MAC-JLP-1
March 30, [illegible]

NON-LINEARITIES IN AG SYNCHROTRON

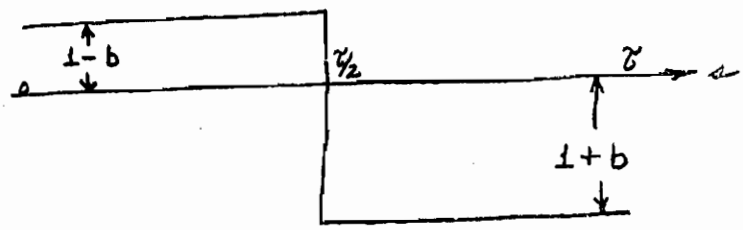
This is an incomplete report on results which have been obtained with the help of the Illinois digital computer. It is intended as an indication of the course of current work, and it is hoped that it may be of use to members of the group in their thinking on the general problem of orbits in the non-linear AG synchrotron.

The existing code provides solutions of the differential equations (MAC-JLP-1)

$$x'' = \left\{ \nu x - \tfrac{1}{3} e \left(x^3 - 3 x y^2 \right) \right\} U(s)$$

$$y'' = \left\{ -\nu y + \tfrac{1}{3} e \left(3 x^2 y - y^3 \right) \right\} U(s) \qquad (1)$$

$U(s) = U(s+\tau) =$



The integration proceeds by the Runge-Kutta method in 20 steps per cycle of the field gradient. Results are presented in tabular or photographic form which has become familiar. Input data in the form of initial conditions for coordinates and momenta are supplied to the computer by means of a punched tape. The integration of the differential equations requires approximately two seconds per cycle, that is, 1/10 second per Runge-Kutta step. Study of a very large accumulation of numerical results has provided more or less empirical information from which certain rather general properties of the non-linear system are beginning to emerge. The writer has been of the opinion that a thorough understanding of the complete system as described by Eqs. (1), which include

the effects of coupling of radial and vertical modes, could not be achieved before the uncoupled system had received considerable attention. As it has turned out, the structure of solutions of uncoupled equations appears to be sufficiently complex that even this is a major task. It is to this latter and simpler problem that the following remarks apply.

It may be of value to explain first the system of scale factors which has been employed in the reduction of Eqs. (1) to a "standard" form. It is clear that units of length may be chosen arbitrarily and independently in the direction parallel to the equilibrium orbit and perpendicular thereto. That is, one may write

$$s = at \qquad x = S\xi \qquad y = S\eta \tag{2}$$

where a and S are scale factors which are at our disposal. The differential equations for ξ and η are

$$\ddot{\xi} = \left\{N\xi - \tfrac{1}{3}E(\xi^3 - 3\xi\eta^2)\right\} v(at)$$

$$\ddot{\eta} = \left\{-N\eta + \tfrac{1}{3}E(3\xi^2\eta - \eta^3)\right\} v(at)$$

$$N = a^2 m \qquad E = a^2 S^2 e$$

One may, for example, choose the period of the driving force, $T = \tau/a$ arbitrarily and also arrange scale factors so as to provide any desired ratio of E to N. The scheme of normalization which has been adopted has been to take T = 1/10 and to choose S in such a way that E = N. The equations then assume the form

$$\ddot{\xi} = N\left\{\xi - \tfrac{1}{3}(\xi^3 - 3\xi\eta^2)\right\} v(at)$$

$$\ddot{\eta} = N\left\{-\eta + \tfrac{1}{3}(3\xi^2\eta - \eta^3)\right\} v(at)$$

in which only the single parameter N appears and the relative strengths of linear and non-linear force are determined by the amplitude of the motion. The rules for converting results of computations to standard form are

$$T = 1/10 \quad , \quad E = N \quad , \quad a = 200h \quad , \quad S = \sqrt{\frac{m}{e}}$$

$$N = (200h)^2 m$$

(h is the Runge-Kutta interval).

By the use of such scale factors it is possible to cover practically the entire range of values of variables and at the same time always present the computer with a problem which conforms to limitations set by the code. So far as the uncoupled system is concerned, a complete and practical solution of the entire system would consist of a series of graphs of the motion for representative values of N within the range of interest. Perhaps the most useful form for such a presentation is provided by the system of phase plots which has been used with such great advantage in the linear theory. For comparison to the linear theory the following table gives the connection between N and the familiar σ of EDC/HSS-1.

