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ON THE MATRIX REPRESENTATION OF $CP_q(n)$ AND COVARIANT q-OSCILLATORS

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Abstract

The matrix representation of the quantum projective space $CP_q(n)$ is given and it is explicitly shown that the elements of the matrix representation of $CP_q(n)$ are the annihilation operators of Pusz-Woronowicz oscillators provided they are rescaled. We also give an explicit construction of Jordan-Schwinger bilinears in terms of q-oscillators where different q-oscillators commute.

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I. INTRODUCTION

Quantum groups appeared first as quantum algebras, i.e., as one parameter deformations of the universal enveloping algebras of classical Lie algebras, in the study of the algebraic aspects of quantum integrable systems [1,2].

Other approaches to quantum groups, in which the objects may be called quantum matrix groups and Hopf algebras dual to the quantum algebras, are developed by Faddeev et al [3], Woronowicz [4] and Manin [5].

A new realization of the quantum group $SU_q(2)$ has been obtained by introducing a q-analogue of the usual harmonic oscillator and using the Jordan-Schwinger mapping [6,7]. Similar deformations of the quantum harmonic oscillator algebra have attracted a lot of attention [8-15]. The quantum group invariant q-oscillator is intimately related to the differential calculus [16,17] on the quantum hyperplane. In the Ref. 16, a formalism of a second quantization procedure based upon the twisted SU(n) group is constructed and the related twisted canonical commutation relations are investigated.

Recently it has been shown that the unitary quantum group $U_q(n)$ can be constructed in terms of $n(n-1)/2$ q-oscillators and n commuting phases [18]. This approach shows that the quantum projective space $CP_q(n)$ can be identified with n q-oscillators. Also Arik et al [19] constructed the quantum Grassmannian manifolds and it has been shown that the coordinates of this quantum manifold can be taken to be mxn dimensional q-oscillators.

The purpose of this work is to show explicitly that the elements of the matrix representations of the quantum projective space $CP_q(n)$ are the covariant annihilation operators of Pusz-Woronowicz oscillators.

In section II, we will give the review of the quantum

group $SU_q(2)$ and the matrix representation of $CP_q(1)$. In section III, using the approach in Ref. 18 we present the matrix representation of the quantum projective space and show that the matrix elements of this representation are the annihilation operators of Pusz-Woronowicz oscillators. We also give an explicit construction of Jordan-Schwinger bilinears in terms of q-oscillators. In section IV, we present a discussion of our results.

II. REVIEW OF THE QUANTUM GROUP $SU_q(2)$

Let us begin with some basic concepts.

The quantum group $GL_q(2)$ is formed by the matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries belong to a C^*-algebra and satisfy the following q-commutation relations

$$\begin{aligned} ab &= qba \ ; & ac &= qca, \\ bd &= qdb \ ; & cd &= qdc, \\ bc &= cb \ ; & ad - da &= (q - q^{-1})bc. \end{aligned} \tag{2.1}$$

Using the above relations one can easily verify the quantum determinant

$$\mathcal{D}_q = ad - qbc$$

belongs to the center of $GL_q(2)$. Hence it is defined to be equal to unity then M belongs to the special quantum group $SL_q(2)$.

The quantum inverse of the quantum matrix M is given by

$$M^{-1} = \mathcal{D}_q^{-1} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}. \tag{2.2}$$

We define the unitary quantum group $U_q(2)$ by

$$M^* = (M^T)^{-1} \tag{2.3}$$

where M^T is the ordinary transpose of M and

$$M^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}.$$

The matrices in the quantum subgroup $SU_q(2)$ of $U_q(2)$ are characterized by the condition (2.3) with the further condition $\mathcal{D}_q = 1$. After a little algebraic manipulation we deduce that $SU_q(2)$ consists of all matrices of the form

$$M = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} \tag{2.4}$$

where

$$ab = qba \quad , \quad ab^* = qb^*a \ , \quad bb^* = b^*b \tag{2.5}$$

$$aa^* + q^2bb^* = 1,$$

$$a^*a + bb^* = 1. \tag{2.6}$$

Since the matrix elements of $GL_q(2)$ belong to a C^* algebra, they are reverse-multiplicative and therefore by the relations (2.5) the deformation parameter q must be a real number. Thus we can take the parameter q such that $0 < q < 1$.

