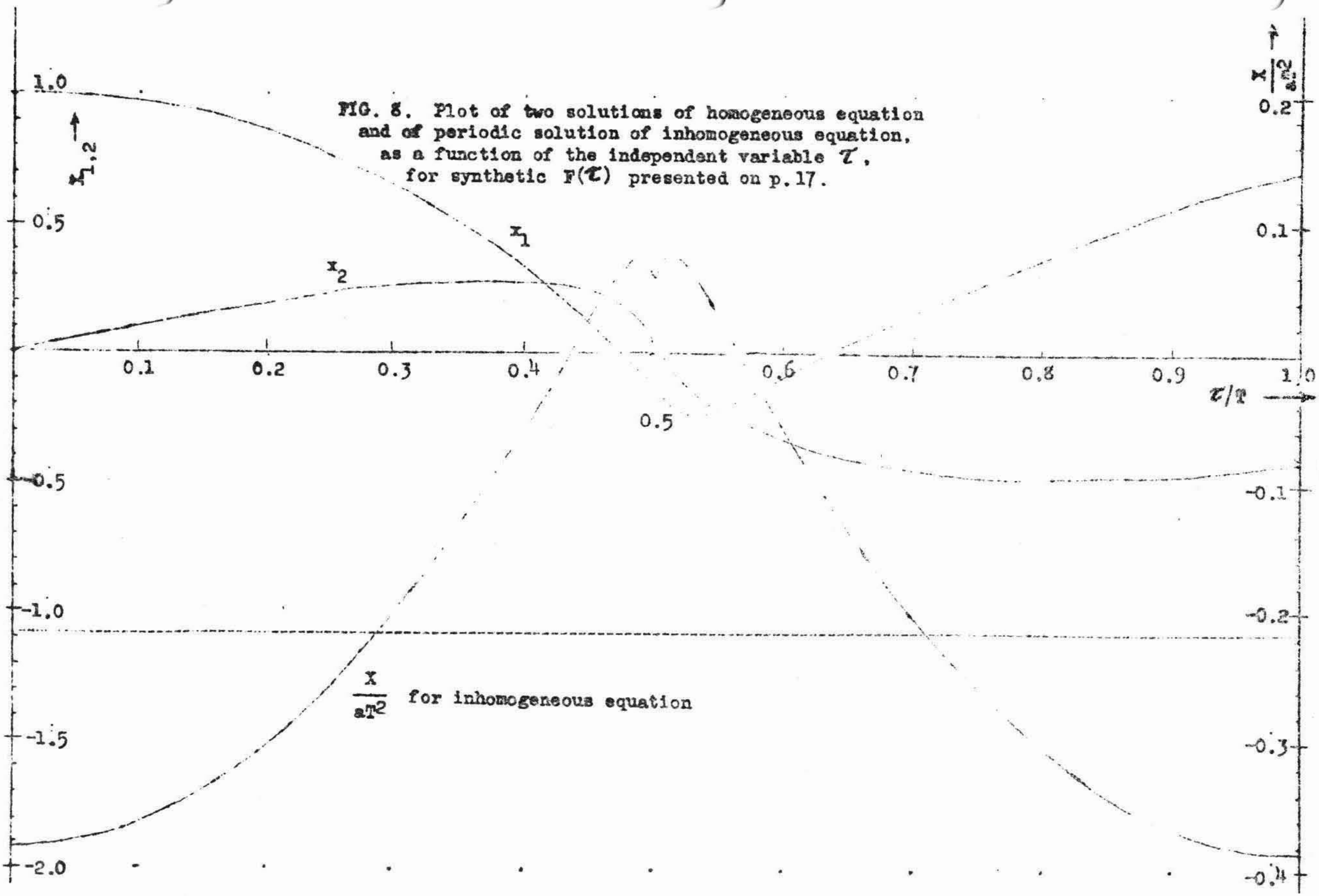


FIG. 8. Plot of two solutions of homogeneous equation and of periodic solution of inhomogeneous equation, as a function of the independent variable τ, for synthetic $F(\tau)$ presented on p. 17.

MURA NOTES

L. Jackson Laslett
14 August 1954
(Stencil cut 22 Nov. 1954)

RETABULATION OF SPACE-CHARGE EFFECTS IN THE AGS

[The material in these notes was contained in a letter, dated 14 August 1954, sent to Dr. Kerst at Madison. It is reproduced here for convenient reference by the MURA technical group.]

A simple discussion of space-charge effects in the alternate-gradient synchrotron was given in LJL(MAC)-2 (January, 1954). It now appears that attention may be focused on single-turn injection, but that the value $\Delta = 4.0$ cm assumed for that case in the example of LJL(MAC)-2 may be excessive.

It appears desirable, therefore, to present a revised table based on single-turn injection and with Δ taken as 1.5 cm (radius). We take, in this example, $n = 345$ (cf. EDC-12, p. 6, in which the value 348 for n is discussed). The table is constructed by use of eq. (5) of LJL(MAC)-2 and, because of the more conservative assumptions, contains results approximately 1/10 as large as those cited previously.

