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EQUATIONS OF MOTION IN THE A.G.S.

A Review

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1. Introduction

The equations governing the trajectories in an alternate-gradient synchrotron have been derived, using a special system of coordinates and giving attention to coupling terms, in an Appendix to a report by Courant and Snyder (EDC/HSS-1, pp. 51 ff.). Reference has been made to the results of this analysis in EDC-3/MAC-2 and the subject has been reviewed in notes prepared at AIL by M. Hamermesh. The character of non-linear terms which may be caused deliberately to enter into the equations has been discussed by Powell (MAC-JLP-1,2).

It is the purpose of the present note to review in a simple way for internal use a derivation of these results for cases in which the equilibrium orbit is sufficiently smooth that it may in effect be considered circular. The extent to which substantial non-linear terms imply marked modification of the magnet poles will also be noted.

2. Use of the Principle of Least Action

Since it is the differential equations for the trajectories which are of interest, rather than equations for the motion in time, it is appropriate to consider use of the principle of least action. This principle is convenient to apply in the present instance, since the magnetic field (if static) does no work on the particle and the magnitude of the mechanical momentum may be considered constant.

We accordingly write

$$\delta \int (\bar{p} + e\bar{A}) \cdot \overline{ds} = 0, \tag{1}$$

where $\bar{p}$ represents the mechanical momentum and $\bar{A}$ is the vector potential. In cylindrical coordinates this assumes the form

$$\delta \int \left\{ p\,(r^2 + r'^2 + z'^2)^{1/2} + e\,(rA_\phi + r'A_r + z'A_z) \right\} d\phi = 0, \tag{2}$$

with primes designating differentiation with respect to the azimuth angle.

By application of Euler's equations, one finds for the z-motion

$$\frac{d}{d\phi}\left\{ \frac{p\,z'}{(r^2 + r'^2 + z'^2)^{1/2}} + eA_z \right\} - e\left[r\frac{\partial A_\phi}{\partial z} + r'\frac{\partial A_r}{\partial z} + z'\frac{\partial A_z}{\partial z} \right] = 0,$$

$$p\,\frac{d}{d\phi}\left\{ \frac{z'}{(r^2 + r'^2 + z'^2)^{1/2}} \right\} - e\left[r\frac{\partial A_\phi}{\partial z} + r'\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial \phi} - r'\frac{\partial A_r}{\partial r} \right] = 0, \quad \text{or}$$

$$\frac{d}{d\phi}\left\{ \frac{z'}{(r^2 + r'^2 + z'^2)^{1/2}} \right\} - \frac{e}{p}\left[r'B_\phi - rB_r \right] = 0. \tag{3}$$

Similarly, for the r-motion,

$$\frac{d}{d\phi}\left\{\frac{pr'}{(r^2+r'^2+z'^2)^{1/2}}+eA_r\right\}-\frac{pr}{(r^2+r'^2+z'^2)^{1/2}}$$

$$-\ e\left[A_\phi+r\frac{\partial A_\phi}{\partial r}+r'\frac{\partial A_r}{\partial r}+z'\frac{\partial A_z}{\partial r}\right]=0\,,$$

$$p\,\frac{d}{d\phi}\left\{\frac{r'}{(r^2+r'^2+z'^2)^{1/2}}\right\}-p\,\frac{r}{(r^2+r'^2+z'^2)^{1/2}}-e\left[\begin{array}{l}(A_\phi+r\,\partial A_\phi/\partial r)\ +z'\,\partial A_z/\partial r\\ -\,z'\,\partial A_r/\partial z\ -\partial A_r/\partial\phi\end{array}\right]=0\,,$$

$$\frac{d}{d\phi}\left\{\frac{r'}{(r^2+r'^2+z'^2)^{1/2}}\right\}-\frac{r}{(r^2+r'^2+z'^2)^{1/2}}-\frac{e}{p}\left[r\,B_z-z'\,B_\phi\right]=0\,. \quad (4)$$

In the absence of currents within the vacuum chamber, $\nabla\times\bar{B}=0$ and we write $\bar{B}=-\nabla V$. Since $\nabla\cdot\bar{B}=0$, $\nabla^2V=0$. If no currents thread the orbit, V will then be a single-valued harmonic function. Although expressions for vector potentials for possible use in (2) are given, for example, by Smythe ("Static and Dynamic Electricity", Ed. II, Sect. 7.05), it is considered more expedient to work with fields derived in this way from a scalar potential, for the following reasons:

(i) the vector potential components are inter-related,

(ii) the completeness of the assumed set of vector potential components may be more difficult to establish, and

(iii) the use of the scalar potential can automatically be employed to eliminate fields which would imply currents threading the orbit.

In the usual design of an ideal machine, neglecting azimuthal variations, it is appropriate for a first-order analysis to take

$$V = B_{oo}\left[z - n\frac{\Delta r}{r_o}z\right]. \quad (5)$$

The expression (5), containing second-order terms, gives fields which are satisfactory through first-order and satisfies Laplace's equation in zero order.

3. Potential Function

Harmonic potential functions, suitable for use with cylindrical coordinates, are obtainable by separation of variables and are listed, for example, in Smythe (op. cit., Sect. 5.291). Smythe's expressions involve Bessel functions $J_m(kr)$ and $N_m(kr)$, hyperbolic functions of kz, and circular functions of $m\phi$. The parameter m is restricted to integral values if single-valued azimuth functions are desired. In addition some specialized forms may be introduced: $r^{\pm m}\cdot(z \text{ or } 1)\cdot(\sin m\phi \text{ or } \cos m\phi)$, $(\ln r \text{ or } 1)\cdot(z \text{ or } 1)$.

Accordingly, expressions of interest for the radial function might include

$$\frac{J_m(kr)N_m(kr_o)-J_m(kr_o)N_m(kr)}{J_m'(kr_o)N_m(kr_o)-J_m(kr_o)N_m'(kr_o)} \quad \text{and} \quad \frac{J_m(kr)N_m'(kr_o)-J_m'(kr_o)N_m(kr)}{J_m(kr_o)N_m'(kr_o)-J_m'(kr_o)N_m(kr_o)}\,.$$

For work with functions of the above form it would be convenient to expand them about the radius r_o of the equilibrium orbit, as may be done conveniently by reference to the differential equation for these functions.