We define the * operation as follows.

$$(a)^* = a^+ \quad \text{and} \quad (a^+)^* = a.$$

Eliminating bb^* from the equations in (2.6) results in

$$\alpha\alpha^* - q^2\alpha^*\alpha = 1 - q^2. \tag{2.7}$$

We will take this equation to define our q-oscillator[18] where the operator α^* is the creation and its hermitean conjugate α is the annihilation operator. Note that the normalization of the oscillators are chosen such that in the $q \longrightarrow 1$ limit this operators commute and give c-numbers. The spectrum of the q-oscillator defined by equation (2.7) is completely fixed for $0 < q < 1$ and it shown that [18]

$$\alpha^*\alpha = 1 - q^{2N} = [N]_q \tag{2.8}$$

where N denotes the number operator. Also one has [6]

$$\begin{aligned} \alpha q^{rN} &= q^{r(N+1)}\alpha \\ \alpha^* q^{rN} &= q^{r(N-1)}\alpha^* . \end{aligned} \tag{2.9}$$

Moreover, one obtains ($n = 0,1,2,...$)

$$\begin{aligned} [\alpha,\alpha^{*(n+1)}] &= [n+1]_q \alpha^{*n} q^{2N} \\ [\alpha^{(n+1)},\alpha^*] &= [n+1]_q \alpha^n q^{2N} \end{aligned} \tag{2.10}$$

where

$$[n]_q = 1 - q^{2n} .$$

Now, if we define B by

$$B = (1 - q^2)^{-1/2} q^{-N/2} \alpha \tag{2.11a}$$

so that

$$B^{+} = (1 - q^{2})^{-1/2}\alpha^{*}q^{-N/2} \tag{2.11b}$$

where $N^{+} = N$. Then (2.7) becomes

$$BB^{+} - qB^{+}B = q^{-N}. \tag{2.12}$$

This is the form the one dimensional q-oscillator which has been conventionally used in various works by many authors (see, for example, Ref.s 6,9). Note that although for $0 < q < 1$ equations (2.7) and (2.12) are related by just a rescaling, in the $q \to 1$ limit they are fundamentally different. In this limit the oscillator creation and annihilation operators in (2.7) are commutative so that α and α^{*} can be considered as commuting complex numbers.

It shown that the parametrization of a $SU_q(2)$ matrix in terms of a q-oscillator plus a commuting phase [18] :

$$M = M(\alpha)\chi(\alpha) \tag{2.13}$$

where

$$M(a) = \begin{bmatrix} \alpha & (1-\alpha^{*}\alpha)^{1/2} \\ -q(1-\alpha^{*}\alpha)^{1/2} & \alpha^{*} \end{bmatrix} \tag{2.14}$$

and

$$\chi(\alpha) = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}. \tag{2.15}$$

By this construction the representation given in (2.13) is unique and due to Ref. 18, we can take the matrix in

(2.14) as a representative of the quantum projective space $CP_q(1)$ since the quantum isotropy group [20] of $CP_q(1)$ is $U_q(1) \cong U(1) = S[U(1)\times U(1)]$.

III. THE QUANTUM PROJECTIVE SPACE $CP_q(n)$ AND COVARIANT q-OSCILLATORS

The unitary quantum group $U_q(n)$ will be defined as the set of nxn matrices $M=(m_{ij})$ obeying (2.3). Explicitly, if $M\in U_q(n)$ we have

$$\sum_{k=1}^{n} m_{ik}^{*} m_{jk} = \delta_{ij} = \sum_{k=1}^{n} m_{ki} m_{kj}^{*} , \qquad (3.1)$$

$$\sum_{k=1}^{n} q^{k-1} m_{ik} \Delta_q(A_{jk}) = 0, \quad i\neq j$$

where the matrix elements of every 2x2 submatrix A_{rs} of M obey the commutation relations (2.2) and for $r \neq i < l$ and $s \neq j < k$

$$\Delta_q(A_{rs}) = m_{ij} m_{kl} - q m_{il} m_{jk}.$$

The quantum determinant $\mathcal{D}_q$ of the matrix M is defined recursively:

$$\sum_{j=1}^{n} (-q)^{j-i} m_{ij} \Delta_q(A_{ij}) = 1. \qquad (3.2)$$

The quantum determinant $\mathcal{D}_q$ has the property that it commutes with all the matrix elements of M, and therefore

$\mathcal{D}_q$ is a central element.

Now we consider the quantum group $SU_q(3)$. It can be shown that any element of $SU_q(3)$ can be expressed uniquely as [18]

$$M = M_{12}(\alpha_1)M_{23}(\alpha_2)M_{12}(\alpha_3)\chi_{12}(\alpha)\chi_{23}(\beta) \tag{3.3}$$

where

$$M_{12}(\alpha_1) = \begin{bmatrix} M(\alpha_1) & 0 \\ 0 & 1 \end{bmatrix} \tag{3.4}$$

$$M_{23}(\alpha_2) = \begin{bmatrix} 1 & 0 \\ 0 & M(\alpha_2) \end{bmatrix} \tag{3.5}$$

with (2.14) and

$$\chi_{12}(\alpha) = \begin{bmatrix} \chi(\alpha) & 0 \\ 0 & 1 \end{bmatrix} \tag{3.6}$$

$$\chi_{23}(\beta) = \begin{bmatrix} 1 & 0 \\ 0 & \chi(\beta) \end{bmatrix} \tag{3.7}$$

with (2.15). The matrix M in Equ. (3.3) contain three q-oscillators and with their hermitean conjugates they satisfy q-oscillator relation (2.7) whereas different α_i commute; α and β are phases which are central.