$R_o = 8600$ cm $\Delta = 1.5$ cm radius

$$\delta n = \pm 0.07\, n^{1/2} = \pm 0.07\, (345)^{1/2} = \pm 1.3$$

Kinetic Energy at Injection (Mev)	Total Charge (Coulombs)	Inj. Current, 1 rev. (ma)	Particles for 50 percent capture
4	9.6×10^{-9}	0.49	3.0×10^{10}
5	12.0	0.68	3.7
6	14.4	0.90	4.5
10	24.2	1.95	7.5
15	36.5	3.6	11.4
20	49.1	5.5	15.3
30	74.7	10.2	23.4
40	101	15.8	31.6
50	128	22.2	40

24 18 Requisite Energy Tolerance at Injection.

L. J. Laslett (LJL) Dtd. (MURA NOTES) 6/21/54

A summary is given of the theoretical basis for estimating the energy homogeneity desirable in the beam injected into an A. G. synchrotron. Some numerical examples are given for proton A. G. synchrotrons of 50 Mev injection energy.

L. Jackson Laslett
21 June 1954
(stencil cut 22 Nov. 1954)

REQUISITE ENERGY TOLERANCE AT INJECTION

[The material contained in these notes was enclosed in a letter, dated 21 June 1954, sent to Dr. Kerst at Madison. It is reproduced here for convenient reference by the MURA technical group.]

Motivation:

As mentioned in previous notes [May-June, 1954], the question has been raised concerning the requisite energy tolerances at injection and concerning the estimates of this quantity reputedly made at the Brookhaven National Laboratory.

Method of Approach:

The approach may be considered to be similar in principle to that of K. Johnsen [CERN proton-synchrotron lectures, Sect. III-3 (October, 1953)], as reported earlier [previous notes and LJL(MAC)-3 (February, 1954)]. In this approach the requirement considered is that the initial momentum spread shall be no greater than that acceptable into synchrotron phase oscillations, but, in the interests of efficient capture, a more conservative specification may be imposed. Only injection for a single turn or less is considered.

Derivation of Formulas:

The equation of phase oscillations in the steady state, with ϕ representing the electrical phase angle, may be written [cf. LJL(MAC)-3, and references therein]:

$$\ddot{\phi} + A^2 (\sin\phi - \sin\phi_0) = 0 ,$$

where

$$A^2 = \omega_0^2 h(-\gamma) \frac{eV_0}{2\pi} \frac{1}{E_s} \quad \text{and} \quad -\gamma = (\frac{E_0}{E})^2 - \frac{\alpha}{1 + \frac{\Sigma L}{2\pi R}} .$$

This equation has the first integral

$$\dot{\phi}^2 = 2A^2 [\cos\phi + \cos\phi_0 - (\pi - \phi - \phi_0) \sin\phi_0]$$

and

$$\dot{\phi}_{max}^2 = 4A^2 [\cos\phi_0 - (\frac{\pi}{2} - \phi_0) \sin\phi_0] .$$

For $\phi_0 = \frac{\pi}{6}$,

$$\dot{\phi}_{max}^2 = 4A^2 [\frac{\sqrt{3}}{2} - \frac{\pi}{6}]$$

$$= 1.37 A^2 ;$$

$$\dot{\phi}_{max} = 1.17 A .$$

For efficient capture, however, we may write

$$\dot{\phi}_{max} = 1.17 f A ,$$

where f is a numerical factor which might be taken to be about 0.3 (see Fig. 1).

The associated momentum variation is

$$\frac{dp}{p} = \frac{1}{\gamma} \frac{d\omega}{\omega} = \frac{1}{\omega\gamma h} \dot{\phi} = \frac{1}{\beta\omega_0 \gamma h} \dot{\phi}$$

$$= \frac{1.17 f}{\beta} \sqrt{\frac{1}{h|\gamma|} \frac{eV_0/2\pi}{E_s}} .$$

The required peak R.F. voltage per turn, V_o, is estimated by

$$\frac{1}{2\pi}\,(\delta E/\mathrm{Turn}) \doteq \frac{R[1+\frac{\Sigma L}{2\pi R}]\,\dot{B}_f}{c\cdot(\mathrm{acc.\ time})}$$

$$\frac{eV_o}{2\pi} \doteq \frac{R[1+\frac{\Sigma L}{2\pi R}]\,E_f}{c\cdot(\mathrm{acc.\ time})\cdot\sin\phi_o}$$

Numerical Values:

We include in the table below results computed for (i) a machine similar to that planned by the CERN group, (ii) a machine of the type believed under design at Brookhaven, and (iii) a similar machine with characteristics discussed in the mid-west group. The values listed for the CERN machine are based on what appears to be Johnsen's assumption of $f = 1$ and $|\gamma| \doteq 1$; the value of $eV_o/2\pi$ for the CERN machine is estimated from the quoted $\dot{B}$ = 12 kilogauss/sec and leads to results similar to those stated by Johnsen in the absence of frequency error. The value taken for $eV_o/2\pi$ in the BNL case is obtained from the formula of the preceding section as

$$\frac{85.34 \times 1.5 \times 25\times 10^9}{3\times 10^8 \times 1 \times 0.5} = 21.4\times 10^3 \quad \mathrm{ev/radian},$$

for $\phi_o = \frac{\pi}{6}$, although this value seems somewhat less than that suggested by the Brookhaven ADD minutes No. 62 (May 12, 1954). 50 Mev injection is assumed in each case.