Suitable series, written through terms of fourth order, would thus include:

$$R_1(r) = k\Delta r - \frac{1}{2(kr_o)}(k\Delta r)^2 - \frac{1}{6}\left[1 - \frac{m^2+2}{(kr_o)^2}\right](k\Delta r)^3 + \frac{1}{24}\left[\frac{2}{kr_o} - 6\frac{m^2+1}{(kr_o)^3}\right](k\Delta r)^4 , \tag{6a}$$

$$R_2(r) = 1 - \frac{1}{2}\left[1 - \frac{m^2}{(kr_o)^2}\right](k\Delta r)^2 + \frac{1}{6}\left[\frac{1}{kr_o} - \frac{3m^2}{(kr_o)^3}\right](k\Delta r)^4 + \frac{1}{24}\left[1 - \frac{2m^2+3}{(kr_o)^2} + \frac{m^2(m^2+11)}{(kr_o)^4}\right](k\Delta r)^4 . \tag{6b}$$

These series, which satisfy Bessel's equation through terms of second order, are seen to approach the expansions for $\sin k\Delta r$ and $\cos k\Delta r$, respectively, when kr_o is large (in comparison both to unity and to m). Similarly, for $k = 0$, the series become the expansions of $\frac{1}{2}(kr_o/m)[(r/r_o)^m - (r_o/r)^m]$ and of $\frac{1}{2}[(r/r_o)^m + (r_o/r)^m]$, respectively; with <u>both</u> m and k equal to zero, R_1 represents $kr_o \ln r/r_o$.

The complexity of selecting a sufficiently general potential function by means similar to those indicated above suggests that it is preferable to follow the procedure of Powell (MAC-JLP-1) and write the potential directly as a power series in the quantities $x \equiv \Delta r/r_o$ and $y \equiv z/r_o$:

$$\frac{V}{B_{oo}} = z + r_o\Big[2\phi + Ax + By + Cx^2 + Dxy + Ey^2 + Fx^3 + Gx^2y + Hxy^2 + Iy^3 + Jx^4 + Kx^3y + Lx^2y^2 + Mxy^3 + Ny^4\Big], \tag{7}$$

through terms of fourth order, with ϕ and the coefficients A, B, ...N functions of the azimuth angle ϕ.

It has been suggested by Powell that it is adequate to require the potential arising from the terms within the square brackets (7) to satisfy Laplace's equation in rectangular coordinates; if we do not make this approximation at this point, however, we require instead that

$$(1+x)^2\frac{\partial^2 V}{\partial x^2} + (1+x)\frac{\partial V}{\partial x} + (1+x)^2\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial \phi^2} = 0 . \tag{8}$$

If, following Powell's notation, we introduce the coefficients

$$\left.\begin{array}{lll} \alpha = A , & a = C - E , & d = -4L , \text{ and} \\ \beta = B , & b = I - \frac{G}{3} , & e = \frac{3}{2}(K - M) , \\ n = -D , & c = F - \frac{H}{3} , & \end{array}\right\} \tag{9a..h}$$

the coefficients of (7) are determined by Laplace's equation (8) to yield the potential

$$
\begin{aligned}
\frac{V}{r_o B_{oo}} = {} & y + 2\phi \\
& + \alpha x + \beta y + \frac{a - \phi'' - (\alpha/2)}{2} x^2 - n xy - \frac{a + \phi'' + (\alpha/2)}{2} y^2 \\
& + \frac{6c - a + (3/2)\alpha - \alpha'' + 5\phi''}{12} x^3 - \frac{6b - n + \beta''}{4} x^2 y \\
& - \frac{6c + a - (3/2)\alpha + \alpha'' - 5\phi''}{4} xy^2 + \frac{6b + n - \beta''}{12} y^3 \\
& + \frac{d - 3c + (5/2)a - (15/4)\alpha + 5\alpha'' - (33/2)\phi'' - a'' + \phi^{iv}}{24} x^4 \\
& + \frac{4e + 3b - (3/2)n + (5/2)\beta'' + n''}{12} x^3 y - \frac{d}{4} x^2 y^2 \\
& - \frac{4e - 3b + (3/2)n - (5/2)\beta'' - n''}{12} x y^3 \\
& + \frac{d + 3c + (a/2) - (3/4)\alpha + a'' + \alpha'' - (5/2)\phi'' + \phi^{iv}}{24} y^4 . \qquad (10)
\end{aligned}
$$

This potential may be used to obtain expressions for the field-components valid through terms of third order.

$$
\begin{aligned}
-\frac{B_\phi}{B_{oo}} = {} & (1+x)^{-1} \frac{\partial\left(\frac{V}{r_o B_{oo}}\right)}{\partial \phi} \\
= {} & 2\phi' + (\alpha' - 2\phi') x + \beta' y \\
& + \frac{a' - (5/2)\alpha' + 4\phi' - \phi^{iii}}{2} x^2 - (n' + \beta') xy - \frac{a' + (\alpha'/2) + \phi^{iii}}{2} y^2 \\
& + \frac{6c' - 7a' + (33/2)\alpha' - \alpha^{iii} - 24\phi' + 11\phi^{iii}}{12} x^3 \\
& + \frac{-6b' + 5n' + 4\beta' - \beta^{iii}}{4} x^2 y \\
& + \frac{-6c' + a' + (5/2)\alpha' - \alpha^{iii} + 7\phi^{iii}}{4} xy^2 + \frac{6b' + n' - \beta^{iii}}{12} y^3 , \qquad (11a)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{B_z}{B_{oo}} = {} & \frac{\partial\left(\frac{V}{r_o B_{oo}}\right)}{\partial y} \\
= {} & 1 + \beta - nx - \left(a + \phi'' + \frac{\alpha}{2}\right) y \\
& - \frac{6b - n + \beta''}{4} x^2 - \frac{6c + a - (3/2)\alpha + \alpha'' - 5\phi''}{2} xy + \frac{6b + n - \beta''}{4} y^2 \\
& + \frac{4e + 3b - (3/2)n + (5/2)\beta'' + n''}{12} x^3 - \frac{d}{2} x^2 y \\
& - \frac{4e - 3b + (3/2)n - (5/2)\beta'' - n''}{4} xy^2 \\
& + \frac{d + 3c + (a/2) - (3/4)\alpha + a'' + \alpha'' - (5/2)\phi'' + \phi^{iv}}{6} y^3 , \qquad (11b)
\end{aligned}
$$

$$-\frac{B_r}{B_{oo}} = \frac{\partial\left(\frac{V}{r_o B_{oo}}\right)}{\partial x}$$

$$= \alpha + \left(a - \phi'' - \frac{\alpha}{2}\right)x - n\,y$$

$$+ \frac{6c - a + (3/2)\alpha - \alpha'' + 5\phi''}{4}\,x^2 + \frac{-6b + n - \beta''}{2}\,xy$$

$$- \frac{6c + a - (3/2)\alpha + \alpha'' - 5\phi''}{4}\,y^2$$

$$+ \frac{d - 3c + (5/2)a - (15/4)\alpha + 5\alpha'' - (33/2)\phi'' - a'' + \phi^{iv}}{6}\,x^3$$

$$+ \frac{4c + 3b - (3/2)n + (5/2)\beta'' + n''}{4}\,x^2y - \frac{d}{2}\,xy^2$$

$$- \frac{4c - 3b + (3/2)n - (5/2)\beta'' - n''}{12}\,y^3 \,. \qquad (11c)$$

The correctness of these field components may be checked by use of the condition $\nabla\cdot\bar{B} = 0$. The field components (11a,b,c) are found to satisfy

$$(1+x)\left(\frac{\partial B_z}{\partial y} + \frac{\partial B_r}{\partial x}\right) + B_r + \frac{\partial B_\phi}{\partial \phi} = 0$$

through terms of second order.

