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EDC-3
MAC-2

LINEAR COUPLING BETWEEN VERTICAL AND HORIZONTAL OSCILLATIONS

E. D. Courant, Brookhaven National Laboratory
and Midwest Accelerator Conference, Madison, Wisconsin
August 28, 1953

If the median surface of an accelerator is not everywhere plane, or if azimuthal fields are present, coupling is introduced between horizontal and vertical oscillations; the equations of motion have the form

$$x'' = -K_1(\theta)x + Q(\theta)z' + \left(M(\theta) + \tfrac{1}{2}Q'(\theta)\right)z$$
$$z'' = -K_2(\theta)z - Q(\theta)x' + \left(M(\theta) - \tfrac{1}{2}Q'(\theta)\right)x \qquad (1)$$

(See Brookhaven report EDC/HSS-1, pp. 37 ff.). It is shown in EDC/HSS-1 that this coupling may lead to resonance if

$$\cos\sigma_x = \cos\sigma_z \qquad (2)$$

where σ_x and σ_z are the phase shifts of the horizontal and vertical oscillations in one revolution. Condition (2) is fulfilled whenever either the <u>sum</u> or the <u>difference</u> of the vertical and horizontal oscillation frequencies equals an integral multiple of the frequency of revolution. Twiss has shown in the report of the Cambridge Design Study Group that, under certain restrictive conditons, this coupling actually results in instability only if the <u>sum</u> of the two frequencies is integral, while if the difference is integral there is merely an interchange of energy between the two modes. In this note it will be shown that this conclusion remains valid in a more general case.

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We shall first present a plausibility argument and then a more rigorous proof.

For simplicity, let us omit the terms in Q and Q' in (1). The equations (1) can, in fact, be transformed so as to eliminate Q and Q' provided $\int_0^{2\pi} Q\, d\theta = 0$, i.e. if there is no current linking the orbit. We replace the alternating gradients K_1 and K_2 by 'equivalent' harmonic oscillator terms ν_1^2 and ν_2^2, obtaining the equations

$$\begin{aligned} x'' + \nu_1^2 x &= M(\theta)\, z \\ z'' + \nu_2^2 z &= M(\theta)\, x \end{aligned} \qquad (3)$$

and replace x and z on the right hand side of (3) by their unperturbed values: $x = X e^{i\nu_1\theta} + \bar{X} e^{-i\nu_1\theta}$; $z = Z e^{i\nu_2\theta} + \bar{Z} e^{-i\nu_2\theta}$. Furthermore, M (θ) may be written as a Fourier series: $M(\theta) = \sum a_k e^{ik\theta}$:

$$\begin{aligned} x'' + \nu_1^2 x &= Z \sum_{k=-\infty}^{\infty} a_k e^{i(k+\nu_2)\theta} + \bar{Z} \sum a_k e^{i(k-\nu_2)\theta} \\ z'' + \nu_2^2 z &= X \sum a_k e^{i(k+\nu_1)\theta} + \bar{X} \sum a_k e^{i(k-\nu_1)\theta} \end{aligned} \qquad (4)$$

with $\bar{a}_k = a_{-k}$; a bar over a quantity denotes its complex conjugate. A difference resonance arises if, for some integer k, $\nu_1 - \nu_2 = k$ while a sum resonance arises if, for some k, $\nu_1 + \nu_2 = k$.

Now suppose we are at a difference resonance. Omitting all except the resonant terms, we have

$$\begin{aligned} x'' + \nu_1^2 x &= a_k Z e^{i\nu_1\theta} + \bar{a}_k \bar{Z} e^{-i\nu_1\theta}, \\ z'' + \nu_2^2 z &= \bar{a}_k X e^{i\nu_2\theta} + a_k \bar{X} e^{-i\nu_2\theta} \end{aligned} \qquad (5)$$

The terms in $e^{i\nu_1\theta}$ and $e^{-i\nu_1\theta}$ in the first of equations (5) lead, respectively, to changes in the amplitudes X and $\bar{X}$; similarly the first and second terms in the second equation lead to increases in Z and $\bar{Z}$, respectively:

$$X = X_0 + \frac{a_2 Z}{2i\nu_1}\theta, \quad Z = Z_0 + \frac{\bar{a}_2 X}{2i\nu_2}\theta. \tag{6}$$

Squaring the absolute values of X and Z in (6), multiplying the results, respectively, by ν_1 and ν_2, and adding, we find

$$\nu_1|X|^2 + \nu_2|Z|^2 = \nu_1|X_0|^2 + \nu_2|Z_0|^2 \tag{7}$$

if terms of the order θ^2 are neglected. This means that at a difference resonance the amplitudes remain bounded, although the amplitude of one mode can increase at the expense of the other.