	CERN	BNL	MURA			
Inj. Energy, Kinetic	50	50	50 Mev			
Inj. Energy, Total	0.99	0.99	0.99 Gev			
β	0.314	0.314	0.314			
$\|\gamma\|$	~ 1	0.89	0.89			
$eV_o/2\pi$	23.1×10^3	21.4×10^3	18.7×10^3 ev/radian			
h	38	12	16		36	
f	1	0.3	1	0.3	1	0.3
$\Delta p/p$	±0.29 %	±0.16 %	±0.43 %	±0.13 %	±0.29 %	±0.086 %
$\Delta E/E_{Kin.}$	±0.56 %	±0.31 %	±0.84 %	±0.25 %	±0.56 %	±0.17 %

Summary:

It appears from the foregoing table that, for many of the designs presently under consideration, it would be adequate to inject from a LINAC 50 Mev protons whose energy spread was definitely within ±0.3 percent. Energy stability, from pulse to pulse, is also highly desirable. This tolerance may be compared with that expected for the Minnesota design, as reported by L. H. Johnston [Bull. APS, 1954 meeting in Washington, D.C., paper C1], namely $E_{Kin} = 50 \pm 0.2$ Mev or $\Delta E/E_{Kin} = \pm 0.4$ percent.

The Brookhaven group have suggested privately the objective of ±0.1 percent, although the ADD minutes No. 57 (March 16, 1954) suggest the requirement of ±1/2 percent in energy. It is believed that ±1/3 percent in energy represents the present thinking of the group. Attention must also be given to the "emittance" of the beam, and the desirability of a beam of 1 milliampere from the injector has been suggested.

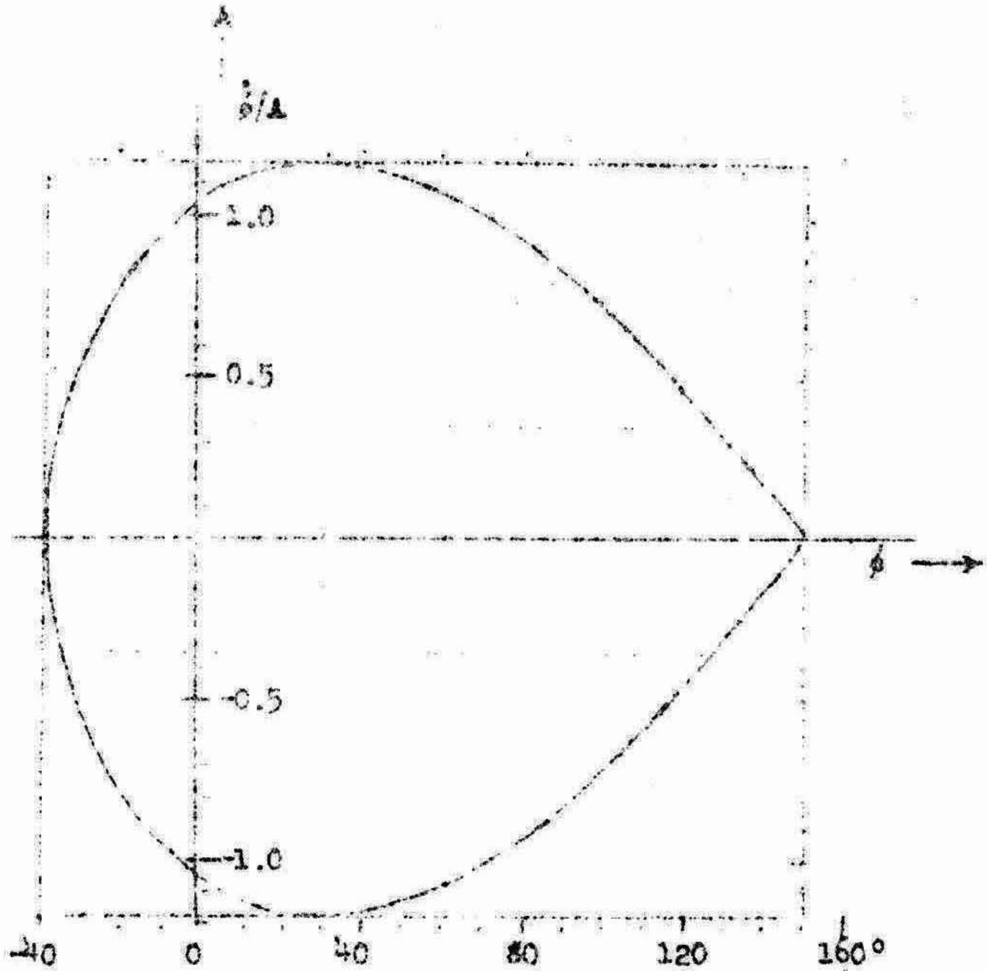


FIG. 1. Plot of region for acceptance into phase oscillations.

MURA NOTES

L. Jackson Laslett
23 December 1954
(Sequel to Notes of
28 October and Addendum
of 22 November)

FURTHER REMARKS CONCERNING STABLE ORBITS WITH NEGATIVE MOMENTUM COMPACTION

1. Implications of Symon's Smooth-Approximation Technique:

The implications of the smooth-approximation technique in regard to the attainment of stable orbits with negative momentum compaction were discussed with E. Courant and K. Symon at a recent MURA meeting (18 December 1954). The smooth-approximation essentially replaces the differential equation for the trajectory by

$$X'' + k^2 X = a ,$$

where the constant a is introduced as a measure of the fractional momentum error and k^2 is a positive constant in the stable case. To the extent that this approximation represents the motion adequately,

$$\alpha \equiv \frac{\langle X \rangle}{a} \doteq \frac{1}{k^2}$$ and is clearly positive.