4. The Equations of Motion

The field components (11a,b,c) may be substituted into (3) and (4) to obtain the equations governing the trajectories. We restrict ourselves at present to particles possessing the equilibrium momentum; i.e., $p = p_o = eB_{oo}r_o$. Since it shall be assumed that x and y are no greater than of order $\frac{1}{n} \ll 1$, it is permissible to replace

$$\frac{d}{d\phi}\left\{\frac{z'}{(r^2+r'^2+z'^2)^{1/2}}\right\} \text{ by } \frac{y''}{1+x} \quad \text{and} \quad \frac{d}{d\phi}\left\{\frac{r'}{(r^2+r'^2+z'^2)^{1/2}}\right\} \text{ by } \frac{x''}{1+x}\,.$$

To the same order we put $\dfrac{r}{(r^2+r'^2+z'^2)^{1/2}} \cong 1$.

One then obtains for the axial motion

$$\frac{y''}{1+x} - \left[\frac{r'}{r_o}\frac{B_\phi}{B_{oo}} - \frac{r}{r_o}\frac{B_r}{B_{oo}}\right] = 0\,,$$

$$y'' + (1+x)^2\frac{B_r}{B_{oo}} - (1+x)\,x'\,\frac{B_\phi}{B_{oo}} = 0\,, \qquad (12)$$

$$
\begin{aligned}
y'' + ny \;=\; & \alpha + (a + \tfrac{3}{2}\alpha - \phi'')\,x \\
& + (\tfrac{3}{2}c + \tfrac{7}{4}a + \tfrac{3}{8}\alpha - \tfrac{\alpha''}{4} - \tfrac{3}{4}\phi'')\,x^2 - (3b + \tfrac{3}{2}n + \tfrac{\beta''}{2})\,xy \\
& - (\tfrac{3}{2}c + \tfrac{a}{4} - \tfrac{3}{8}\alpha + \tfrac{\alpha''}{4} - \tfrac{5}{4}\phi'')\,y^2 \\
& + (\tfrac{d}{6} + \tfrac{5}{2}c + \tfrac{11}{12}a - \tfrac{3}{8}\alpha + \tfrac{\alpha''}{3} - \tfrac{a''}{6} - \tfrac{5}{4}\phi'' + \tfrac{\phi^{iv}}{6})\,x^3 \\
& + (e - \tfrac{21}{4}b - \tfrac{3}{8}n - \tfrac{3}{8}\beta'' + \tfrac{n''}{4})\,x^2y \\
& - (\tfrac{d}{2} + 3c + \tfrac{a}{2} - \tfrac{3}{4}\alpha + \tfrac{\alpha''}{2} - \tfrac{5}{2}\phi'')\,xy^2 \\
& - (\tfrac{e}{3} - \tfrac{b}{4} + \tfrac{n}{8} - \tfrac{5}{24}\beta'' - \tfrac{n''}{12})\,y^3 \\
& - x'\,(2\phi' + \alpha' x + \beta' y - n' xy)\,, \qquad (13)
\end{aligned}
$$

in which terms through third order in x, y are retained in the part of the expression not involving x'. Since the terms involved in the coefficient of x' will presumably not be introduced intentionally, we keep explicitly in this coefficient only terms through first order in x, y, and the term $-n'xy$ which is necessarily present.

Similarly, for the radial motion,

$$
x'' - (1+x) - (1+x)^2 \frac{B_z}{B_{oo}} + (1+x)\,y' \frac{B_\phi}{B_{oo}} = 0\,, \qquad (14)
$$

$$
\begin{aligned}
x'' + (1 - n + 2\beta)\,x \;=\; & -\beta + (a + \phi'' + \tfrac{\alpha}{2})\,y \\
& + (\tfrac{3}{2}b + \tfrac{7}{8}n - 1 - \beta + \tfrac{\beta''}{4})\,x^2 \\
& + (3c + \tfrac{5}{2}a + \tfrac{\alpha}{4} + \tfrac{\alpha''}{2} - \tfrac{\phi''}{2})\,xy - (\tfrac{3}{2}b + \tfrac{n}{4} - \tfrac{\beta''}{4})\,y^2 \\
& - (\tfrac{e}{3} - \tfrac{11}{4}b - \tfrac{5}{8}n - \tfrac{7}{24}\beta'' + \tfrac{n''}{12})\,x^3 \\
& + (\tfrac{d}{2} + 6c + 2a - \alpha + \alpha'' - 4\phi'')\,x^2y \\
& + (e - \tfrac{15}{4}b - \tfrac{n}{8} - \tfrac{\beta''}{8} - \tfrac{n''}{4})\,xy^2 \\
& - (\tfrac{d}{6} + \tfrac{c}{2} + \tfrac{a}{12} - \tfrac{\alpha}{8} + \tfrac{a''}{6} + \tfrac{\alpha''}{6} - \tfrac{5}{12}\phi'' + \tfrac{\phi^{iv}}{6})\,y^3 \\
& + y'\,(2\phi' + \alpha' x + \beta' y - n' xy)\,, \qquad (15)
\end{aligned}
$$

to the same order as eq. (13) for the axial motion.

It is noted from eqns. (14) and (12) that the coefficients of y' in (15) and of $-x'$ in (13) are identically the same expression $\partial(V/r_oB_{oo})/\partial\phi$.

By way of comparison, the potential cited by Powell (MAC-JLP-1, p. 3), if employed in accordance with his procedure, leads to the equations:

$$
\begin{aligned}
y'' + ny = {} & \alpha + (a - \phi'')\,x + \tfrac{1}{4}(6c - \alpha'')\,x^2 \\
& - \tfrac{1}{2}(6b + \beta'')\,xy - \tfrac{1}{4}(6c + \alpha'')\,y^2 \\
& + \tfrac{1}{6}(d - a'' + \phi^{iv})\,x^3 + \tfrac{1}{4}(n'' + 4e)\,x^2y \\
& - \tfrac{1}{2}d\,xy^2 + \tfrac{1}{12}(n'' - 4e)\,y^3 \\
& - x'\,(2\phi' + \alpha' x + \beta' y - n' xy + \ldots)
\end{aligned}
\qquad \text{(i)}
$$

and

$$
\begin{aligned}
x'' + (1-n)x = {} & -\beta + (a + \phi'')\,y + \tfrac{1}{4}(6b + \beta'')\,x^2 \\
& + \tfrac{1}{2}(6c + \alpha'')\,xy - \tfrac{1}{4}(6b - \beta'')\,y^2 \\
& - \tfrac{1}{12}(n'' + 4e)\,x^3 + \tfrac{1}{2}d\,x^2y \\
& - \tfrac{1}{4}(n'' - 4e)\,xy^2 - \tfrac{1}{6}(d + a'' + \phi^{iv})\,y^3 \\
& + y'\,(2\phi' + \alpha' x + \beta' y - n' xy + \ldots)\,.
\end{aligned}
\qquad \text{(ii)}
$$

[Evidently the coefficient of $n''y^3$ in (i) and of $n''x^3$ in (ii) was miswritten on p.5 of Powell's report; of comparable note are the terms $\frac{1}{4}n''x^2y$ in (i) and $-\frac{1}{4}n''xy^2$ in (ii), which also appear in (13) and (15) respectively.]

