NON-LINEARITIES IN THE AG SYNCHROTRON

The equation for the orbit of a particle in a magnetic field may be written

$$p\frac{d\vec{T}}{ds} = \vec{T}\times\vec{H}$$

where p = momentum in "$H\rho$" units

s = arc length of orbit

$\vec{T}$ = unit tangent to orbit

$\vec{H}$ = magnetic field

The equations of motion are equivalent to the variational principle

$$\delta\int_{P}^{P'}\left\{\vec{A}\cdot d\vec{r} + p|d\vec{r}|\right\} = 0$$

where $\vec{A}$ = vector potential and P and P' are any two points of the orbit. If $P = P'$ this means that closed orbits are curves enclosing maximum flux for given total length. Such closed orbits exist, at least if the momentum is not too large, in any magnetic field. The discussion of synchrotron orbits therefore divides itself into two parts: (1) What are the closed orbits in the field? (2) Is motion in the neighborhood of these "equilibrium orbits" stable? The stability question is the one of interest here.

Suppose that the equilibrium orbit lies in the neighborhood of a circle of unit radius. The particle may be located relative to this circle by means of co-ordinates θ, x, y where θ is the azimuthal position, $r = 1+x$ is the radius and y is the displacement perpendicular to the plane of the circle. The element of displacement is

$$d\vec{r} = \vec{\theta}(1+x)d\theta + \vec{r}\,dx + \vec{y}\,dy$$

and taking θ as the independent variable, the variational

principle is $\delta \int L \, d\theta = 0$, where

$$L = p\sqrt{(1+x)^2 + x'^2 + y'^2} + (1+x)A_\theta + A_x x' + A_y y'$$

and primes denote differentiation with respect to θ . The equilibrium orbit will be in the neighborhood of the circle provided

$$H_y = p + \text{small terms}$$

and therefore, with a special choice of gauge, we may write

$$A_\theta = -\frac{p}{e}(1+x) + A_y$$

where A_x, A_y and A_z are to be regarded as small compared to A_θ.

Now we know that x and y are of order $1/n$, and that x' and y' are of order $1/\sqrt{n}$, where $n = \frac{R}{H}\frac{\partial H}{\partial K}$. In an AG machine n is large, and we may approximate L by

$$L \approx p(1+x) + \frac{p}{2}(x'^2 + y'^2) - \frac{p}{2}(1+x)^2 + A_z + A_x x' + A_y y'$$

or, dropping a constant term,

$$L = \frac{p}{2}(x'^2 + y'^2) - \frac{p}{2}x^2 + A_z + A_x x' + A_y y'$$

The quantities A_x , A_y and A_z describe the magnetic field exclusive of the main guide field, and are therefore zero on the equilibrium orbit. It is a good approximation to treat x , y and θ as cartesian coordinates so far as A_x , A_y and A_z are concerned, and we have approximately

$$H_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial \theta}, \qquad H_y = p + \frac{\partial A_x}{\partial \theta} - \frac{\partial A_z}{\partial x}, \qquad H_\theta = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

Maxwell's equations require $\nabla \times \nabla \times \vec{A} = 0$, and the A's may be expanded in powers of x and y , the coefficients being functions of θ . In each order in x and y there are two independent vectors $\vec{A}$ which may be arranged to have $\nabla^2 \vec{A} = 0$ by a suitable choice of the gauge. It is perhaps simpler to make this expansion in terms of a scalar potential. To fourth order in x and y this scalar potential is

$$
\begin{aligned}
\psi/\rho = {} & 2\varphi + \alpha x + \beta y + \tfrac{1}{2}(a-\varphi'')x^2 - n x y - \tfrac{1}{2}(a+\varphi'')y^2 \\
& + \tfrac{1}{12}(6c-\alpha'')x^3 - \tfrac{1}{4}(6b+\beta'')x^2 y - \tfrac{1}{4}(6c+\alpha'')x y^2 + \tfrac{1}{12}(6b-\beta'')y^3 \\
& + \tfrac{1}{24}\{d-(a-\varphi'')''\}x^4 + \tfrac{1}{12}(n''+4e)x^3 y - \tfrac{1}{4} d x^2 y^2 \\
& + \tfrac{1}{12}(n''-4e)x y^3 + \tfrac{1}{24}\{d+(a+\varphi'')''\}y^4 + O(x^5)
\end{aligned}
$$