For $n_1 = |n_2| = n$ and equal sector-lengths, the compaction would be given at the center of the "necktie" ($n = \pi^2/T^2$) by [KRS(MURA)-1, Eq. (51)]

$$\alpha \doteq \frac{1}{k^2} = \frac{48}{n^2 T^2} = \frac{48/\pi^2}{n} = \frac{4.8634}{n} , \quad \text{or}$$

$$= (48/\pi^4)\, T^2 = 0.4928\, T^2 .$$

This result is seen to be in close agreement with the exact result for this case:

$$\alpha = \frac{\exp(\pi/2)}{n} = \frac{4.8105}{n} , \quad \text{or}$$

$$= 0.4874\, T^2 .$$

We may expect the smooth-approximation to fail for betatron wavelengths shorter than the period of the magnet structure [KRS (MURA)-1, p.10] -- in such cases the arguments based on the smooth-approximation do not apply and we may then encounter stable operation with $\alpha < 0$.

2. Stable Motion with Negative α in a Conventional A-G Synchrotron:

At the MURA meeting of December 18 it was pointed out by E. Courant that it is possible with "patch" operation to have stable motion with negative momentum compaction in a conventional A-G synchrotron [square-wave n].

In the case that $n_1 = |n_2| = n \quad (\xi = 1)$

$$n\alpha = \frac{1}{\psi_1 + \psi_2}\left[\psi_1 - \psi_2 + \frac{8}{\mathrm{Ctnh}(\psi_2/2) - \mathrm{ctn}(\psi_1/2)}\right]$$

$$\cos\sigma_r = \mathrm{Cosh}\,\psi_2 \cos\psi_1 \qquad \text{and}$$

$$\cos\sigma_v = \mathrm{Cosh}\,\psi_1 \cos\psi_2 \;,$$

with $\psi_1 = \sqrt{n}\,\theta_1, \quad \psi_2 = \sqrt{n}\,\theta_2, \quad$ and $\quad n \gg 1.$

In the first figure the regions of positive and negative α are illustrated, as a function of ψ_1, ψ_2, for this case, together with a representation of the regions of stability. As was demonstrated in Sect. 1 of the October 28 Notes, all the boundaries $\alpha = \infty$ coincide with certain of the curves for which $\cos\sigma_r = +1$. It is seen from the figure that negative α can be attained with stability by operation in selected patches.

For $\psi_1 = 3\pi/2 = 4.7124$ and $\psi_2 = 7\pi/2 = 10.9956$,

$$\sigma_r = 3\pi/2 \;, \qquad \sigma_v = 7\pi/2 \;,$$

$$\theta_1 = \frac{3\pi}{2\sqrt{n}} = 0.3\,T\,, \quad \theta_2 = \frac{7\pi}{2\sqrt{n}} = 0.7\,T\,, \text{ and the period is } T = \frac{5\pi}{\sqrt{n}}.$$

Then $\quad n\alpha = \frac{1}{15.708}\left[-6.2832 + \frac{8}{2.00003}\right] = -0.1454 \quad$ or

$$\alpha = -0.1454/n \;.$$

The periodic solution to the inhomogeneous equation for this case is illustrated in Figure 2. It is noted that, as in Figure 7 of the October 28 Notes, this solution crosses the axis -- i.e., crosses the (periodic) orbit for the "equilibrium" particle.

3. Aperture Requirements for the Preceding Example ($\psi_1 = \frac{3\pi}{2}, \psi_2 = \frac{7\pi}{2}$):

The maximum (radial) displacement resulting from an angular scattering δ occurring at the most unfavorable location is computed by the formula [see, f. ex., the BNL report of Courant and Snyder, EDC/HSS-1, p. 15]

$$\text{displ.} = \frac{|B|_{max}}{|\sin\sigma_r|} R\delta$$

$$= \beta_{max} R\delta \;,$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the transfer matrix for one period of the (radial) motion. In the example of Sect. 2, the maximum value of $|B|$ is obtained at $\pm T/10$ from the center of a radially focusing sector and is given by