On the other hand, at a sum resonance we have, for some k, $\nu_1 + \nu_2 = k$. If we again pick out the resonant terms from (4) and omit all others, we have

$$\begin{aligned} X'' + \nu_1^2 X &= Z\bar{a}_2 e^{-i\nu_1\theta} + \bar{Z}a_2 e^{i\nu_1\theta}, \\ Z'' + \nu_2^2 Z &= X\bar{a}_2 e^{-i\nu_2\theta} + \bar{X}a_2 e^{+i\nu_2\theta} \end{aligned} \tag{8}$$

so that

$$X = X_0 + \frac{a_2\bar{Z}}{2i\nu_1}\theta; \quad Z = Z_0 + \frac{a_2\bar{X}}{2i\nu_2}\theta. \tag{9}$$

Thus

$$\begin{aligned} |X|^2 &= |X_0|^2 + (a_2\overline{XZ} - \bar{a}_2 XZ)/2i\nu_1 \\ |Z|^2 &= |Z_0|^2 + (a_2\overline{XZ} - \bar{a}_2 XZ)/2i\nu_2 \end{aligned} \tag{10}$$

Therefore, in this case, the only quantity that is constant is

$$\nu_1 |X|^2 - \nu_2 |Z|^2$$

which means that if one amplitude increases so does the other, whether the amplitudes actually increase or decrease depends on the sign of $(a_k \overline{ZX} - \bar{a}_k XZ)$.

The above arguments indicate that the sum resonances lead to instability, while the difference resonances lead to stability.

The proof for an AG machine, using matrix algebra and not making any assumption about replacing the AG oscillators with harmonic oscillators, goes as follows:

We assume first that the machine consists of two parts, one of which has a perfect median plane, while the median plane of the other part is tilted through an angle φ. Twiss, in the Cambridge report, assumes that each of these parts consists of a whole number of unit cells; we do not make this assumption here.

Suppose that the transformation matrices for the untilted part are X_1 and Z_1 for horizontal and vertical motion, and for the tilted part X_2 and Z_2. We define the matrices $A = X_1 X_2$, $B = Z_1 Z_2$, $C = X_2^{-1} Z_2$. (11)

At either a sum or difference resonance we have

$$Sp\, A = Sp\, B = 2 \cos \alpha. \tag{12}$$

It can be shown that the resonance leads to stability or instability according to whether the quantity

$$K = Sp\left[2(1-C) - A(1-C)B(1-C^{-1})\right] \tag{13}$$

is positive or negative.

We now write the matrices in the following form:

$$\begin{aligned} A &= \cos\varphi + \phi \sin\varphi \\ B &= \cos\varphi + \psi \sin\varphi \\ C &= \cos\gamma + \phi \sin\gamma \\ C^{-1} &= \cos\gamma - \phi \sin\gamma \end{aligned} \tag{14}$$

where $\not{a} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$, etc. The σ's here are the Pauli spin matrices. It will be noted that the trace of matrices of the form $\not{a}$ is zero, and that $a_1^2 + a_2^2 + a_3^2 = -1$; if $|\cos\varphi| < 1$, a_1, a_3, b_1, and b_3 are real, while a_2 and b_2 are imaginary. In our case $|\cos\varphi| < 1$, but no assumption need be made about $\cos\gamma$. If a_1, a_2, a_3 are regarded as the components of a vector, we have the following relations:

$$\begin{aligned} &\mathrm{Sp}\ \not{a}\not{b} = 2\underline{a}\cdot\underline{b}, \\ &\mathrm{Sp}\ \not{a}\not{b}\not{c} = -\mathrm{Sp}\not{a}\not{c}\not{b} = 2i(\underline{a}\times\underline{b}\cdot\underline{c}) \\ &\mathrm{Sp}\ \not{a}\not{c}\not{b}\not{c} = 4(\underline{a}\cdot\underline{c})\ (\underline{b}\cdot\underline{c}) + 2\ (\underline{a}\cdot\underline{b}). \end{aligned} \tag{15}$$

