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Smooth Solution to One-Dimensional AG Orbits with Cubic Forces

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In the present note, we investigate the solutions of the smooth approximation equations for one-dimensional alternating gradient orbits with cubic forces. The results will perhaps be of interest as an indication of the validity of certain approximation procedures for arriving at rough estimates of orbit characteristics.

The equation of motion for an AG orbit on the necktie diagonal, with cubic forces, is

$$x'' = (nx + \tfrac{1}{3}ex^3)g(s), \tag{1}$$

where we will take g(s) to be a square wave of unit amplitude and period S. The equation of motion for the smooth approximation is derived in Ref. (1) (Eq. (43)) and is

$$x'' = -\omega_0^2\left(x + \frac{4}{3}\frac{e}{n}x^3 + \frac{1}{3}\frac{e^2}{n^2}x^5\right), \tag{2}$$

where

$$\omega_0^2 = \frac{n^2 S^2}{48}. \tag{3}$$

The potential energy for this motion is

$$V(x) = \omega_0^2\left(\frac{1}{2}x^2 + \frac{1}{3}\frac{e}{n}x^4 + \frac{1}{18}\frac{e^2}{n^2}x^6\right). \tag{4}$$

The potential energy and phase plot for $e > 0$ are shown in Fig. 1, and for $e < 0$ in Fig. 2. The general character of the various types of motion is clear from the figures. Near the stable fixed points, the motion $x(s)$ is sinusoidal and isochronous. At larger amplitudes, in Fig. 1, the period decreases. The function $x(s)$ approaches a triangular wave, while $x'(s)$ approaches a square wave at very high amplitudes, as can be seen from the plot of $V(x)$ which shows that nearly all the acceleration occurs at the turning points in the oscillation. In Fig. 2, large amplitude motions within the eyes have longer periods. The period approaches infinity near the separatrix and the function $x(s)$ is a square wave in the center eye and a series of pulses in the right and left hand eyes. For motions outside the separatrix the period decreases with amplitude and at large amplitudes $x(s)$ is a triangular wave.

In order to arrive at an approximate solution of Eq. (2), we notice that in all cases the motion is periodic and except for motions very close to the separatrix in the side eyes in Fig. 2, we can represent $x(s)$ reasonably well by its fundamental component. For oscillations about $x = 0$, a fair approximation to the motion will be

$$x = A \cos \omega s. \tag{5}$$

We will therefore substitute the expression (5) for x in the right side of Eq. (2) and integrate:

$$x = \frac{\omega_0^2 A}{\omega^2}\left(1 + \frac{e}{n}A^2 + \frac{5}{24}\frac{e^2}{n^2}A^4\right)\cos \omega s$$

$$+ \frac{1}{27}\frac{\omega_0^2}{\omega^2}\frac{e}{n}A^3\left(1+\frac{5}{16}\frac{e}{n}A^2\right)\cos 3\omega s + \frac{1}{1200}\frac{\omega_0^2}{\omega^2}\frac{e^2}{n^2}A^5\cos 5\omega s. \tag{6}$$

We now require that the coefficients of cos ω s in Eqs. (6) and (5) agree (principle of harmonic balance):

$$\omega = \omega_0\sqrt{1 + \frac{e}{n}A^2 + \frac{5}{24}\frac{e^2}{n^2}A^4} \tag{7}$$

This gives a formula for frequency vs. amplitude which has at any rate the correct qualitative behavior. One will have some reservations about the application of the principle of harmonic balance at large amplitudes, particularly near the separatrices in Fig. 2, for although the fundamental component in a square or triangular wave is the largest component and the Fourier series converges fairly rapidly, this is certainly not true for the second derivatives of these waves. However, a little experimenting suggests that when Eq. (5) is substituted in the right side of Eq. (2), with proper frequency and amplitude, the resulting expression for

the force has at least the right general behavior. Eq. (6) predicts that when $e/n < 0, \omega = 0$ at an amplitude

$$A = 1.192\sqrt{-\frac{e}{n}}, \quad 1.838\sqrt{-\frac{e}{n}} \quad . \tag{8}$$

The smaller value corresponds to motion on the separatrix around the center eye, and is only 10% smaller than the correct value

$$A = \frac{4}{\pi}\sqrt{-\frac{e}{n}} = 1.273\sqrt{-\frac{e}{n}} \tag{9}$$

for the amplitude of the fundamental component of a square wave of amplitude $\sqrt{-\frac{e}{n}}$. Near $\omega = 0$, the coefficients of the higher harmonics in Eq. (6) become infinite, although the amplitudes of the harmonics of the true square wave solution do not.