Figure 1. Regions of Stability and of Positive and Negative Compaction

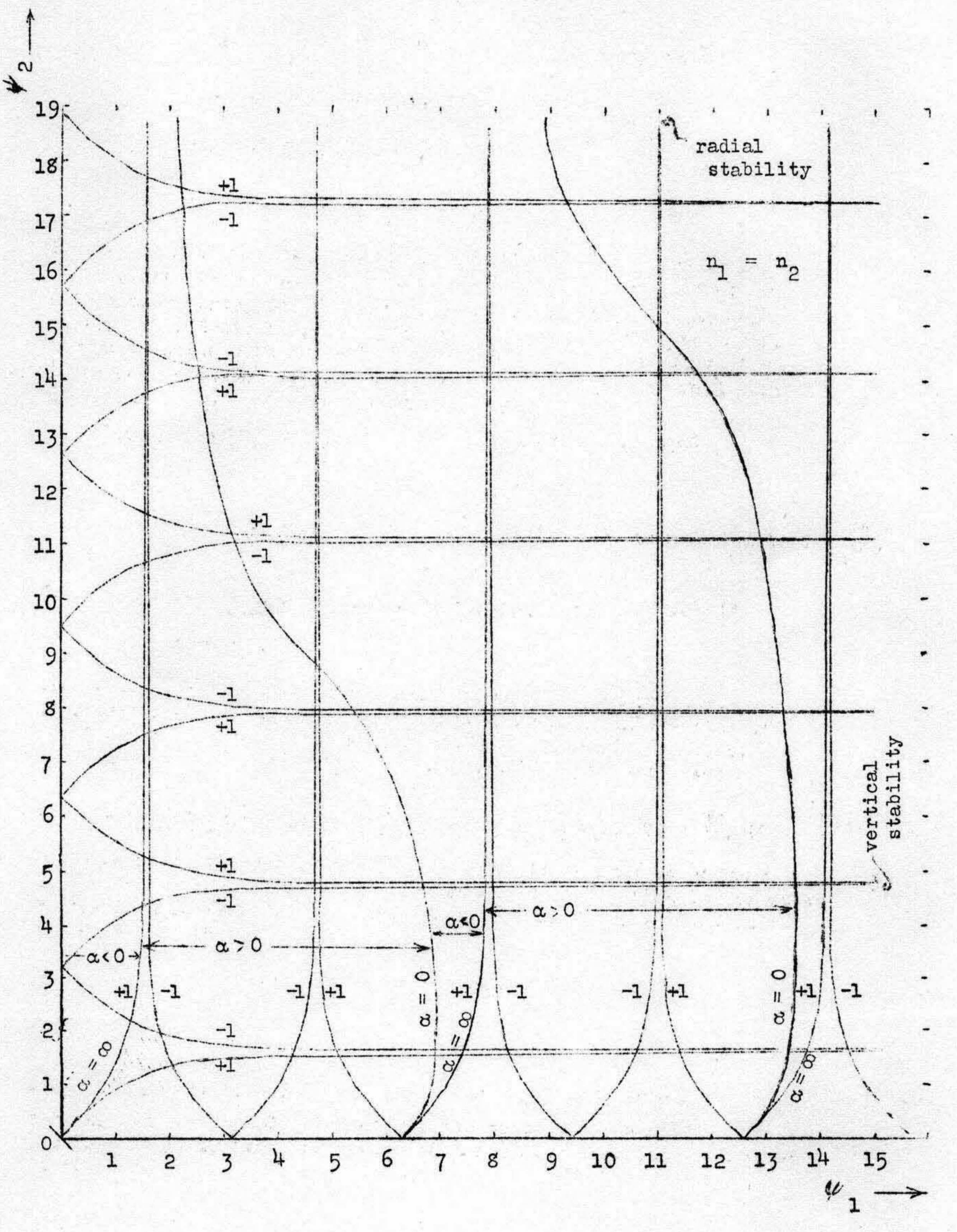
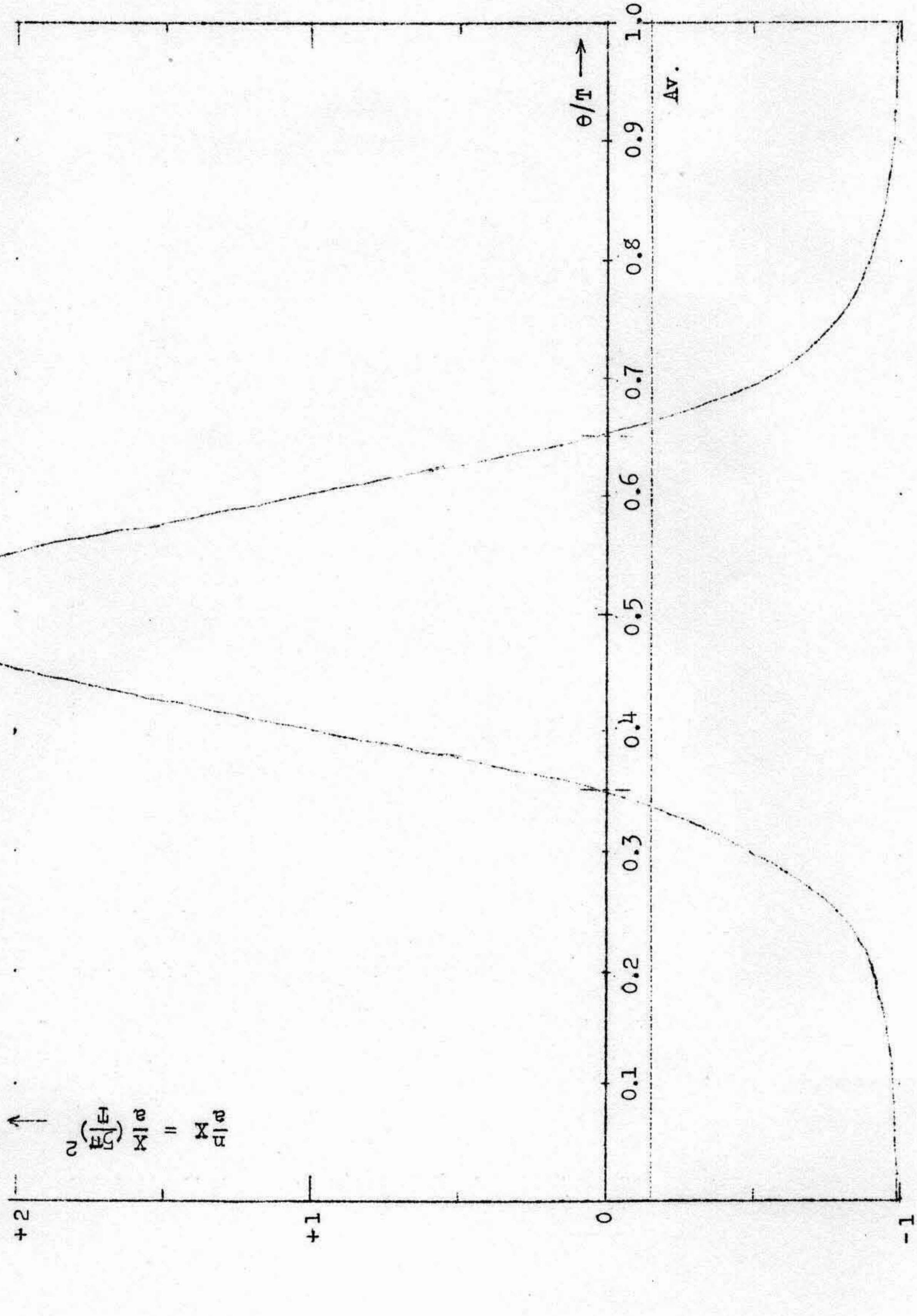


