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POSSIBLE INSTABILITY FROM MOMENTUM ERRORS IN THE A.G.S.

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1. Introduction

As J. B. Adams[1] has pointed out, particles with momentum different from the equilibrium momentum are presented in a synchrotron with a different n-value. Since during the relatively slow synchrotron oscillations the particle momenta have values differing from the equilibrium value, it appears that resonances may thereby be reached in the A.G.S. It is the purpose of the present note (i) to review (§2) the effective change in n due to momentum errors, (ii) to indicate (§3) the changes in n which appear tolerable, and (iii) to examine (§4) the magnitudes of the momentum errors which may be expected to arise from synchrotron oscillations in a typical case.

2. Differential Equations for Betatron Oscillations

A simple derivation of the differential equation for the radial motion of a particle with momentum p differing from the equilibrium momentum p_o commences with the force equation

$$m\ddot{r} - \frac{pv}{r} = - eB_{oo} \left[1 - n \frac{r - r_o}{r_o} \right] v , \qquad \text{or}$$

$$\frac{pv}{r^2} r'' - \frac{pv}{r} = - eB_{oo} \left[1 - n \frac{\Delta r}{r_o} \right] v ,$$

primes indicating differentiation with respect to the azimuthal angle, quantities of second order being neglected, and the mass being taken as constant since the magnetic field does no work on the particle.

where $\vec{p}$ represents the mechanical momentum and $\vec{A}$ is the vector potential. Considering a magnetic field of constant gradient independent of azimuth, as might be expected to hold within an individual sector, we neglect the impulsive terms of the character discussed by Powell[3] and employ a vector potential $A_\phi(r,Z)\hat{e}_\phi$ for which

$$A_\phi(r,Z) = -B_{oo}\left[\frac{1+n}{2}r - \frac{nr^2}{3r_o} + \frac{nZ^2}{2r_o} + \text{terms of order } \frac{nZ^4}{r^2 r_o}\right];$$

thus

$$B_r \equiv -\partial A_\phi/\partial Z = B_{oo}\, nZ/r_o \qquad \text{and}$$

$$B_Z\Big|_o \equiv (1/r)\,\partial(rA_\phi)/\partial r\Big|_o = -B_{oo}\left[1 - n\frac{r-r_o}{r_o}\right]$$

at points in the median plane.

Since the magnetic field does no work on the particle, we treat $|\vec{p}|$ as a constant and our action principle assumes the form

$$\delta\int\left\{(r^2 + r'^2 + Z'^2)^{1/2} - \frac{p_o}{pr_o}\left[\frac{1+n}{2}r^2 - \frac{nr^3}{3r_o} + \frac{nZ^2}{2r_o}r\right]\right\}d\phi = 0,$$

where primes again designate differentiation with respect to the angular coordinate ϕ and where $p_o \equiv eB_{oo}r_o$.

The differential equations for the trajectory result from application of Euler's equations

$$\frac{d}{d\phi}\left(\frac{\partial F}{\partial r'}\right) - \frac{\partial F}{\partial r} = 0 \qquad \text{and}$$

$$\frac{d}{d\phi}\left(\frac{\partial F}{\partial Z'}\right) - \frac{\partial F}{\partial Z} = 0\ ,$$

where F represents the integrand of the foregoing integral.

For the r-motion we accordingly obtain

$$\frac{d}{d\phi}\left[\frac{r'}{(r^2 + r'^2 + Z'^2)^{1/2}}\right] - \frac{r}{(r^2 + r'^2 + Z'^2)^{1/2}} +$$

$$\frac{p_o}{p}\left[(1+n)\frac{r}{r_o} - n\left(\frac{r}{r_o}\right)^2 + \frac{n}{2}\left(\frac{Z}{r_o}\right)^2 + \ldots\right] = 0\ .$$

$$|\delta n| \cong 0.2 \sqrt{n} \tag{3}$$

when operating near the center of the diagram ($n = N_s^2/16$).[4]

The tolerable momentum spread is thus found by the relation

$$n \left| \frac{\Delta p}{p_o} \right| = 0.2 \sqrt{n}, \qquad \text{whence}$$

$$\left| \frac{\Delta p}{p_o} \right| = 0.2/\sqrt{n} \quad , \tag{4}$$

or an acceptable variation of roughly ± 1 per cent for n = 400. Use of larger n-values would imply still smaller tolerable variations in momentum.