It would at this point appear reasonable to adopt the following viewpoint concerning the functions in (13) and (15) and in the corresponding equations (i) and (ii):

With care in the design and fabrication of the magnet structure one endeavors to secure

α and β of order $1/n$, or less;

a, b, and c of order no greater than n;

d and e may be made deliberately of order n^3 if significant cubic terms are desired;

ϕ' no greater than of order $\sqrt{n}$ and ϕ'' no greater than order n;

the rate of variation of the function n with azimuth will be limited by the pole-face discontinuity being removed from the median plane by roughly $r_0/|n|$ -- this may result in each differentiation increasing the magnitude of the function by no more than one order of n and, in such cases, the extent of the region in which the derivatives are large will be short compared to a sector length;

the derivatives of the remaining functions will be subject to limitations similar to those applying to n; and, finally,

x' and y' will be of order $1/\sqrt{n}$.

If the foregoing conditions apply, the equations (13) and (15), or (i) and (ii), reduce within a sector to the similar, and simpler, form

$$y'' + ny = -2\phi' x' + (a - \phi'') x + \frac{e}{3}(3x^2y - y^3) - \frac{d}{6}(3xy^2 - x^3), \tag{16a}$$

$$x'' + (1-n)x = +2\phi' y' + (a + \phi'') y + \frac{e}{3}(3xy^2 - x^3) + \frac{d}{6}(3x^2y - y^3); \tag{16b}$$

if these conditions are <u>not</u> fullfilled, recourse might be had to eqs. (13) and (15).

The linear terms in these equations (16a,b) fall into the form of eqs. (1) of EDC-3/MAC-2, which originated in the Brookhaven report EDC/HSS-1 (pp. 37 ff.) of Courant and Snyder, with $M = a$ and $Q = 2\phi'$.

In concordance with an observation of Courant (EDC-3/MAC-2, p. 2), the terms involving x' and y' in (16a,b) may be removed by a suitable transformation. For this purpose we introduce the quantities u and v as new dependent variables*:

$$\begin{aligned} x &= u \cos\phi + v \sin\phi, \\ y &= -u \sin\phi + v \cos\phi. \end{aligned} \tag{17}$$

(It is noted that, with this substitution, the bounds on x and y are similar to those on u and v in that, in particular, $x^2+y^2 = u^2+v^2$.)

Our equations (16a,b) then become:

$$\begin{aligned} v'' &+ v\left[\phi'^2 + \frac{1}{2} + (n - \frac{1}{2}) \cos 2\phi - a \sin 2\phi\right] \\ &= u\left[a \cos 2\phi + (n - \frac{1}{2}) \sin 2\phi\right] \\ &+ (3u^2v - v^3)\left[\frac{e}{3} \cos 4\phi + \frac{d}{6} \sin 4\phi\right] \\ &- (3uv^2 - u^3)\left[\frac{d}{6} \cos 4\phi - \frac{e}{3} \sin 4\phi\right], \end{aligned} \tag{18a}$$

$$\begin{aligned} u'' &+ u\left[\phi'^2 + \frac{1}{2} - (n - \frac{1}{2}) \cos 2\phi + a \sin 2\phi\right] \\ &= v\left[a \cos 2\phi + (n - \frac{1}{2}) \sin 2\phi\right] \\ &+ (3uv^2 - u^3)\left[\frac{e}{3} \cos 4\phi + \frac{d}{6} \sin 4\phi\right] \\ &+ (3u^2v - v^3)\left[\frac{d}{6} \cos 4\phi - \frac{e}{3} \sin 4\phi\right]. \end{aligned} \tag{18b}$$

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*This transformation was suggested to the writer by discussions with Professors Langenhop, Maple, and Thielman of the Department of Mathematics, Iowa State College. As has been pointed out by Powell (private communication), this corresponds physically to a rotation of axies at a rate proportional to the longitudinal field $-2\phi' B_{oo}$.

The terms involving n' and n'' in (13) and (15), or in (i) and (ii), will play a dominant rôle at the transition between sectors. The equations would then assume the form

$$y'' = +\frac{n''}{12}y^3 + \frac{n''}{4}x^2y + n'xy\,x' , \tag{19a}$$

$$x'' = -\frac{n''}{12}x^3 - \frac{n''}{4}xy^2 - n'xy\,y' . ** \tag{19b}$$

The discontinuities which result, according to these equations, in passage between adjacent sectors have been examined by Powell (MAC-JLP-2) and by Snyder and Sternheimer (HSS/RMS-1), presumably with the condition in mind that the transition occupies an arc-length comparable with the aperture:

$$\Delta y' = \left[-\frac{x_o^2+y_o^2}{4}\langle y'\rangle + \frac{x_oy_o}{2}\langle x'\rangle\right]\cdot\Delta n , \tag{20a}$$

$$\Delta x' = \left[\frac{x_o^2+y_o^2}{4}\langle x'\rangle - \frac{x_oy_o}{2}\langle y'\rangle\right]\cdot\Delta n , \tag{20b}$$

$$\Delta y = \left[\frac{y_o^3}{12} + \frac{x_o^2y_o}{4}\right]\cdot\Delta n , \tag{20c}$$

$$\Delta x = -\left[\frac{x_o^3}{12} + \frac{x_oy_o^2}{4}\right]\cdot\Delta n . \tag{20d}$$

Powell has checked this result by appeal to the condition that the Jacobian of the transformation from $-\epsilon$ to $+\epsilon$ shall be unity. A second check of the results for $\Delta x', \Delta y'$ may be obtained by noting $dH/d\phi = \partial H/\partial\phi$; by forming the partial derivative of Powell's Hamiltonian, considering it as the total derivative, and integrating it through the transition, one may compare the result so found with that resulting from the values of $\Delta p_x, \Delta p_y$ (or $\Delta x', \Delta y'$) found previously. In the present case the results appear to be consistent with

$$\Delta H = \left[(1/4)(x_o^2 + y_o^2)(x_o'^2 - y_o'^2) - (1/2)(x_o^2 - y_o^2)\right]\cdot\Delta n .$$

In conclusion it should be remarked that neglecting relatively small terms in a differential equation does not necessarily lead to solutions which remain close to the correct form. Thus the terms α and $-\beta$ (in (13) or (i) and in (15) or (ii)) -- or the terms $\alpha\cos\phi - \beta\sin\phi$ which would then enter in (18a) and $-\beta\cos\phi - \alpha\sin\phi$ in (18b) -- represent periodic forcing functions which, even when small, may have grave consequences under resonant (or very-nearly resonant) conditions (EDC/HSS-1, Sect. 6a).