Figure 2. Plot of Periodic Solution of Inhomogeneous Equation for the case of $n_1 = n_2$, $\psi_1 = 3\pi/2$, $\psi_2 = 7\pi/2$.



$$B_{max} = -\frac{\exp(7\pi/2)}{\sqrt{n}} .$$

Thus, since $\sin\sigma_r = -1$,

$$\text{displ.} = \frac{\exp(7\pi/2)}{\sqrt{n}} R\delta = \frac{59611}{\sqrt{n}} R\delta , \quad \text{or}$$

$$= 3795\, TR\delta .$$

The vertical aperture requirements for this case appear somewhat less severe.

The excessive radial displacement met in this example is in contrast to the typical result for $\psi_1 = \psi_2 = \pi/2$:

$$\text{displ.} = \frac{\exp(\pi/2)}{\sqrt{n}} R\delta = \frac{4.8105}{\sqrt{n}} R\delta = 1.5312\, TR\delta .$$

It is thus seen that the amplitude in the patch operation is over three or four orders of magnitude larger than in normal structures of comparable period or (alternatively) similar n-value. One expects, in addition, that resonances will be closely spaced in patch operation -- these features have been noted elsewhere [MURA-LWJ-6] by L. W. Jones in connection with the Mark III FFAG configuration.

4. Aperture Requirements Formulated on the Phase-Amplitude Representation of Solutions to Hill's Equation:

For solutions written in the form $w e^{\pm i\phi}$ or

$$w \sin\phi , \qquad w \cos\phi$$

[EDC-15], with $w^2\phi' = K^2$, the matrix element B for transfer through one period is [see expressions for $U(\tau, \tau_0)$ and $P(\tau)$ at the bottom of p. 14 and the top of p. 15 of the Appendix to the October 28 Notes]:

$$B = \frac{w^2 \sin\sigma}{K^2}$$

$$= \frac{\sin\sigma}{\phi'} .$$

The maximum displacement resulting from an angular scattering δ is, then,

$$\text{displ.} = \frac{B_{max}}{\sin\sigma} R\delta = \frac{w^2_{max}}{K^2} R\delta \quad \text{or}$$

$$= \frac{1}{\phi'_0} R\delta ,$$

where ϕ'_0 is evaluated at the position of maximum w or minimum ϕ'.

5. Aperture Requirements for the Examples of the October 28 MURA Notes:

(a) For the example of Sect. 4 of the October 28 Notes,

$$w_{max} = 1.8 \qquad \text{and}$$
$$K^2 = 1.056(2\pi/T) = 6.635/T \;;$$

hence, by the results of the preceding section,

$$\frac{w^2_{max}}{K^2} = 0.4833\,T \qquad \text{and}$$
$$\text{radial displ.} = 0.4833\,T\,R\delta \,.$$

(b) For the example of the Addendum (Nov. 22), in which step-wise constant values of F were introduced, the matrix element B for transfer of the radial motion through one period (commencing, as in sub-section 5a, at the center of a focusing region) is believed to be

$$B_r = -\,0.4107\,T$$

and

$$\sin\sigma_r = -\,0.6956\,.$$

Hence

$$\text{radial displ.} = \frac{B}{\sin\sigma}R\delta = \frac{0.4107\;T}{0.6956}R\delta = 0.5904\,T\,R\delta\,.$$

(c) For the vertical aperture requirements of this example, unfortunately, the situation does not appear to be as favorable. If we write the transfer matrix for one period of vertical motion, referred to the center of the vertically focusing region at $\tau = 0.5\,T$, we appear to find

$$B_v = 25.63\,T$$

while

$$\sin\sigma_v = 0.9724\,.$$

Hence

$$\text{vert. displ.} = \frac{B}{\sin\sigma}R\delta = \frac{25.63\;T}{0.9724}R\delta = 26.4\,T\,R\delta\,.$$

It is not clear, however, that this feature of the vertical motion found in the present instance is in any essential way characteristic of stable motion with negative momentum compaction.