In a practical case, many of the terms in ψ are negligible, but they are all written down here for completeness' sake. The quantities φ , α , β , a , b , c , d , e , are all periodic functions of the azimuth. If the unit circle is a possible orbit, α and β are zero. In the general case α and β appear as inhomogeneous terms in the equations of motion. The functions n , a and φ are understood best by writing down the fields to first order

$$
\begin{aligned}
H_x &= \frac{\partial \psi}{\partial x} = \rho\{\alpha - n y + (a-\varphi'')x\} \\
H_y &= \rho + \frac{\partial \psi}{\partial y} = \rho\{1 + \beta - n x - (a+\varphi'')y\} \\
H_\theta &= \frac{\partial \psi}{\partial \theta} = \rho\{2\varphi' + \alpha' x + \beta' y\}
\end{aligned}
$$

n is the field index evaluated on the unit circle, a describes a tilt of the lines H_x = const, H_y = const, the slope of the

former being α/n. φ describes the tangential field, which has no effect on the motion in the equilibrium orbit, but makes itself felt as a "Larmor" rotation of the xy plane about the equilibrium orbit. The magneto motive force is $\oint H_\theta d\theta = 2p\oint \varphi' d\theta = 0$ so that no current links the orbit, which was already implied in writing $\vec{H}$ as a gradient.

These fields describe the system in the linear approximation. The Lagrangian is (taking $p = 1$ for convenience)

$$L = \tfrac{1}{2}(x'^2 + y'^2) + \alpha y - \beta x - \tfrac{1}{2}(1-n)x^2 - \tfrac{1}{2} n y^2 + a x y - \varphi'(x'y - y'x)$$

and the equations of motion

$$x'' + (1-n)x - 2\varphi' y' - (a + \varphi'')y = -\beta$$
$$y'' + n y + 2\varphi' x' - (a - \varphi'')x = \alpha$$

are the basis for the linear theory (EDC/HSS - 1).

Current interest centers on the effects of non-linear forces of third order in x and y. If we neglect all other higher terms as well as a and φ the Lagrangian is

$$L = \tfrac{1}{2}(x'^2 + y'^2) + \alpha y - \beta x - \tfrac{1}{2}(1-n)x^2 - \tfrac{1}{2} n y^2$$
$$- \frac{e}{12}(x^4 - 6x^2y^2 + y^4) + \frac{d}{6}(x^3 y - x y^3) - \frac{n''}{24}(x^4 - y^4)$$

where d and e are functions of θ determined by magnet design. The term in n'' represents the effect of the abrupt change in n in going from a focussing to a defocussing sector. Since n is approximately a step-function, n'' is approximately the derivative of a δ-function. The orbit equations are

$$x'' + (1-\mu)x = -\beta - \tfrac{1}{3}e(x^3 - 3xy^2) + \tfrac{1}{2}d(x^2y - \tfrac{1}{3}y^3) - \frac{\mu''}{6}x^3$$

$$y'' + \mu y = \alpha + \tfrac{1}{3}e(3xy^2 - y^3) - \tfrac{1}{2}d(xy^2 - \tfrac{1}{3}x^3) + \frac{\mu''}{6}y^3$$

It is clear that cubic terms are accompanied by very strong coupling terms which may have an important influence on the stability of the non-linear system. This is being investigated on the Illiac by Snyder, et al. Initial plans are to study the system

$$x'' + (1-\mu)x = -\tfrac{1}{3}e(x^3 - 3xy^2)$$

$$y'' + \mu y = \tfrac{1}{3}e(3xy^2 - y^3)$$

i.e., to neglect the effect of μ'' and to assume $d = 0$. The Brookhaven group has already reported results for the equation

$$x'' + (1-\mu)x = -\tfrac{1}{3}ex^3$$

which is the special case $y = 0$. In this case the motion is stable for cubic force of the order 45% or less of the linear force.

The cubic effects which have been omitted in the above, as well as quadratic terms, will be investigated by Illiac in the future. Also, the effects of inhomogeneous terms, in particular, Kerst's " δ-function bump" will be studied.

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