In order to obtain approximate solutions for motion around the right and left eyes in Fig. 2, we substitute the expression

$$x = B + A\cos\omega s \tag{10}$$

in Eq. (2), and require that the constant terms and the terms in $\cos\omega s$ shall balance. The resulting equations can be solved for B and ω in terms of A, with the result:

$$B^2 = 2 + \frac{5}{2}\frac{e}{n}A^2 + \sqrt{1 + 4\frac{e}{n}A^2 + \frac{35}{8}\frac{e^2}{n^2}A^4}, \tag{11}$$

$$\omega = \omega_0\sqrt{1 + \frac{e}{n}A^2 + \frac{5}{24}\frac{e^2}{n^2}A^4 - 4B^2 + \frac{5}{3}B^4 - \frac{5}{2}\frac{e}{n}A^2B^2} \tag{12}$$

Numerical solutions to Eq. (1) have been obtained on the Illiac computer, for the case

$$n = N, \quad e = \pm N, \quad S = 0.1, \tag{13}$$

which is still perfectly general, since e/n and S can be adjusted to any values simply by a change of scale. Formula (7) gives for the betatron wavelength, with the values (13),

$$\lambda = \frac{2\pi}{\omega S} = \frac{\lambda_0}{\sqrt{1 \pm A^2 + \frac{5}{24} A^4}}, \tag{14}$$

where

$$\lambda_0 = \frac{800\,\pi\sqrt{3}}{N} \tag{15}$$

The betatron wavelength is here expressed in units of the sector length S. The betatron wavelengths given by Formula (14) are compared in Tables 1 and 2 with those given by the Illiac solutions, for various values of A. In the Illiac solutions, A was taken to be the maximum amplitude of oscillation. At large amplitudes this is not exactly correct since the solutions are not sinusoidal, but the difference is probably not significant in comparison with the approximations made in arriving at Eq. (14). In order to compare the Illiac solutions for motions around the side eyes in Fig. 2, we read from the Illiac results the maximum and minimum values x_1 and x_2 of x, during the oscillation, and define

$$B = \frac{1}{2}(x_1 + x_2), \quad A = \frac{1}{2}(x_1 - x_2). \tag{16}$$

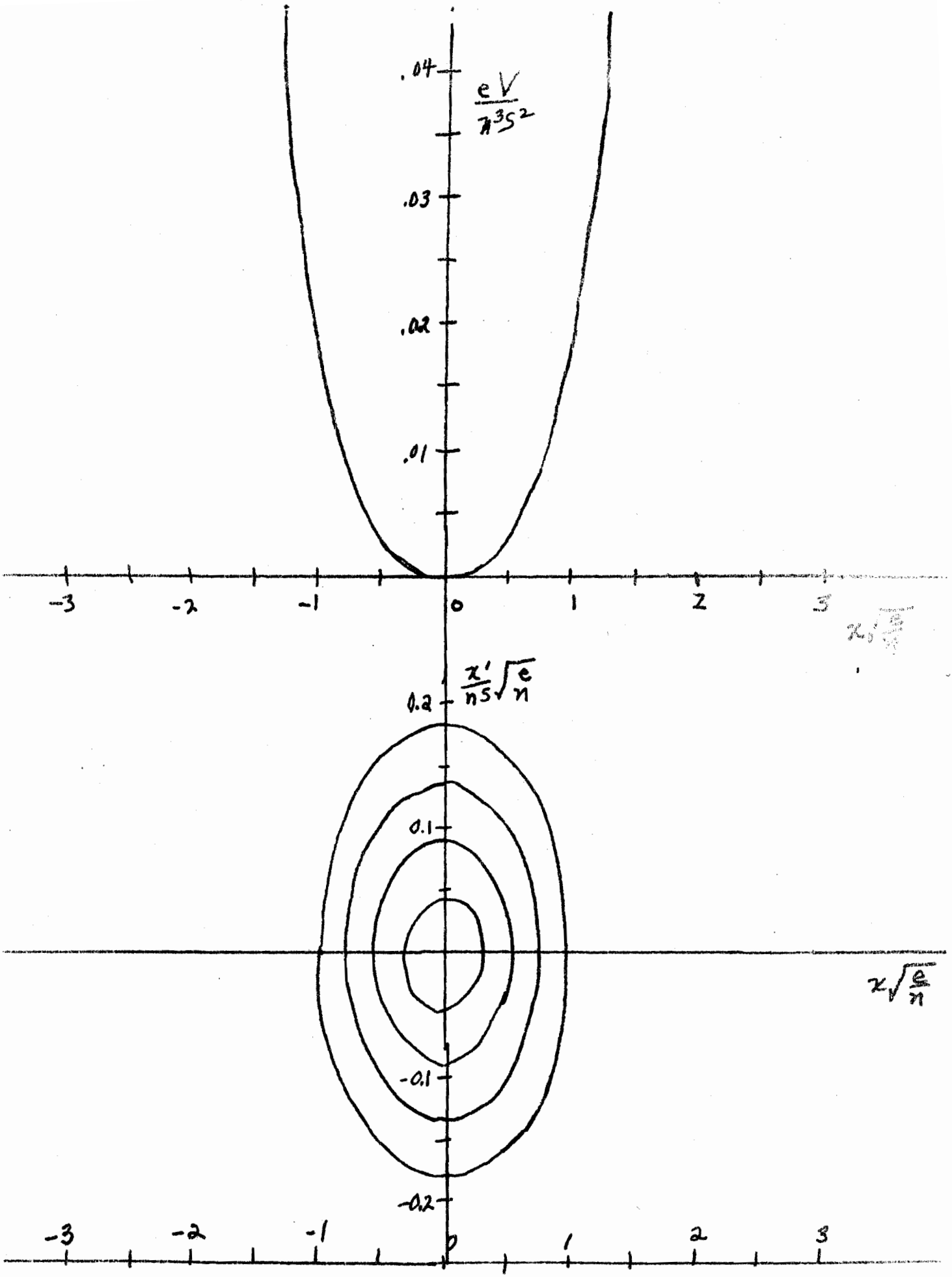
This again is not strictly correct unless the solution has the form (10), but is sufficiently good for the purpose at

