MAC-DWK 4

# APPROXIMATE CALCULATION OF NONLINEAR LOCK-IN AT $\sigma = \pi$

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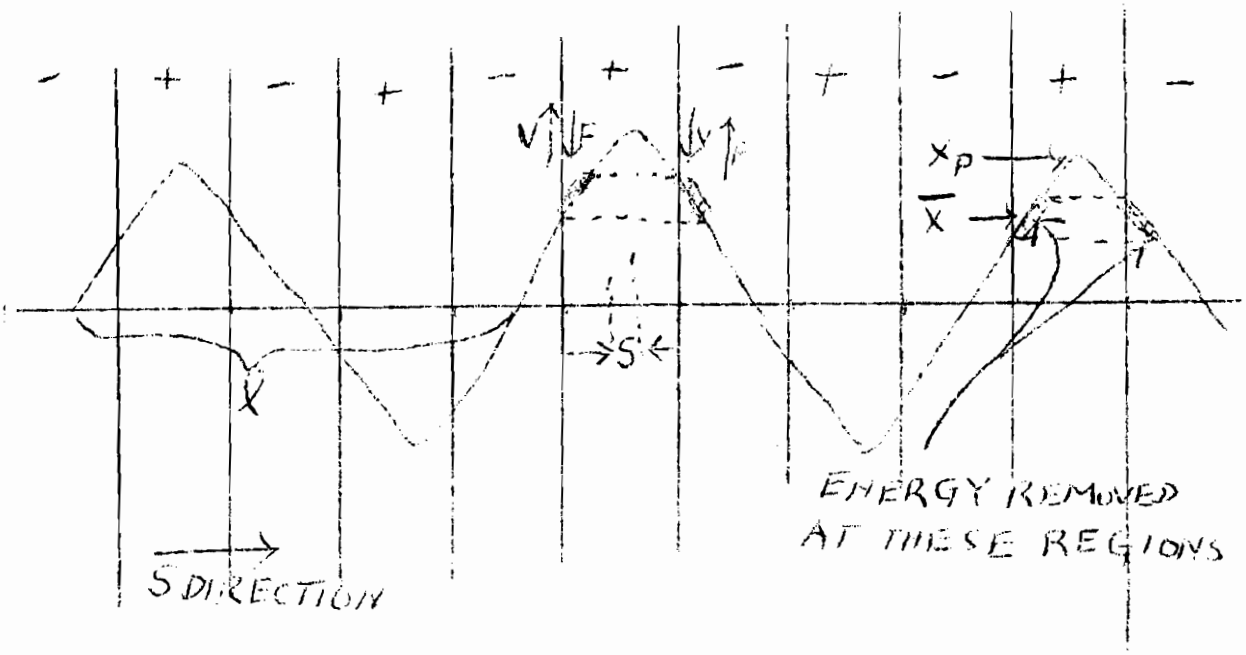
Oscillations fixed in position in an A.G.S. have been observed in computational work on non-linear restoring forces at Brookhaven and at Illinois, and the Michigan analog has demonstrated them. A study of the characteristics of this motion has not gone very far on the computer at Illinois, but the one dimensional and the two dimensional (with coupling) cases have been found. The regular features of this hunting and throbbing motion have led to the following primitive attempt to calculate the general features of the motion.

The process[1] described here has the non-linear force opposing the linear force. The computer shows that wave form is closely triangular with rounded peaks. If the initial conditions are equilibrium conditions, a closed oscillatory orbit is formed which is a second closed orbit in the machine. Oscillations about this additional equilibrium orbit occur if initial conditions are not just right, and they take the form of a slight throbbing of the amplitude at a subharmonic frequency and a slight phase hunting of the peaks about the equilibrium phase which is in the center of a focussing sector.

1. MAC-8 and memo-H.R. Crane "Locked-in Motion" (Craniac).




In the following estimate only small displacements are allowed.

$\bar{F}$ = average restoring force in regions where energy is added or subtracted from oscillation

$\bar{X}$ = average displacement in this region

$V$ = velocity of particle

$F_p$ = force at peak

$X_p$ = displacement at peak

$S$ = displacement of peak in circumferential direction

$\ell$ = wave length

$\eta$ = pure number of wave lengths

$\omega_0$ = equilibrium circular frequency of oscillation

$\delta X_p$ = stray of peak from equilibrium amplitude

$f$ = fraction of peak force which is from cubic term

If $S > 0$, energy is removed because two dark-end regions, each of length 2S on each crest, have force and velocity opposite. All other regions of the wave produce no net energy change. In one wave length a length 8 S contributes to energy change. At peak this is all potential. to raise the peak by $dx_p$ work must be done:

$$F_p\, dx_p = -8\bar{F}\frac{d\bar{x}}{ds}S \qquad \text{per wave length}$$

$$(1)\quad \frac{dx_p}{d\eta} = -8\frac{\bar{F}\bar{x}}{F_p}S$$

If the frequency of oscillation is not the equilibrium frequency,

$$(2)\quad \frac{dS}{\ell} = -\frac{d\omega}{\omega_0} \quad \text{PER } \ell$$

If $\omega \neq \omega_0$ the displacement, S, continues to change. But if forces are non-linear,

$$(3)\quad dS = -\frac{\ell}{\omega_0}\frac{d\omega}{dx_p}\delta x_p \text{ PER } \ell \text{ OR } \frac{dS}{d\eta} = -\frac{\ell}{\omega_0}\frac{d\omega}{dx_p}\delta x_p$$