** For the terms considered here, $-\frac{V}{B_{oo}r_o} = -\frac{xy}{2}\sum_{k=0}(-)^k\frac{x^{2k}+y^{2k}}{(2k+1)!}\frac{d^{2k}n}{d\phi^{2k}}$

and has been carried through the terms of fourth order in x, y.

5. Field Plots

With reasonably large values of the index n, the distinction between a suitable three-dimensional azmuith-independent potential and a similar two-dimensional potential becomes negligible in constructing field plots within the region of interest. One may compare, for example, the z-component of the focusing field (exclusive of the guide-field) as given by potentials of the form

$$\text{(i)} \quad \frac{n}{k^2 r_o} \frac{N_o(kr_o)J_o(kr) - J_o(kr_o)N_o(kr)}{N_o(kr_o)J_1(kr_o) - J_o(kr_o)N_1(kr_o)} \sinh kz \;;$$

$$\text{(ii)} \quad \frac{n}{k}\left[-\frac{\Delta r}{r_o} + \frac{1}{2}\left(\frac{\Delta r}{r_o}\right)^2 - \frac{2-(kr_o)^2}{6}\left(\frac{\Delta r}{r_o}\right)^3\right] \sinh kz \;, \quad \text{from (6a),}$$

the resultant field-component $-\partial V/\partial z$ being proportional to (11b) when $\cosh kz$ is expanded and suitable values are assigned to the constants $\left[n = kr_o, \quad b = -kr_o/6, \quad e = \frac{1}{2}((kr_o)^3 - (kr_o))\right]$;

$$\text{(iii)} \quad \text{the two-dimensional form} \quad -\frac{1}{k} \sin k\Delta r \sinh kz \,.$$

A comparison of the field-values resulting from these expressions is given in the following table:

$$\frac{(-B_z) - B_{oo}}{B_{oo}}$$

for $kr_o = n = 400$, $\Delta r/r_o = \mp 1/n$

$n\ (\Delta r/r_o)$	(i) 3-dimensional	(ii) Series (6a)	(iii) 2-dimensional
-1	$(0.8426)\cosh\frac{nz}{r_o}$	$(0.8346)\cosh\frac{nz}{r_o}$	$-\sin(-1)\cosh\frac{nz}{r_o} = (0.8415)\cosh\frac{nz}{r_o}$
$+1$	$(-0.8405)\cosh\frac{nz}{r_o}$	$(-0.8321)\cosh\frac{nz}{r_o}$	$-\sin(1)\cosh\frac{nz}{r_o} = (-0.8415)\cosh\frac{nz}{r_o}$.

The two-dimensional result is seen to differ from the three-dimensional form by less than two parts per thousand in this example.

It is of interest to sketch the field associated with situations in which the field at the median plane exhibits a dependence on r of the type currently under discussion in the Mid-West Conference. We use two-dimensional potential functions for convenience and consider the region of interest to extend to $x \equiv \Delta r/r_0 = \pm 1/n$, $y \equiv z/r_0 = \pm 1/n$. Non-linearities of amount characterized by $(kr_0)^2 = \pm n^2$ $\left[e = \pm \frac{n^3}{2}\right]$ will be considered; these values will imply cubic forces as great as one-half the linear forces for certain points within the region of interest (see following table).

(x,y)	$\text{Lim}\ \dfrac{(n^3/6)(3x^2y - y^3)}{ny}$
$(0.5/n,\ 0)$	0.125
$(0.75/n,\ 0)$	0.281
$(1/n,\ 0)$	0.500

The figures which have been constructed are listed below, with $X \equiv nx$, $Y \equiv ny$, and $Z = X + iY$:

Fig. 1: Conventional Quadrupole Field --

$$W = Z - Z^2/2 \qquad U = X - \frac{X^2 - Y^2}{2}$$

$$\left|\frac{dW}{dZ}\right|_{Y=0} = \left|1-Z\right|_{Y=0} = 1 - X \qquad V = Y - XY \ ;$$

Fig. 2: Sinusoidal Field in Median Plane --

$$W = Z + \cos Z \qquad U = X + \cos X \cosh Y$$

$$\left|\frac{dW}{dZ}\right|_{Y=0} = \left|1 - \sin Z\right|_{Y=0} \qquad V = Y + \sin X \sinh Y \ ;$$

$$= 1 - \sin X$$

Fig. 3: Hyperbolic Field in Median Plane --

$$W = Z - \cosh Z \qquad U = X - \cosh X \cos Y$$

$$\left|\frac{dW}{dZ}\right|_{Y=0} = \left|1 - \sinh Z\right|_{Y=0} \qquad V = Y - \sinh X \sin Y \ ;$$

$$= 1 - \sinh X$$

Fig. 4: Cubic field in opposition to linear field in median plane --

$$W = Z - \frac{Z^2}{2} + \frac{Z^4}{24} \qquad U = X - \frac{X^2-Y^2}{2} + \frac{X^4-6X^2Y^2+Y^4}{24}$$

$$\left|\frac{dW}{dZ}\right|_{Y=0} = 1 - X + \frac{X^3}{6} \qquad V = Y - XY + \frac{XY(X^2-Y^2)}{4} \;;$$

Fig. 5: Cubic field supplementing linear field in median plane --

$$W = Z - \frac{Z^2}{2} - \frac{Z^4}{24} \qquad U = X - \frac{X^2-Y^2}{2} - \frac{X^4-6X^2Y^2+Y^4}{24}$$

$$\left|\frac{dW}{dZ}\right|_{Y=0} = 1 - X - \frac{X^3}{6} \qquad V = Y - XY - \frac{XY(X^2-Y^2)}{6} \;.$$

It is notable the extent to which the introduction of non-linear fields in the median plane necessitates a marked modification of the theoretical pole faces exterior to the useful aperture. By means of a plotting board at Iowa State College, Dr. Zaffarano is undertaking to examine the non-linear fields which can be produced by reasonable poles of finite extent (with or without the use of a neutral-pole structure), ignoring for the present the influence of the coils which generate the magnetomotive force in practice.