N	σ	N	σ
0	0	358.97	30°
200	16°36'	531.38	45°
400	33°31'	695.32	60°
600	51°10'	986.96	90°
800	70° 9'	1213.32	120°
1000	91°30'	1296.42	135°
1200	117°54'	1357.08	150°
1400	169°13'	1406.41	180°
1500	unstable		

$$\cos\sigma = \cos\psi \cosh\psi \quad , \qquad \psi = .05\sqrt{N}$$

It is not practical at the present time to present a complete summary of computer results. For purposes of illustration, however, we shall review a particular case, namely,

$N = 400$, $T = 1/10$, $b = 0$ (i.e., the forces are equal, but opposite in focussing and defocussing parts of a sector.)

The method of presentation adopted is based upon the observation that the differential equation induces in the phase plane a transformation of the form

$$\gamma: \quad \begin{aligned} \xi(t+T) &= f(\xi(t), \dot{\xi}(t), t) \\ \dot{\xi}(t+T) &= g(\xi(t), \dot{\xi}(t), t) \end{aligned} \qquad \frac{\partial(f,g)}{\partial(\xi,\dot{\xi})} = 1$$

which is continuous and area-preserving. This transformation is conveniently represented by a system of vectors with origins at the initial points $(\xi(t), \dot{\xi}(t))$ and termini at the points $(\xi(t+T), \dot{\xi}(t+T))$ corresponding to the same position in the next sector. The transformation depends continuously upon the choice of position t of initial points in the sector. This continuous variation in the phase plot from point to point in the sector, however, preserves all topological properties of the transformation. Insofar as these are concerned, the initial point may be chosen in whatever way is convenient. In the present instance the first point in a y-focussing sector has been chosen as origin since the computer begins its calculations at this point. It would perhaps be of some advantage to modify things in such a way as to allow more latitude in this respect, since phase plots based upon a point of symmetry as reference will in general have a simpler geometrical form. This is, however, not a particularly important consideration.