Differentiating with respect to $\eta$ gives a second order equation

$$(4)\quad \frac{d^2S}{d\eta^2} = -\frac{\ell}{\omega_0}\frac{d\omega}{dx_p}\frac{dx_p}{d\eta} = +\frac{\ell}{\omega_0}\frac{d\omega}{dx_p}\frac{8\bar{F}\bar{x}}{F_p}S \quad \text{IF } \frac{\delta x_p}{x_p} \ll 1$$

Thus if $\frac{d\omega}{dx_1} < 0$, the motion, S, hunts; and if

$\frac{d\omega}{dx_p} > 0$ motion is unstable.

Thus the frequency must decrease as amplitude increases for stability, that is the cubic force must be oppositely directed to the linear force.

$$(5)\begin{cases} S = S_0 \sin \Omega \eta \\ \Omega = \sqrt{\frac{l}{\omega_0}\left(-\frac{d\omega}{dx_p}\right) 8 \frac{\overline{F}\,\overline{x'}}{F_p}} \end{cases}$$

WITH

By observation from Illiac

$$(6)\begin{cases} l\overline{x'} = 6x_p \\ \overline{F}/F_p \cong .6 \quad \text{SINCE} \quad \frac{\overline{x}}{x_p} \cong .6 \end{cases}$$

So

(7)

$$\Omega = 5.4\sqrt{\frac{x_p}{\omega_0}\left(-\frac{d\omega}{dx_p}\right)}$$

with a non-A.G. non-linear oscillator we know

$$\frac{d\omega}{\omega} = \frac{3}{8} f$$

By Illiac Tape #46 and #49

$$\frac{d\omega}{\omega} = 0.58 f \quad \text{FOR } f = \sim 2\%$$

$$\frac{d\omega}{\omega} = 0.85 f \quad \text{FOR } f \sim 0.3\%$$

arbitrarily taking $\frac{d\omega}{\omega} = \frac{2}{3} f$ at tip of tie and using our

differential equation $X'' = \pm m X \mp \frac{e}{3} X^3$ (Powell MAC-JLP-1)

(8) $\frac{d\omega}{dx} = -\frac{4}{3} f \frac{\omega_0}{x_p}$ so

(9) $\Omega = 6.3\sqrt{f}$

The number of wavelengths per throb is $\eta_w = \frac{2\pi}{\Omega}$

(10) $\begin{cases} \eta_{WOW} = \frac{1}{\sqrt{f}} \\ \eta_{OBS} = \frac{.78}{\sqrt{f}} \end{cases}$

(This is 1.3 times bigger than the observed on Tape 46 and 49).

Thus the subharmonic throb generated by the cubic is $\sim \sqrt{f}$

The amplitude of the throb can be estimated from (3)

$$\delta x_p = \frac{dS}{d\eta}\left(\frac{l}{\omega_0}\right)\left(-\frac{d\omega}{dx_p}\right) = \left(-\Omega S_0 \cos \Omega \eta\right) \Big/ \frac{l}{\omega_0}\left(-\frac{d\omega}{dx_p}\right)$$

BY (8)

(11) $$\frac{\delta x_p}{x_p} = \frac{3}{4}\frac{S_0}{l}\frac{\Omega}{f}\cos\Omega\eta = 4.7\frac{S_0}{l\sqrt{f}}\cos\Omega\eta$$

BY (6)

$$\frac{x_p}{l} S_0 = \frac{\overline{x' S_0}}{6} = \frac{1}{6}\Delta x_l$$ WHERE $\Delta x_l$ = DEVIATION

from equilibrium initial condition at extremity of S oscillation.

so

(12) $$\frac{\delta x_p}{\Delta x_i} = \frac{.8}{\sqrt{F}} \cos \Omega \eta$$

Combining (12) and (10) maxima are

(13) $$\frac{\delta x_p}{\Delta x_i} = .8\, \eta_{wow}$$ ——— using computed $\eta$ or using observed $\eta_w$

$$\frac{\delta x_p}{\Delta x_i} = \eta_{obs}$$

Only the throb frequency has had an experimental test with the result that $\eta$ was observed to be 78% of calculated for two cases to date. If $\delta x_p$ is not much less than $x_p$, $\frac{d\omega}{dx}$ is not constant during the oscillation and the differentiation (4) has a term which tends to remove the restoration to equilibrium for negative $\delta x_p$. The oscillation can damp down toward zero amplitude and then grow up again with reverse phase. This action is observed in both non-linear and linear cases.