4. Expected Magnitude of Momentum Errors

A. Differential Equation for Synchrotron Oscillation

Oscillations of particle momentum will be associated with the synchrotron phase-oscillations. The differential equation for the phase-oscillations is customarily[5-7] written

$$\frac{d}{dt}\left(\frac{E_s \dot{\phi}}{-\gamma}\right) = \left(\frac{c}{r_o}\right)^2 h \frac{eV_o}{2\pi} (\sin \phi_o - \sin \phi)$$

$$= \omega_o^2 \; h \frac{eV_o}{2\pi} (\sin \phi_o - \sin \phi), \tag{5}$$

where ω_o is the correct (synchronous) angular speed for a hypothetical particle with velocity c; ϕ is the electrical phase angle; h, the harmonic number; E_s, the energy of the (synchronous) particle; V_o, the peak radio-frequency E.M.F.; and $\gamma \equiv -(d\omega/\omega_s)/(dp/p_s)$. (6)

The equivalent alternative from

$$\frac{d}{dt}\left(\frac{E-E_s}{\omega_s}\right) = \frac{eV_o}{2\pi} (\sin \phi - \sin \phi_o) \tag{7}$$

has also been cited and derived in a separate treatment by Courant.[5,9]

If we select $\phi_o = \pi/6$ radians (the range of oscillation then being $-39° \lesssim \phi \leq 150°$ and encompassing 52 1/2 per cent of a full revolution),

$$(\dot{\phi})^2_{max} = 1.37\ A, \tag{14a}$$

$$(\dot{\phi})_{max} = 1.17\ A^{1/2}. \tag{14b}$$

Thus, following injection, a wide range of phase oscillation amplitude will imply values of $\dot{\phi}$ as large as

$$(\dot{\phi})_{max} = 1.17\ \omega_o \sqrt{h\,|\gamma|\ \frac{eV_o/2\pi}{E}}, \tag{15}$$

with the associated momentum spread given by

$$\left(\frac{\Delta p}{p}\right)_{max} = \frac{1}{h\ |\gamma|}\ \frac{(\dot{\phi})_{max}}{\beta\ \omega_o}$$

$$= 1.17\ \frac{1}{\beta}\sqrt{\frac{1}{h\,|\gamma|}\ \frac{eV_o/2\pi}{E}}\ . \tag{16}$$

If dp/dt for the synchronous particle is constant, as in a magnetic field which increases linearly with time,

$$dE/Turn = 2\pi R\ \left[1 + \frac{\Sigma L}{2\pi R}\right]\frac{dp}{dt} \tag{17}$$

and, for a phase angle ϕ_o of 30 degrees,

$$eV_o/2\pi = 2R\ \left[1 + \frac{\Sigma L}{2\pi R}\right]\frac{dp}{dt}$$

$$\cong \frac{2R\ \left[1 + \frac{\Sigma L}{2\pi R}\right]E_f}{c\ .\ \text{acceleration time}} \tag{18a}$$

$$= \frac{2 \times 86.50 \times 1.3 \times 25 \times 10^9}{3 \times 10^8 \times 1}$$

$$= 18.7 \times 10^3 \text{ ev in a typical case.} \tag{18b}$$

simple argument suggests that the momentum oscillations may become quite large at that time. Thus, for adiabatic changes of small-amplitude phase oscillations, the amplitude "damps" as $(|\gamma|/E)^{1/4}$ when the R.F. voltage V_o remains constant. The frequency of phase oscillation varies as $(|\gamma|/E)^{1/2}$, so we then expect that $(\phi)_{max}$ varies as $(|\gamma|/E)^{3/4}$ and $\Delta p/p$ as $(1/\beta)(1/|\gamma|)^{1/4}(1/E)^{3/4}$.

To determine the nature of the momentum oscillations at the transition energy it is necessary (i) to study the adiabatic damping, following injection, of the initially large-angle phase oscillations and (ii) to employ a special solution for the basic differential equation in the neighborhood of the transition energy where the change of the parameter γ is no longer adiabatic. This latter solution can be matched asymptotically to the result of the adiabatic analysis.

(i) The damping following injection can be evaluated through use of S. C. Wright's observation[11] that the action, $\oint \left(\frac{E\,\phi}{-\gamma}\right) d\phi$, remains constant under adiabatic change of the parameters. In terms of Wright's notation,

$$F(\phi_m, \phi_o) \propto (-\gamma/E_s)^{1/2}$$

and thus, for $\phi_m - \phi_o$ small,

$$\phi_m - \phi_o \approx |\gamma|^{1/4} E_s^{-1/4},$$

as stated above.

From Wright's table (op. cit., p. 3) we have specifically, for our initial amplitude of oscillation,

$$\left(\frac{E_s}{-\gamma}\right)^{1/2}_{inj} \times 4.049 = \left(\frac{E_s}{-\gamma}\right)^{1/2} \frac{\pi}{\sqrt{2}} (\phi_m - \phi_o)^2, \quad \text{or}$$

$$(\phi_m - \phi_o) = 1.35 \left(\frac{E_{inj}}{E} \left|\frac{\gamma}{\gamma_{inj}}\right|\right)^{1/4}. \tag{21}$$