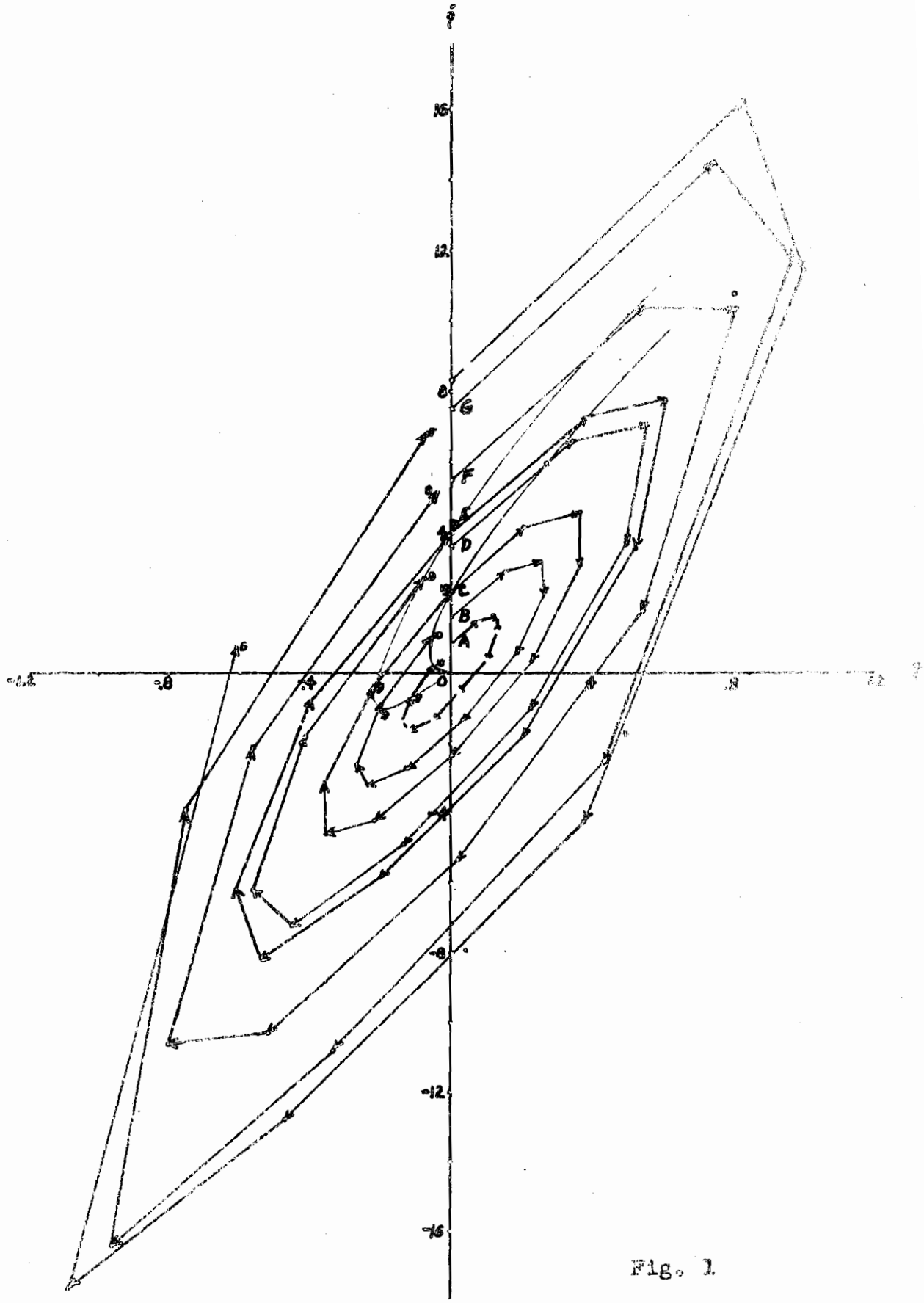


Fig. 1

Curve A in Fig. 1 shows the history of a small amplitude motion which is, therefore, very nearly linear and [illegible] conforms to expectation. Curves B, C, D, etc., illustrate the effect of larger amplitude and correspondingly larger non-linear force. Since this is a "y-motion," the cubic and linear terms add together. The effect of the non-linear term is, [illegible] to strengthen the total force and decrease the betatron wavelength. Thus, in the neighborhood of the origin where $\sigma = [illegible]$ approximately 10-3/4 sectors are required for a complete circuit of the origin. As one proceeds to larger amplitudes a steadily smaller number of sectors is required, and in Curve C the circuit is accomplished in 10 sectors, very nearly. In Curve D the non-linear force is larger still, and less than 10 sectors are required for one betatron oscillation, and in E, F, G, we see the wavelength shortened to 9, 8, 7 sectors, respectively. Slightly beyond G is an apparent "stability limit," and amplitudes larger than this give unstable motions. By a process of successive approximations it is quite easy to establish stability limits with any desired accuracy.

A closer study of the transformation reveals many other interesting features. The sequence of points marked 10 on each of the graphs define a curve which is a map of the vertical axis under the transformation τ^{10}, and those marked 9, under τ^{9} etc. The point of intersection of one of these maps with the vertical axis is seen to be, within the accuracy of present computations, a periodic or "fixed" point. That is to say, the motion which begins with this point, e.g. C, will repeat itself periodically in 10 sectors. The mere fact that the map of a curve under a transformation of this kind intersects the curve is, of course

no indication of periodicity at the point of intersection. (Indeed, the general situation is that illustrated in Fig. 2 in which there is clearly no fixed point.) The same situation is found for T^9, T^8, T^7, and so on, and is indeed a general property of such intersections which occur within regions of stability.

The second interesting fact emerges when one remarks that the points C, E have obviously the periods 20 and 18 sectors respectively. One might, therefore, expect, on perhaps no more than intuitive grounds, that there should be an intermediate point of period 19 sectors. Investigation with the computer shows that this is indeed the case, and the process of doubling and interpolating new fixed points can apparently be carried on indefinitely. One is thus led to the conjecture that the line is densely covered with periodic points, of larger and larger periods of course.

A third and very striking feature of the phase plots is that within stable regions all points derived from a given initial point appear to define a continuous closed curve encircling the origin. That this is so in the linear case is well known, but there is not, to the writer's knowledge, any adequate mathematical proof of this fact for the non-linear system. Also, the calculations seem to show that any radial line exhibits the same sequence of fixed points as the vertical one chosen above for the sake of example, and that each fixed point under $\mathcal{T}^n$ lies on a continuous closed curve, <u>each point</u> of which is fixed uner $\mathcal{T}^n$. Thus in the stable region the motion can apparently be described in terms of a one-parameter family of closed invariant curves, each of which transforms into itself under $\mathcal{T}$ and every point in the stable region is