6. Comment on the Synthesizing of Motions with Negative α:

In Sect. 3 of the October 28 Notes it was pointed out that one could select a form for the function w which, by virtue of the values of its Fourier coefficients, could be seen to result in $\alpha < 0$. With this technique, illustrated in Sect. 4 of the referenced Notes, it appears difficult, however, to secure $\alpha < 0$ without rather large values of σ_r.

Thus, for the (relatively flat-bottomed) function

$$w = 1 + 0.36\cos\phi + 0.1\cos 2\phi$$
$$= 0.9 + 0.36\cos\phi + 0.2\cos^2\phi \;,$$

such that

$$\frac{w_{max}\ (\text{at}\ \cos\phi = 1)}{w_{min}\ (\text{at}\ \cos\phi = -0.9)} = \frac{1.46}{0.738} = 1.98$$

and for which

$$w^3 = 1.21912 + 1.228392 \cos\phi + 0.51459 \cos 2\phi + 0.122364 \cos 3\phi + 0.02472 \cos 4\phi + 0.0027 \cos 5\phi + 0.00025 \cos 6\phi ,$$

the selection of σ_r must be guided by the following Table:

$\frac{\sigma_r}{2\pi}$	$\frac{K^6 T}{\sigma_r} \alpha$
0.80	+ 0.1192
0.81	+ 0.0203
0.82	− 0.0897
0.833...	− 0.2569 .

Hence, to obtain $\alpha < 0$ with the function selected, one must take

$$\frac{\sigma_r}{2\pi} > 0.812 .$$

7. Correction to the November 22 Addendum:

In the November 22 Addendum an example was given of a case involving step-wise constant values of F such that $F = 0$ in the intervals $0.495\,T$ to $0.498\,T$ and $0.502\,T$ to $0.505\,T$. Since the inhomogeneous equation which was considered in connection with that example was