Accordingly,

$$\left(\frac{\Delta p}{p}\right)_{max} = \frac{1}{|\gamma|\, h}\, \Omega \left(\frac{\phi_m - \phi_o}{\omega}\right)$$

$$(\phi_m - \phi_o)_{tr} = \frac{1.39}{h^{1/12}} \left[\frac{0.995 \times (0.939 \times 10^9)^2}{(9.75 \times 10^9)^4} (1.87\times10^4)^{1/2}\ 0.537 \right]^{1/6}$$

$$\times \left(\frac{0.988 \times 10^9}{0.88} \right)^{1/4}$$

$$= 0.112/h^{1/12} \text{ radians.} \tag{25}$$

The maximum momentum spread at the transition energy is determined analogously

$$\left(\frac{\sqrt{3}\ a - b}{\sqrt{a^2 + b^2}} \leqslant 1.93 \right) :$$

$$\frac{(\Delta p/p)_{tr}}{(\Delta p/p)_{adia}} = 1.93 \frac{2\pi^{1/2}}{3^{5/3}\Gamma(\frac{5}{3})} \frac{\beta_{adia}}{\beta_{tr}} K^{1/6} k_{tr}^{-1/6} |\gamma|_{adia}^{1/4}$$

$$\times E_{adia}^{3/4}/E_{tr}$$

$$= 1.21 \frac{\beta_{adia}}{\beta_{tr}} (K/k_{tr})^{1/6} |\gamma|_{adia}^{1/4} E_{adia}^{3/4}/E_{tr}, \tag{26}$$

and, by use of our eq. (22),

$$(\Delta p/p)_{tr} = \frac{1.46}{\beta_{tr}} \left[\frac{\sqrt{h}\ \sqrt{|\cos\phi_o|}/\sin\phi_o}{\beta_{tr}\ \sqrt{eV_o/2\pi}} \right]^{1/6}$$

$$\times \frac{1}{\sqrt{h}} \left(\frac{eV_o}{2\pi} |\cos\phi_o| \right)^{1/2} \frac{E_{inj}^{1/4}}{E_o^{1/3} E_{tr}^{1/3}} \frac{1}{|\gamma_{inj}|^{1/4}}, \tag{27}$$

or, typically,

$$= \frac{1.46}{0.995\ h^{5/12}} \left[\frac{1}{0.995(1.87\times10^4)^{1/2}\ 0.537} \right]^{1/6} (1.62\times10^4)^{1/2}$$

$$\times \frac{(0.988\times10^9)^{1/4}}{(0.938\times10^9)^{1/3}(9.75\times10^9)^{1/3}} \cdot \frac{1}{(0.88)^{1/4}}$$

2. Courant, Livingston, and Snyder, Phys. Rev. 88, 1190-1196 (December 1, 1952).

3. J. L. Powell, MAC-JLP-1 & 2.

4. cf. L. J. Laslett, LJL(MAC)-2, with $\delta n_1 = -\delta n_2 \equiv \delta m$ in the present application; private communication to D. W. Kerst (January 30, 1954).

For operation points remaining on the diagonal of the necktie, the characteristic factor $\exp(\pm ik)$ is given by[2]

$$\cos k = \cos \frac{2\pi\sqrt{n}}{N_s} \cosh \frac{2\pi\sqrt{n}}{N_s}$$

(N_s being the number of sectors, or one-half the number of sector-pairs or periods) and, by differentiation,

$$(\sin k)\ \delta k = \frac{\pi}{N_s\sqrt{n}} \left[\sin \frac{2\pi\sqrt{n}}{N_s} \cosh \frac{2\pi\sqrt{n}}{N_s} - \cos \frac{2\pi\sqrt{n}}{N_s} \times \sinh \frac{2\pi\sqrt{n}}{N_s} \right] \delta n.$$

In the present application we have selected $\delta k = \pm \frac{\pi/4}{N_s/2}$, so

$$\delta n = \pm \frac{\sqrt{n}\ (\sin k)/2}{\sin \frac{2\pi\sqrt{n}}{N_s} \cosh \frac{2\pi\sqrt{n}}{N_s} - \cos \frac{2\pi\sqrt{n}}{N_s} \sinh \frac{2\pi\sqrt{n}}{N_s}}.$$

For points near the center of the necktie, $\frac{2\pi\sqrt{n}}{N_s} = \frac{\pi}{2}$, $\sin k = 1$, and

$$\delta n = \pm \frac{\sqrt{n}}{2 \cosh (\pi/2)} = \pm 0.20\sqrt{n} \quad ;$$

for a point further down the diagonal toward smaller $|n|$, such that, in our notation, $\frac{2\pi\sqrt{n}}{N_s} = 1.254$ [cf. EDC/HSS-1, p. 20],

$\sin k = 0.807$, and one finds

$$\delta n = \pm 0.31\sqrt{n}.$$

$$\frac{2\pi}{\omega}\frac{dE}{dt} = eV_o \sin\phi - e\left[(\dot{\Phi}_Z)_s + 2\pi r_s(\dot{B}_Z)_s \Delta r\right],$$

$$\frac{2\pi}{\omega_s}\frac{dE_s}{dt} = eV_o \sin\phi_s - e(\dot{\Phi}_Z)_s, \text{ and}$$

$$p_s = -e(B_Z)_s r_s.$$

10. cf. N. M. Blachman and E. D. Courant, Rev. Sci. Inst. 20, 596 (1949).

11. S. C. Wright, op cit.[6] Discussion of the adiabatic invariance of the action variable for periodic motion with one degree of freedom is given, for example, by Max Born, "The Mechanics of the Atom", Eng. Trans. by J. W. Fisher, rev. by D. R. Hartree (G. Bell and Sons, Ltd., London, 1927) §10.