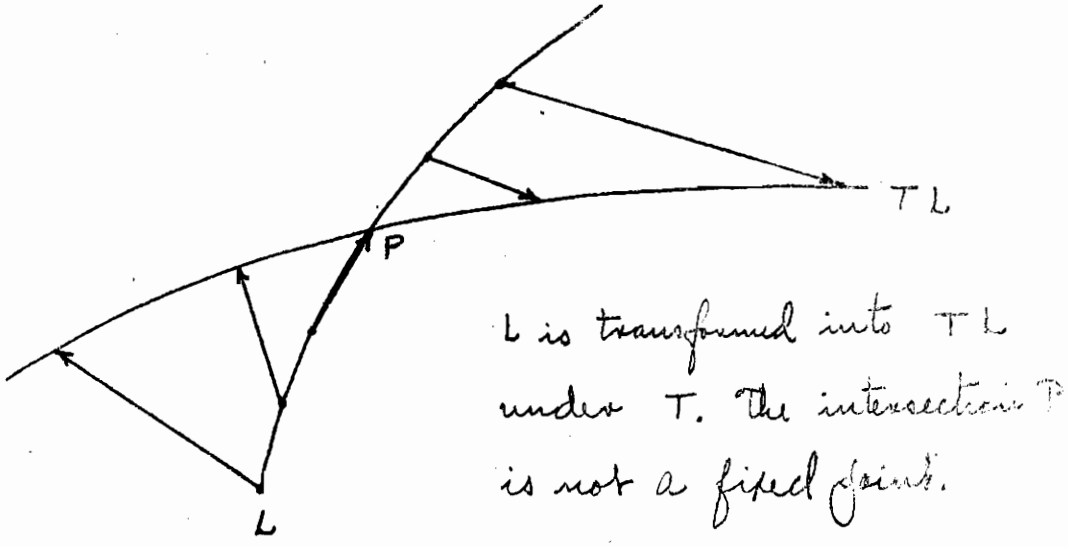

TL
P
L
L is transformed into TL
under T. The intersection P
is not a fixed point.

FIG. 2.

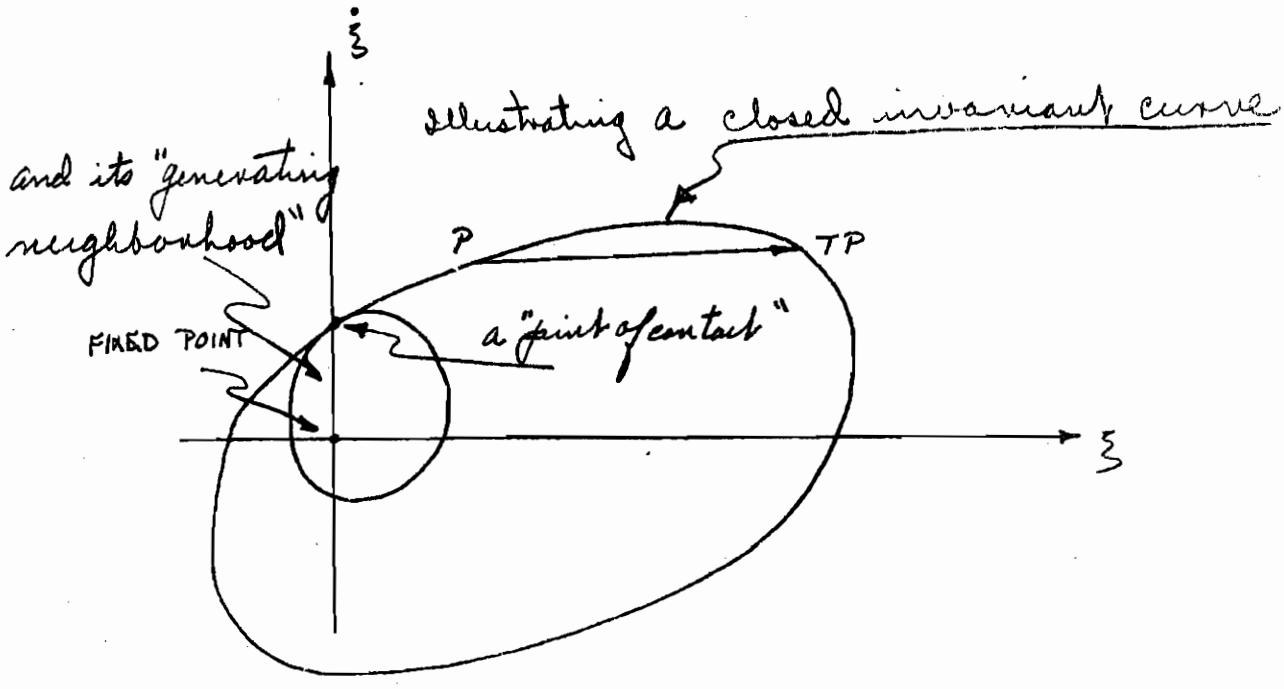

$\dot{\xi}$
illustrating a closed invariant curve
and its "generating neighbourhood"
P
TP
FIXED POINT
a "point of contact"
ξ