Fig. 1

ILLUSTRATION
of
QUADRUPOLE FIELD

Field Linear in Median Plane
(e = 0)

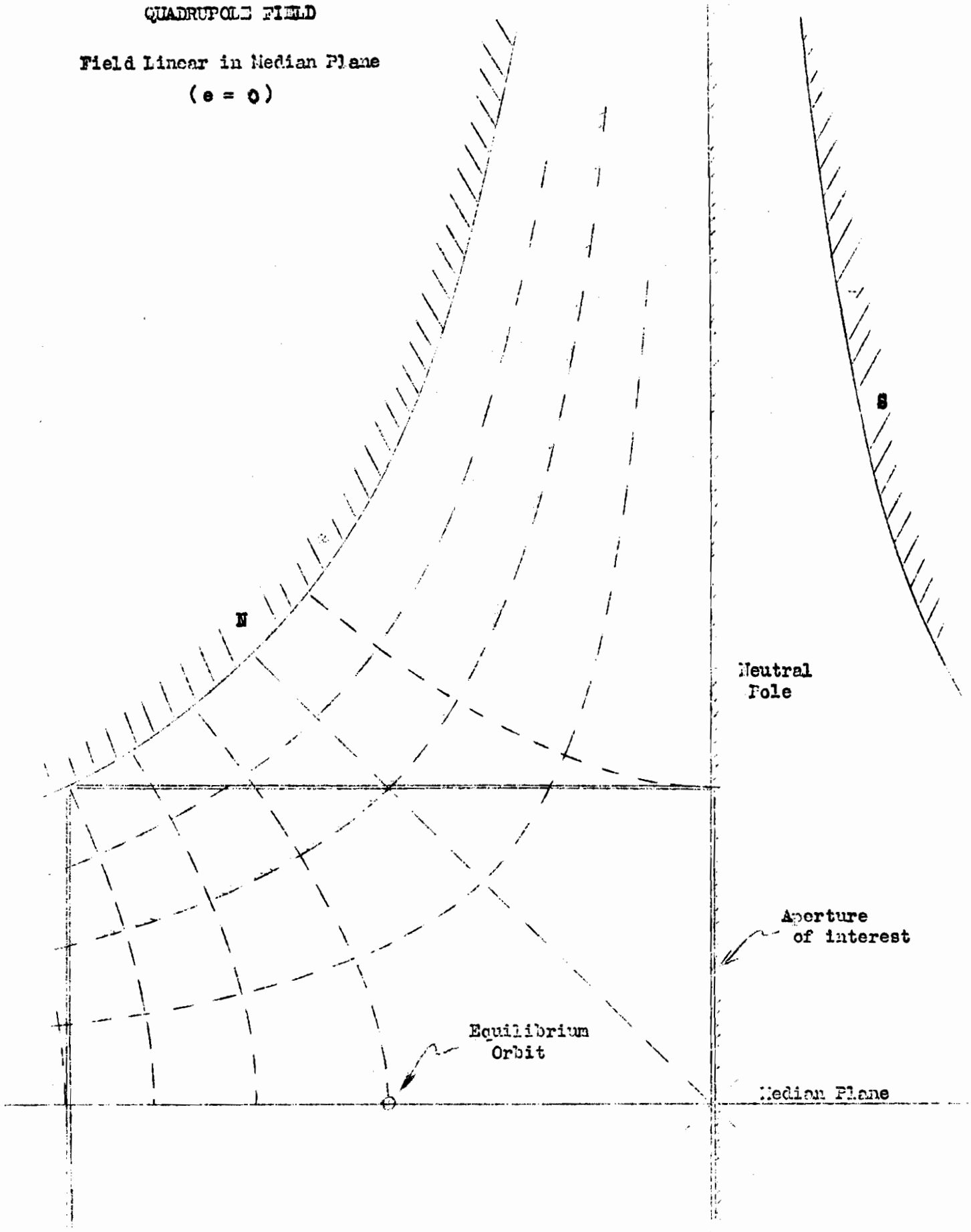


Fig. 2

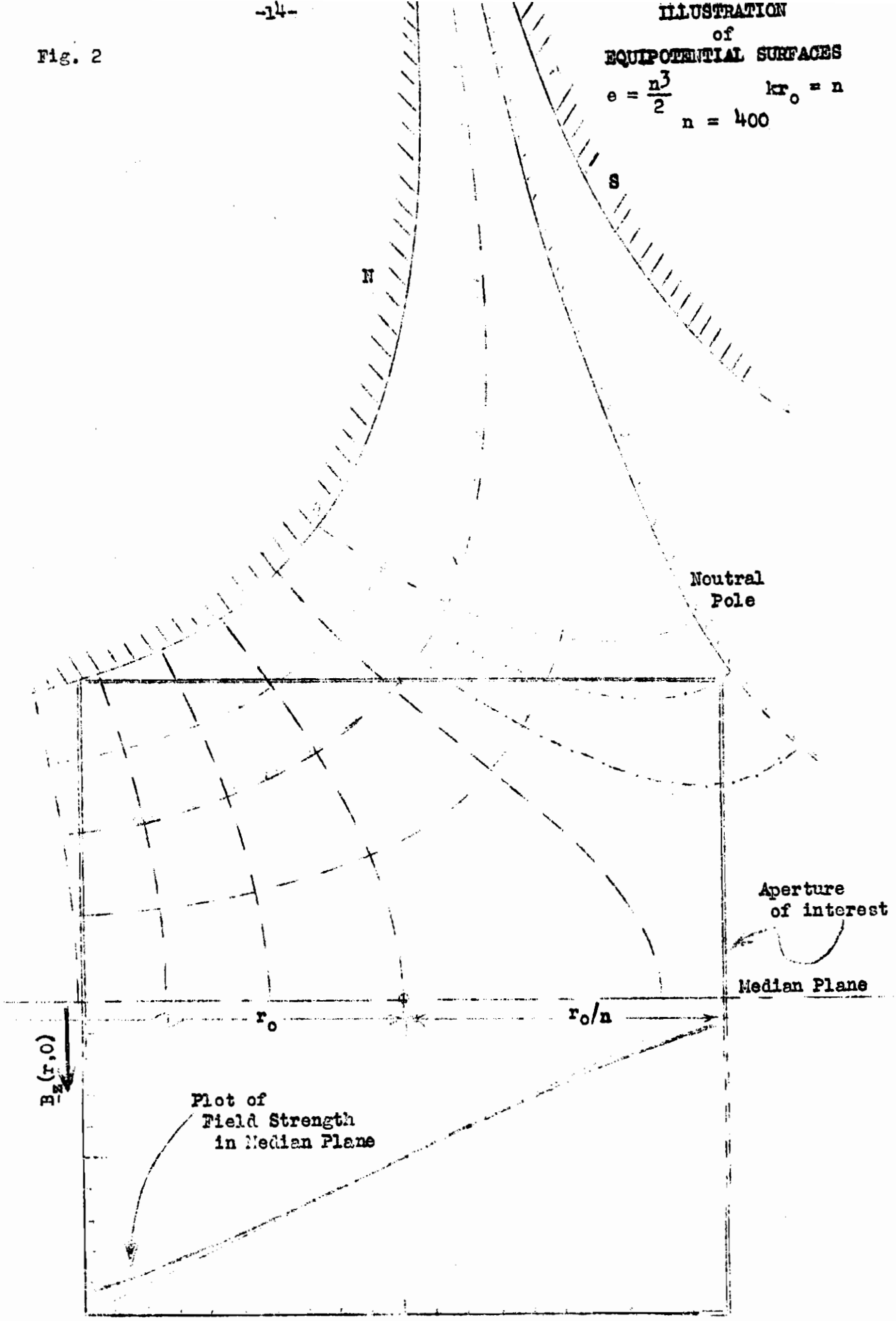


ILLUSTRATION
of
EQUIPOTENTIAL SURFACES
$e = \frac{n^3}{2}$
$kr_o = n$
$n = 400$
S
N
Neutral
Pole
Aperture
of interest
Median Plane
r_o
r_o/n
$B_z(r,0)$
Plot of
Field Strength
in Median Plane

Fig. 3 ILLUSTRATION of EQUIPOTENTIAL SURFACES

$e = -\frac{n^3}{2}$ $kr_o = jn$

(n large)