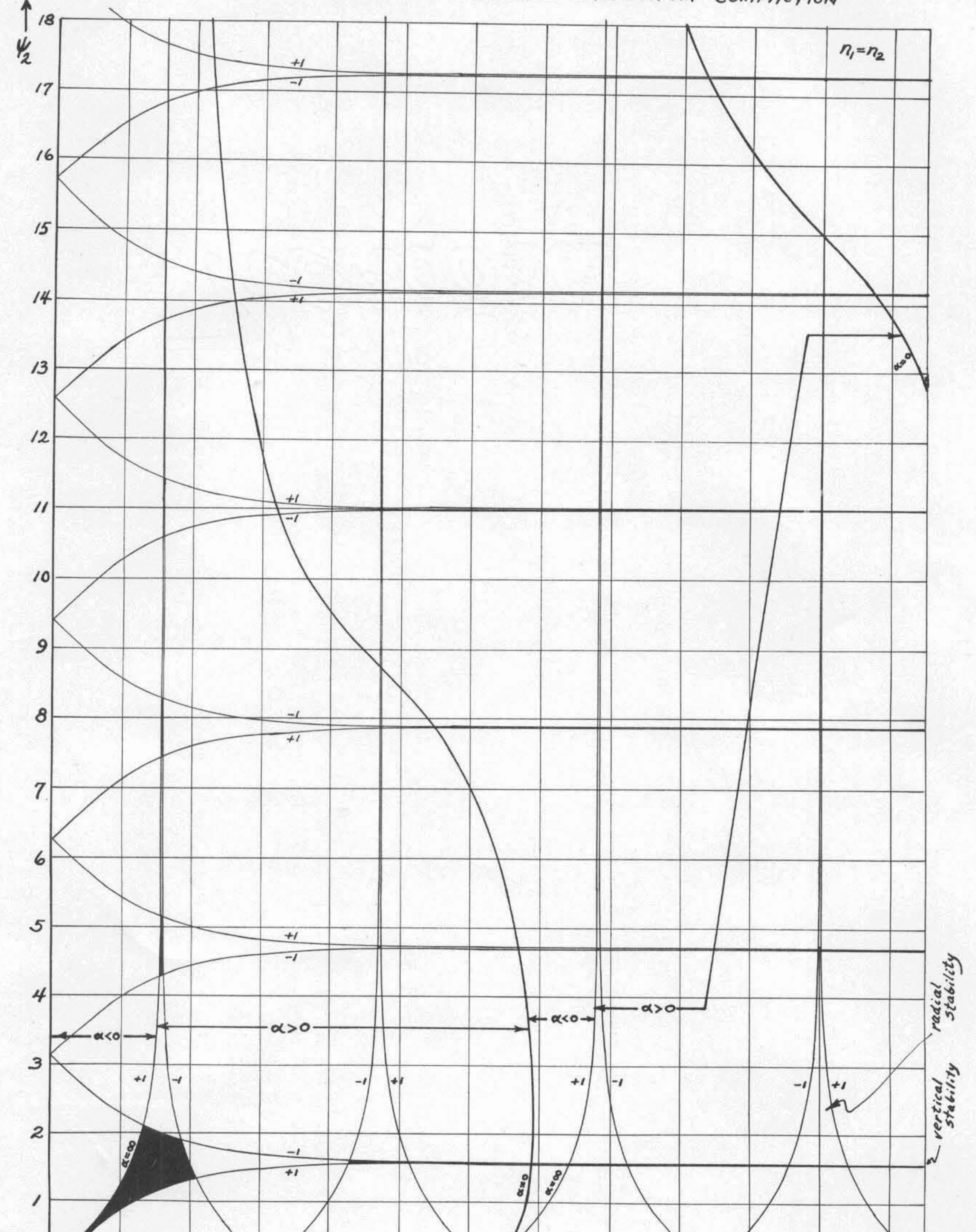
$$X'' + F(\tau)\,X = a$$

and in the intervals for which $F = 0$ was taken to assume the form

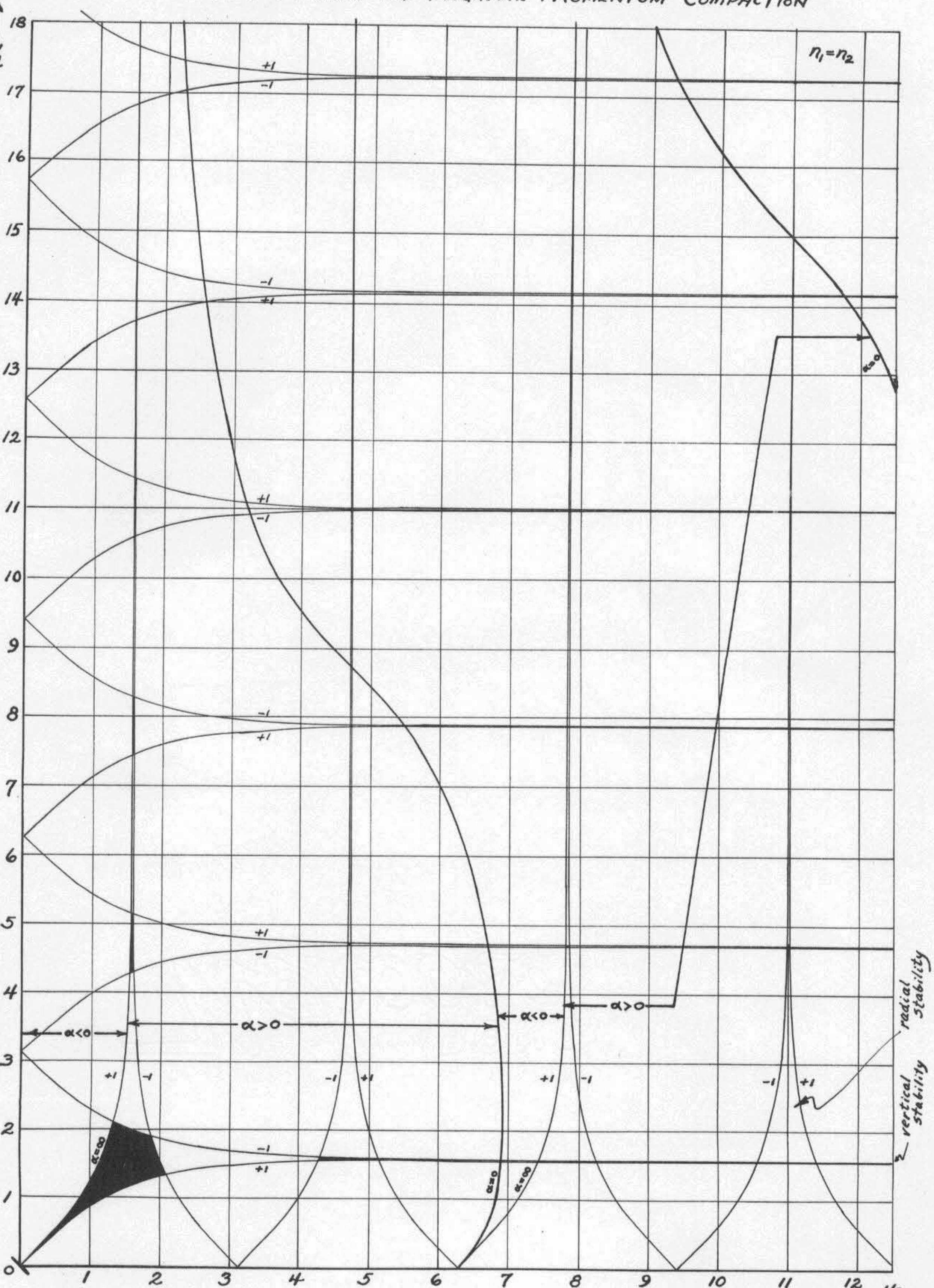
$$X'' = a,$$

it does not appear accurate to term these intervals "Straight Sections". It is believed preferable to regard the intervals for which $F = 0$ in this example as regions of guide field without focusing forces.

REGIONS OF STABILITY
AND OF POSITIVE AND NEGATIVE MOMENTUM COMPACTION



Regions of Stability and of Positive and Negative Momentum Compaction



Regions of Stability
and of Positive and Negative Momentum Compaction