In terms of these vectors, we may write

$$\begin{aligned} K = 4\sin^2\varphi\{&1-\cos\gamma + (\underline{a}\cdot\underline{b})\cos\gamma(1-\cos\gamma) \\ &+(\underline{a}\cdot\underline{c})(\underline{b}\cdot\underline{c})\sin^2\gamma + i(\underline{a}\times\underline{b}\cdot\underline{c})\sin\gamma(1-\cos\gamma)\}. \end{aligned} \tag{16}$$

Let us now write

$$\underline{c} = u\underline{a} + v\underline{b} + iw\ \underline{a}\times\underline{b}. \tag{17}$$

If $|\cos\gamma| < 1$, u, v, w are all real; if $|\cos\gamma| > 1$ they are all imaginary. Using (17), we find

$$\begin{aligned} &(a.c)\ (b.c) = (1+(a.b)^2)uv - (a.b)\ (u^2+v^2), \\ &i(a\times b.c) = w((a.b)^2-1). \end{aligned} \tag{18}$$

Furthermore, since $a^2 = b^2 = c^2 = -1$,

$$u^2 + v^2 = 1 + 2uv(a.b) + w^2((a\cdot b)^2-1). \tag{19}$$

Using all these relations, we find

$$\begin{aligned} K = 4\sin^2\varphi\ \{&(1-a.b)(1-\cos\gamma) - ((a.b)^2-1)(uv+(a.b)w^2)\sin^2\gamma \\ &+((a.b)^2-1)w\sin\gamma\ (1-\cos\gamma\ ) \end{aligned} \tag{20}$$

Now at a difference resonance $(a.b) \leq -1$, while at a sum resonance $(a.b) \geq +1$. This can be seen by comparing the representation (14) of the transformation matrices with the conventional representation in terms of the parameters σ, α, β, given in EDC/HSS-1. For the matrix A,

$$\cos\sigma = \cos\varphi,$$
$$\sin\sigma = \pm\sin\varphi,$$
$$\alpha\sin\sigma = a_3\sin\varphi,$$
$$\beta\sin\sigma = (a_1+ia_2)\sin\varphi.$$

If β is taken to be positive for all matrices with real σ, it follows that $\sin\sigma$ has the same sign in A and B at a difference resonance and the opposite sign at a sum resonance. Therefore at a difference resonance,

$$(a\cdot b) = -1 - \frac{(\alpha_A\beta_B-\alpha_B\beta_A)^2+(\beta_B-\beta_A)^2}{2\beta_A\beta_B} \tag{22}$$

and at a sum resonance

$$(a\cdot b) = 1 + \frac{(\alpha_A\beta_B-\alpha_B\beta_A)^2+(\beta_B-\beta_A)^2}{2\beta_A\beta_B}, \tag{23}$$

where we have used (21) and the relation $a_1^2+a_2^2+a_3^2 = -1$.

Let us first consider a difference resonance, with

$$(a\cdot b) = -(1+\delta), \quad \delta \geq 0.$$

Here

$$K = 4\sin^2\varphi\,\{(2+\delta)(1-\cos\gamma) - \delta(2+\delta)(uv-(1+\delta)w^2)\sin^2\gamma + \delta(2+\delta)w\sin\gamma(1+\cos\gamma)\}. \tag{24}$$

From (19) we obtain the inequality

$$\sin^2\gamma\,(-2\delta uv + 1 + w^2\delta(2+\delta)) = (u+v)^2\sin^2\gamma \geq 0 \tag{25}$$

since u,v,w, and $\sin\gamma$ are either all real or all imaginary. Hence

$$-uv\delta\sin^2\gamma \geq -\tfrac{1}{2}(1+w^2\delta(2+\delta))\sin^2\gamma \tag{26}$$

and

$$K \geq 4\sin^2\varphi\,(2+\delta)\{(1-\cos\gamma)-\tfrac{1}{2}(1-w^2\delta^2)\sin^2\gamma + w\delta(1-\cos\gamma)\}$$
$$= 2\sin^2\varphi\,(2+\delta)(1-\cos\gamma+w\delta\sin\gamma)^2 \geq 0 \tag{27}$$

Similarly, at a sum resonance with $(a.b) = 1+\delta$,

$$K = 4\sin^2\varphi\cdot\delta\,[-(1-\cos\gamma) - (2+\delta)(uv+(1+\delta)w^2)\sin^2\gamma + (2+\delta)w\sin\gamma(1-\cos\gamma)] \tag{28}$$

and uv obeys the inequality

$$-(2+\delta)\, uv \sin^2\gamma \leq \frac{1}{2}\left(1 + \delta(2+\delta) w^2\right) \sin^2\gamma . \tag{29}$$