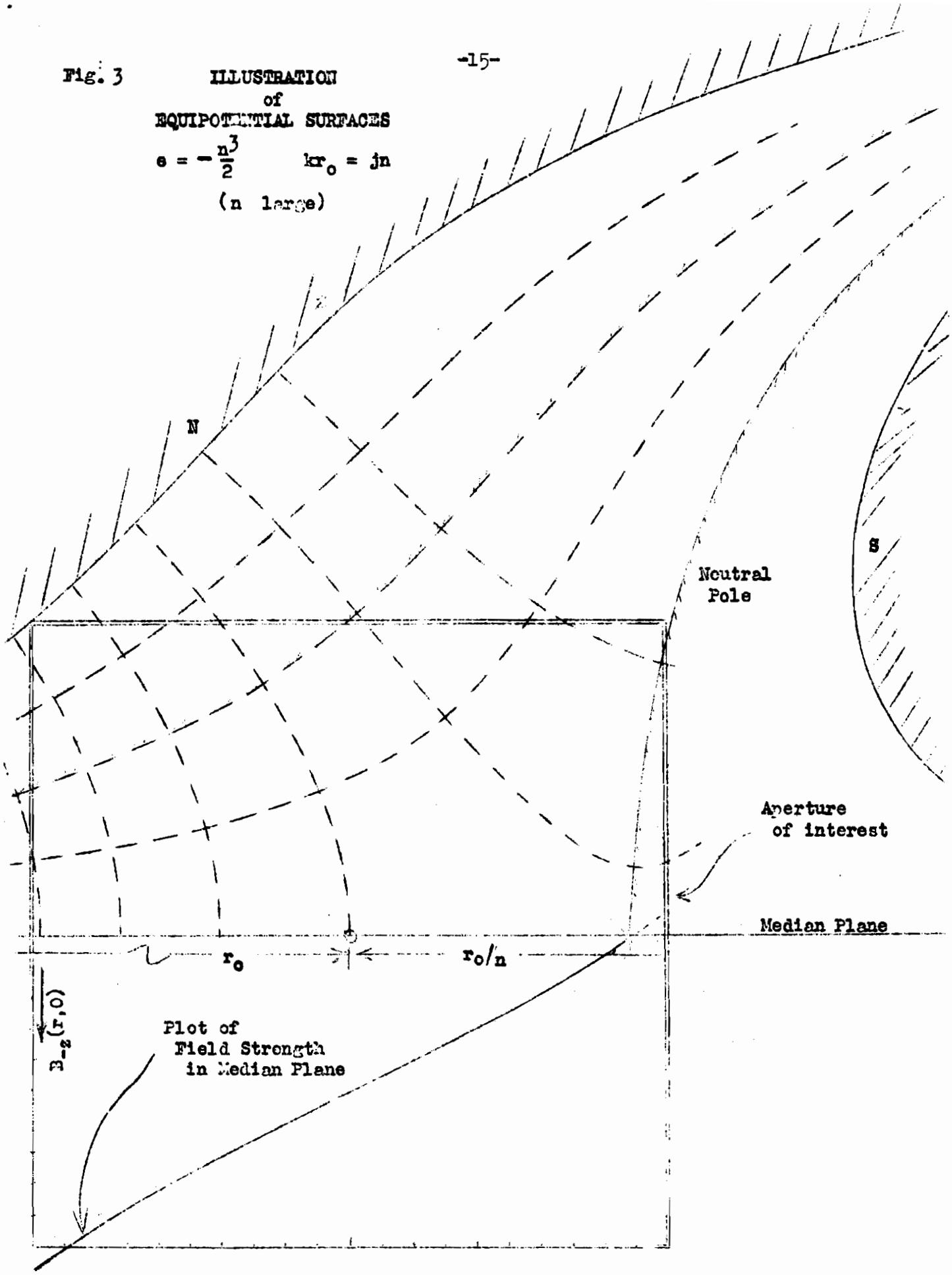


Fig. 4

ILLUSTRATION
of
EQUIPOTENTIAL SURFACES

Cubic Field
in opposition to linear field
in median plane

$$W = Z - \frac{Z^2}{2} + \frac{Z^4}{24}$$

$$\left|\frac{dW}{dZ}\right|_{Y=0} = 1 - X + \frac{X^3}{6}$$