hand. The results are given in Table 3. The Illiac values in Tables 1, 2 and 3 were furnished by John Powell. Beyond the maximum values of A shown in Tables 1 and 2, the Illiac solutions become unstable. No such unstability appears in the smooth approximation equations.

One may conclude from these results that both the smooth approximation equations and the method of solution indicated give a fairly good rough approximation to the alternating gradient orbits inside the stability boundaries.

Reference

1. KRS(MURA)-1, "A Smooth Approximation to the Alternating Gradient Orbit Equations," July 1, 1954.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$x\sqrt{\frac{e}{n}}$
$\frac{x'}{n s}\sqrt{\frac{e}{n}}$
0.2
0.1
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-3
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0
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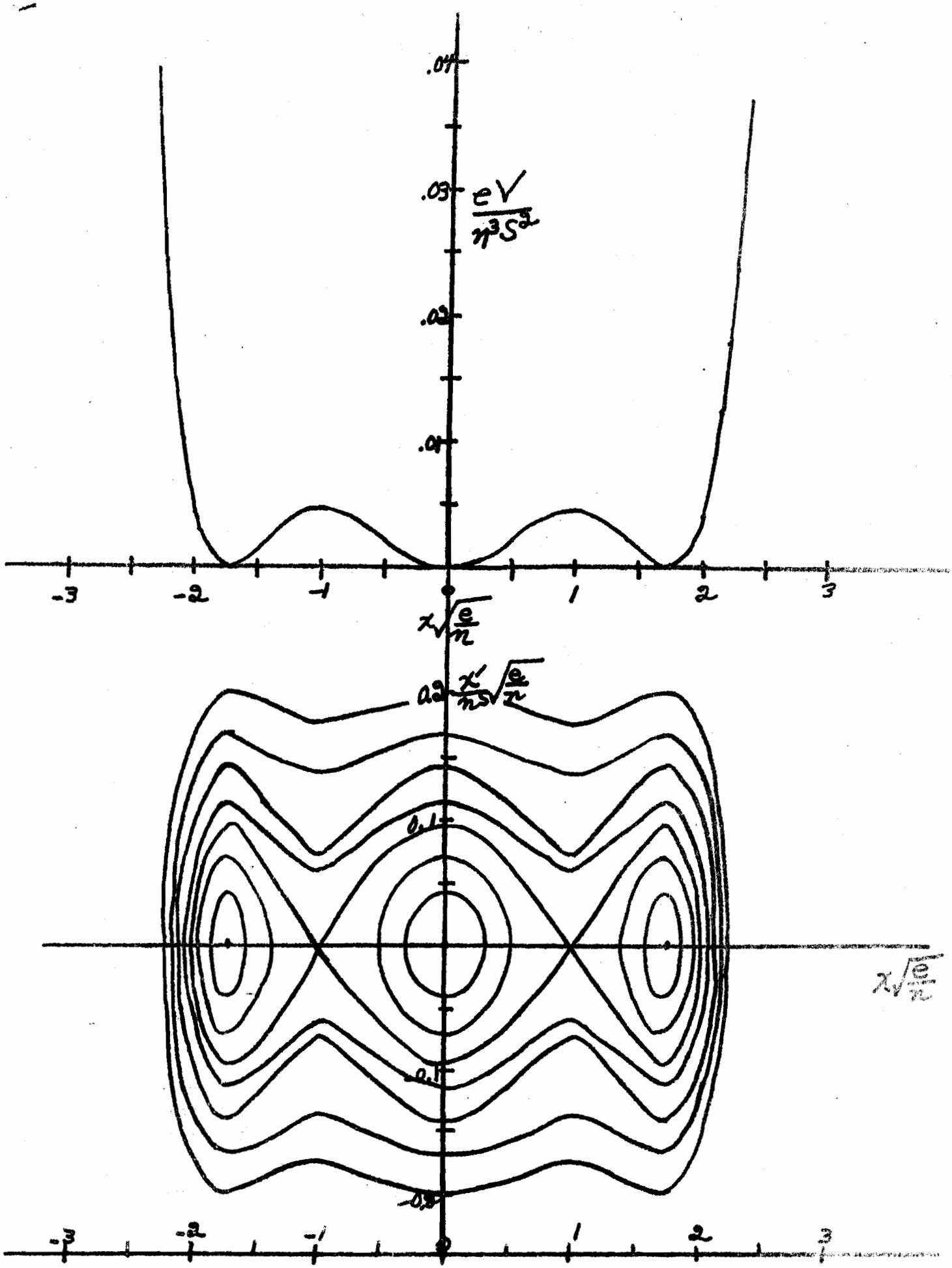
$x\sqrt{\frac{e}{n}}$