This leads to

$$K \leq 4\delta \sin^2\varphi \left[-(1-\cos\gamma) + \frac{1}{2}\left(1 - (2+\delta)^2 w^2\right)\sin^2\gamma + (2+\delta) w \sin\gamma (1-\cos\gamma)\right] \tag{30}$$

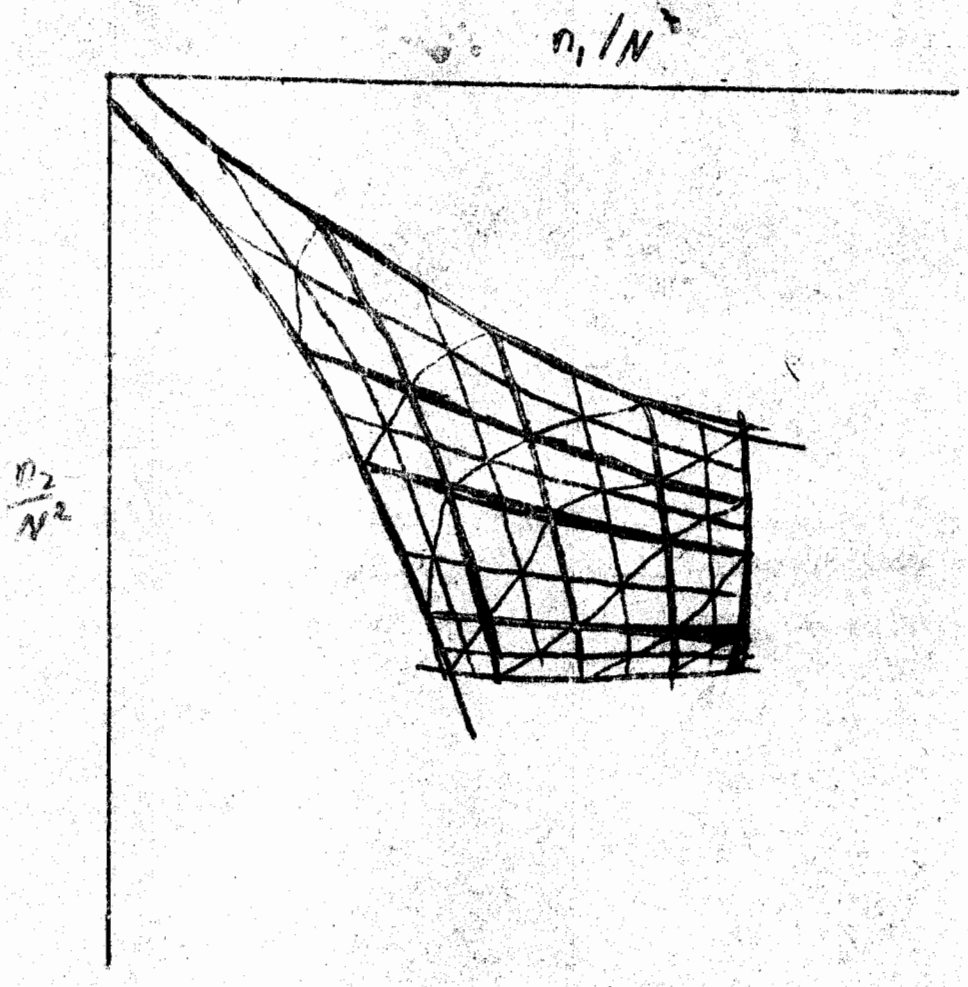
$$= -2\delta \sin^2\varphi \left(1 - \cos\gamma - (2+\delta) w \sin\gamma\right)^2 \leq 0 \tag{31}$$

We thus have the result that K is positive (or zero), and the motion stable, at a difference resonance, while at a sum resonance $K \leq 0$ and the motion is unstable (unless $K = 0$).

So far this result applies only when the coupling takes the form assumed here, namely one part of the machine is tilted with respect to the other parts.

More general forms of coupling can always be expressed as combinations of many perturbations of this form. It is hoped that the theorem proved here for the special case can be extended to the general case; the plausibility argument at the beginning of this note makes this probable.

As a result only half of the parallelograms in the 'necktie' are divided into trianges by the sum resonances, while half of the parallelograms remain intact (see Fig. 1).

Fig. 1

Stability diagram with resonances