FIG. 3

"almost periodic", i.e., every point lies in a neighborhood [illegible] contains arbitrarily many points which are fixed under some [illegible]

When one goes into the region of unstable motion, [illegible] the other hand, these systematics disappear entirely. [illegible] points for an unstable motion have an erratic behavior which gives no indication whatever of any kind of smooth curve. This is in itself not surprising, since it is the property of the curve being closed which is of essential importance. This is illustrated most forcibly, perhaps, by remarking that any curve which joins a point with its image under τ generates under the transformation a curve which is invariant in the sense that it transforms into itself. However, it is not in general a <u>closed</u> curve.

It will perhaps be worthwhile to digress for a moment and describe some rather vague theoretical considerations which, in the writer's opinion, may in the future be developed into some sort of adequate theory. These considerations are based on the fact, well known to mathematicians and pointed out by Snyder at Brookhaven last summer, that in such a stable region there must be at least one closed invariant curve. The reasoning is as follows: Suppose that a point, say p_0, is uniformly stable, that is, that there is a neighborhood R of p_0 such that no image (under τ^n) of any point in the neighborhood lies outside some circle of radius M. Consider now the union of this neighborhood with all of its images under τ^n $(-\infty < n < \infty)$. Under the hypothesis stated this will define a finite region within the circle. Since the transformation is a continuous one, it is clear that the boundary of this finite region must transform into itself under τ, because any neighborhood which contains points of the boundary in its interior contains also points which

are inside and points which are outside the region generated by R, and the image of this neighborhood under τ must, [illegible] be of the same character and thus contain points of the [illegible] It is appreciated, of course, that these are mathematically [illegible] plete arguments, but they are illustrative of the point of view.

Continuing in this intuitive way, one is forced to the conclusion that the boundary not only transforms into itself under τ, but that it must also at some point be in contact with the generating neighborhood, as illustrated in Fig. 3. This is for the reason, of course, that the boundary consists of points which are at least arbitrarily close to images of points in the generating neighborhood. Incidentally, the boundary generated in this way defines a "disc" in the topological sense which must, according to a theorem of Brouwer, contain at least one point which is fixed under T. In practice, of course, this point is almost always the origin.

Suppose now that a small neighborhood which contains the origin has been used to generate an invariant closed curve in the way described above. The closed curve and its generator are in contact at some point. Construct now a new neighborhood which is a little bit larger and contains this point of contact in its interior. Provided the new neighborhood is stable, it will also generate a new boundary which lies outside the first one and is again an invariant closed curve. The process may be continued, and it is clear that the stable region may in this way be covered densely with invariant curves, the outermost of which will be the stability limit. If all the above is true, the remarkable fact relative to the intersection of a line with its map is now understood. Since the point of intersection is

simultaneously the image of a point of the [illegible] and a point of the invariant curve through the original point, it can be none other than the image of the original point itself. Thus the intersection is periodic. The above considerations are certainly not to be taken very seriously until they have been put on some sort of firm mathematical footing. It is the writer's opinion, however, that this should be possible, and work in this direction is going forward.

Since fixed points apparently play so prominent a role in the structure of the theory, it has been found useful to have at hand a classification of fixed points in terms of the transformation in their neighborhoods. This classification has already been carried out by various writers, particularly N. Levinson. The condition that the transformation be area-preserving, however, simplifies the classification over that which has been given in the literature for more general systems. In the neighborhood of a fixed point one may expand the transformation

$$\xi(T) = f(\xi, \dot{\xi})$$
$$\dot{\xi}(T) = g(\xi, \dot{\xi})$$

by Taylors theorem, retaining only first-order terms. One obtains in this way

$$\xi(T) = f_1 \xi + f_2 \dot{\xi} \qquad f_1 = \frac{\partial f}{\partial \xi}, \text{ etc.}$$
$$\dot{\xi}(T) = g_1 \xi + g_2 \dot{\xi}$$