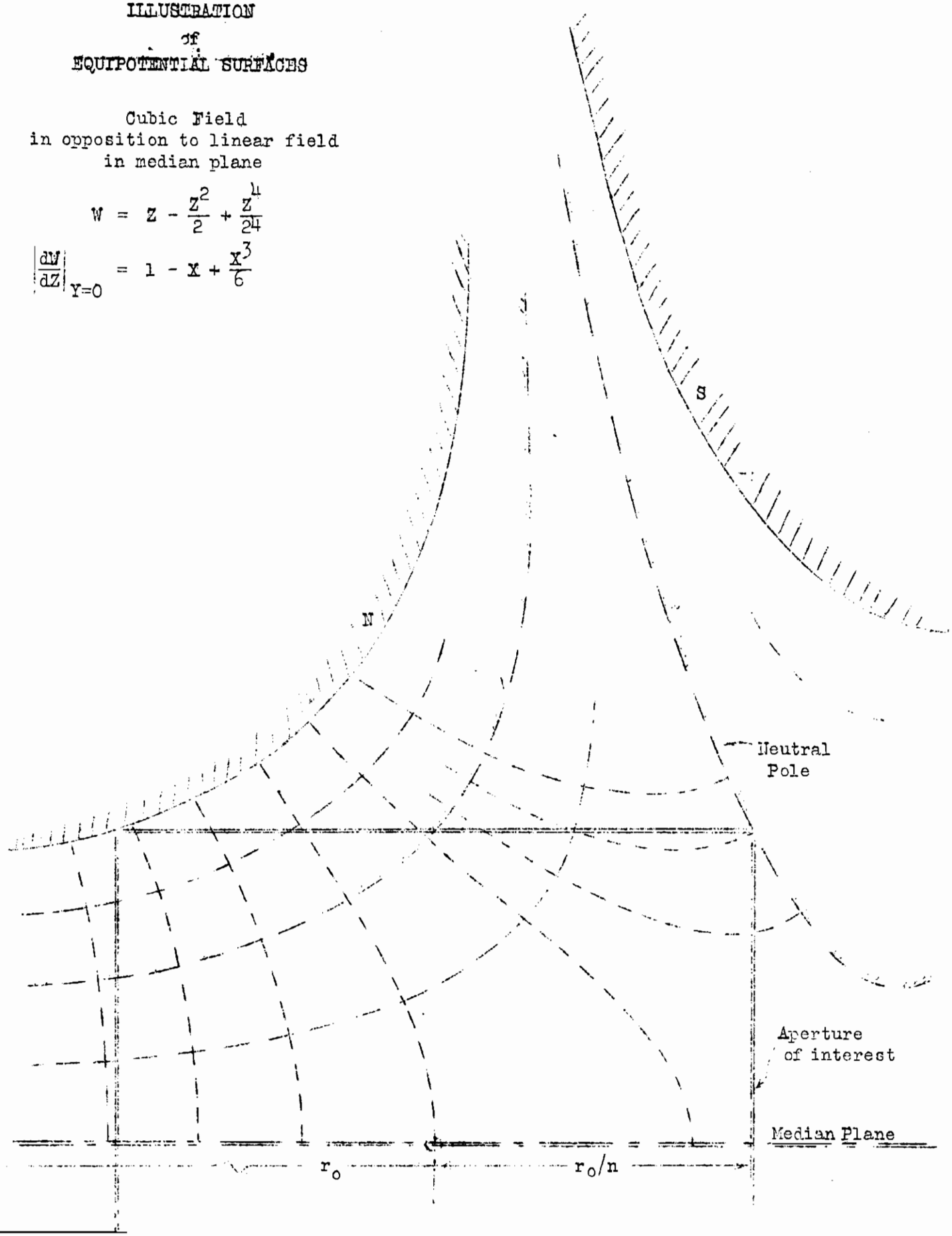
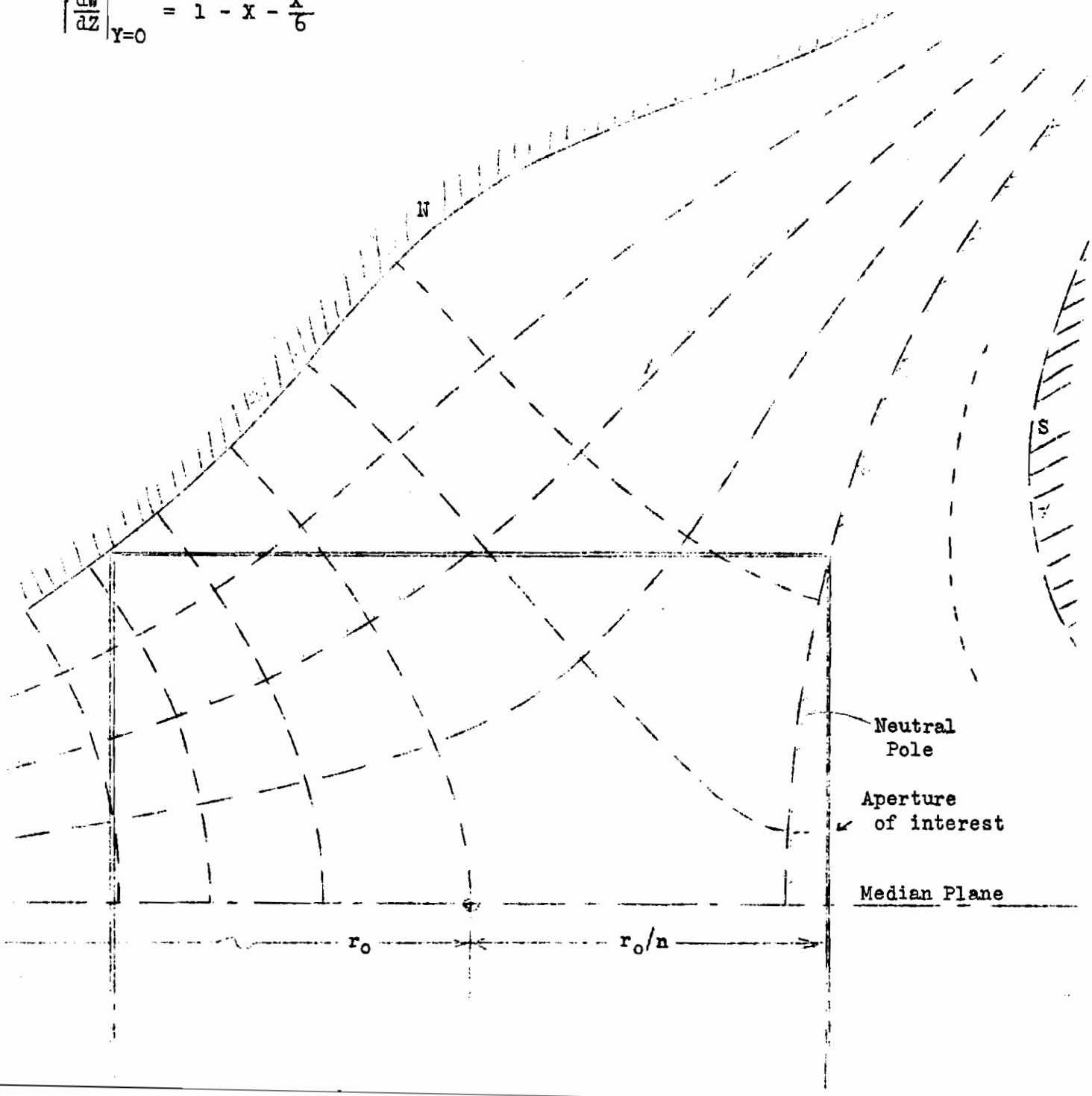


Fig. 5

ILLUSTRATION
of
EQUIPOTENTIAL SURFACES

Cubic Field
supplementing linear field
in median plane

$$W = Z - \frac{Z^2}{2} - \frac{Z^4}{24}$$

$$\left|\frac{dW}{dZ}\right|_{Y=0} = 1 - X - \frac{X^3}{6}$$