Fig. 1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-0.2
-3
-2
-1
1
2
3
$x\sqrt{\frac{e}{\eta}}$

Fig. 2

Table 1

Betatron wavelengths for AG orbits with cubic force aiding the linear force.

Smooth approximation: $\lambda = \lambda_0 / \sqrt{1 + A^2 + \frac{5}{24} A^4}$

N	A	λ_{smooth}	λ_{Illiac}
36	0	120.92	120.83
	3.16	21.46	23.88
	4.43	12.04	14.36
200	0	21.76	21.69
	1.12	13.54	14.37
	1.56	10.04	11.16
	1.79	8.64	9.81
	2.01	7.49	8.85
	2.24	6.48	7.87

Table 2

Betatron wavelengths for AG orbits with cubic force opposing linear force (oscillation about x = 0).

Smooth approximation: $\lambda = \lambda_0 \Big/ \sqrt{1 - A^2 + \frac{5}{24} A^4}$

N	A	λ_{smooth}	λ_{Illiac}
36	0	120.92	120.83
	.95	234	264
	separatrix		
	2.32	94.05	105.4
	2.88	45.58	47.25
	3.16	35.19	35.3
	3.48	27.42	26.74
	3.79	22.22	20.83
	4.11	18.32	16.43
	4.43	15.41	13.00
200	0	21.76	21.69
	.67	28.26	27.57
	1.01	49.06	53.47
	separatrix		
	1.96^+	45.09	110
	1.96^-	45.09	68.26
	1.97	42.76	58.17
	1.98	42.01	50.82
	2.00	37.69	45.23
	2.01	36.24	40.82

Table 3

Betatron wavelengths and centers of oscillation for AG orbits about side eyes with cubic force opposing linear force.

Smooth approximation: $B^2 = 2 - \frac{5}{2}A^2 + \sqrt{1 - 4A^2 + \frac{35}{8}A^4}$,

$$\lambda = \lambda_0 \sqrt{1 - A^2 + \frac{5}{24}A^4 - 4B^2 + \frac{5}{3}B^4 + \frac{5}{2}A^2B^2}.$$

N	A	B_{smooth}	B_{Illiac}	λ_{smooth}	λ_{Illiac}
36	0	1.732	1.732	60.46	60.41(?)
	.205	1.677	1.695	67.8	66.3
	.35	1.566	1.61	89.0	83.5
	.44	1.463	1.55	125.8	114
200	0	1.732	1.732	10.88	10.84(?)
	.15	1.703	1.71	11.55	11.36
	.29	1.619	1.63	13.92	13.95
	.42	1.488	1.54	19.69	21.82