Not all of the coefficients f_1, f_2, g_1, g_2 can be zero because of Liouville's theorem, viz $f_2 g_2 - f_1 g_1 = 1$. The transformation is, therefore, <u>always</u> linear and expressible in non-singular matrix form. Moreover, the area-preserving property is invariant to similarity, i.e., affine transformations. As is well known, the trace and determinant of a 2 X 2 matrix are invariant to such affine transformations. In this case the

determinant is always unity and the trace is the parameter which determines the topological character of the transformation in the neighborhood of the fixed point. Now the theory of such linear transformations has already been expounded at length. It is known that there are exactly five essentially different types of transformation determined by the trace according to the following table.

Case 1)	$-2 < tr A_1 < 2$
2)	$2 < tr A_2$
3)	$tr A_3 < -2$
4)	$tr A_4 = -2$
5)	$tr A_5 = 2$

$$A_1 = \begin{pmatrix} \cos\sigma & -\sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix} \qquad A_2 = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -\cosh\gamma & \sinh\gamma \\ \sinh\gamma & -\cosh\gamma \end{pmatrix} \quad A_4 = \begin{pmatrix} -1 & \alpha \\ 0 & -1 \end{pmatrix} \quad A_5 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

All possible linear transformations may be obtained by similarity transformations of the five matrices given. σ, γ, and α are real non-negative parameters.

Case 1 is the one familiar for stable linear motion. The origin is an isolated fixed point and the transformation in the neighborhood is described in terms of invariant circles. Case 2 corresponds to unstable linear motion in which a point transforms successively into points of the same branch of an hyperbolic arc. The origin is again an isolated fixed point. Case 3 is also an unstable situation in which successive points now alternate between opposite branches of an hyperbola. Case 4 is the limiting case separating 1 from 3 and is exemplified in the linear theory by the "$\sigma = \pi$

motion" in which successive points alternate between two parallel lines. The motion is divergent except for special initial conditions corresponding to the ξ - axis. The origin itself is an isolated fixed point under γ, but every point of the horizontal axis is fixed under γ^2. Case 5, which is of special interest, corresponds to a linear motion with $\sigma = 2\pi$ and forms the boundary line between case 1 and case 2. Here consecutive points lie on the same straight line and the transformation proceeds in equal increments at each step. The origin is again a fixed point, but <u>not</u> isolated. Every point of the ξ - axis is now fixed under γ. It is apparent that this must be the nature of the transformation in the neighborhood of any point of a closed <u>periodic</u> curve of the kind described above. It is understood, of course, that γ^m may be substituted for γ in all of the above discussion. It is a rather useful check on the conjectures made earlier to examine with the computer the details of the transformation in the neighborhood of such a curve. Again within the accuracy of computation, the conjectures are confirmed. That is, a transformation of type 5 is found in the neighborhood of any periodic point in the stable region. In terms of the functions f and g, the equation of such a closed curve of <u>periodic</u> points is simply $f_1 + g_2 = 2$ which fact may be an opening wedge for analytic treatment of the subject.

It is striking that all of the above remarks have practically nothing whatever to do with the fact that γ is generated by the differential equations of motion. Rather, the properties described would seem to apply to more general classes of finite transformations of the plane. However, it may be that every area-preserving transformation may be generated by a differential

equation of the type given although this would be surprising indeed. In this connection, it may be remarked that if the differential equation

$$\ddot{\xi} = F(\xi, t) \qquad F(\xi, t+T) = F(\xi, t)$$

generates the transformation

$$\xi(t+T) = f(\xi(t), \dot{\xi}(t), t)$$

$$\dot{\xi}(t+T) = g(\xi(t), \dot{\xi}(t), t)$$

that then one has, by an easy calculation,

$$f_2 F = g_2 g - \dot{\xi} - (f_3 g_2 - f_2 g_3) \quad , \quad f_3 = \frac{\partial}{\partial t} f(\xi, \dot{\xi}, t)$$

so that there is a one-one correspondence between the differential equation and the "force function" $F(\xi, t)$. It is amusing, and in some ways instructive, to verify this relationship in the linear case.

It is hoped that the above very sketchy outline of a few salient points will suffice as an interim report on a quite extensive program of calculation which is in progress. A complete summary must be post-poned in favor of continuing the program.

J. L. Powell