In equation (3.3) the element $M_{12}(\alpha_3)\chi_{12}(\alpha)\chi_{23}(\beta)$ is an arbitrary element of the quantum isotropy group [20] $SU_q(2)\times U_q(1)$. Hence

$$H = M_{12}(\alpha_1)M_{23}(\alpha_2) \tag{3.8}$$

can be taken as a representative of the quantum projective space $CP_q(2)$ [18].

Explicit form of the matrix H in (3.8) is

$$H = \begin{pmatrix} \alpha_1 & \alpha_1'\alpha_2 & \alpha_1'\alpha_2' \\ -q\alpha_1' & \alpha_1^*\alpha_2 & \alpha_1^*\alpha_2' \\ 0 & -q\alpha_2' & \alpha_2 \end{pmatrix} \tag{3.9}$$

where

$$\alpha_i' = (1 - \alpha_i^*\alpha_i)^{1/2}. \tag{3.10}$$

Now we take

$$A_1 = \alpha_1 \text{ and } A_2 = \alpha_1'\alpha_2 \tag{3.11}$$

where different q-oscillators commute, then one obtains

$$\begin{aligned} A_1A_2 &= qA_2A_1 \\ A_1^+A_2^+ &= q^{-1}A_2^+A_1^+ \\ A_iA_j^+ &= qA_j^+A_i \ , \ i\neq j \ (i,j=1,2) \\ A_1A_1^+ - q^2A_1^+A_1 &= 1 - q^2 \\ A_2A_2^+ - q^2A_2^+A_2 &= (1 - q^2) + (q^2 - 1)A_1^+A_1. \end{aligned} \tag{3.12}$$

These results generalize to $CP_q(n)$ as follows. We arrive at [18]

$$H = \prod_{i=1}^{n} M_{i,i+1}(\alpha_i) \tag{3.13}$$

being identified with the matrix representatives of the n-dimensional quantum projective space $CP_q(n)$. Here each $M_{i,i+1}(\alpha_i)$ is a matrix whose 2x2 diagonal block in the i,i+1 position is a q-oscillator matrix

$$\begin{pmatrix} \alpha & (1 - \alpha^*\alpha)^{1/2} \\ -q(1 - \alpha^*\alpha)^{1/2} & \alpha^* \end{pmatrix} \tag{3.14}$$

and the remaining diagonal elements contain 1, and all elements apart from these are zero. In Equ. (3.13) each a_i is a q-oscillator and their properties are determined by the commutation relations, for i,j = 1,2,...,n

$$[\alpha_i,\alpha_j] = 0 = [\alpha_i^*,\alpha_j^*] , \tag{3.15}$$

$$[\alpha_i,\alpha_j^*] = (1 - q^2)q^{2N}\delta_{ij} \tag{3.16}$$

where δ_{ij} denotes the Kronecker delta.

In the matrix H given by (3.13) instead of taking the (n-1) α_i's as independent one can take the first (n-1) elements of the first row as independent. Then these elements can be shown to be the annihilation operators of Pusz-Woronowicz oscillators [16] provided they are rescaled as follows

$$\begin{aligned} \alpha &= (1 - q^2)^{1/2}A \\ \alpha^* &= (1 - q^2)^{1/2}A^+ . \end{aligned} \tag{3.17}$$

In fact, if we take

$$A_k = (1 - q^2)^{-1/2} \left[\prod_{i=1}^{k-1} \alpha'_i \right] \alpha_k \tag{3.18}$$

for $k = 2,3,\ldots,n$, and

$$A_1 = (1 - q^2)^{-1/2} \alpha_1$$

one can has

$$\begin{aligned} A_i A_j &= q A_j A_i \ , \quad i < j \\ A_i A_j^+ &= q A_j^+ A_i \ , \quad i \neq j \\ A_i^+ A_j^+ &= q A_i^+ A_j^+ \ , \quad i > j \\ A_i A_i^+ - q^2 A_i^+ A_i &= 1 + (q^2 - 1) \sum_{k<i} A_k^+ A_k \ . \end{aligned} \tag{3.19}$$

Now we take

$$\alpha_1 = A_1 . \tag{3.20a}$$

For a representation in a Hilbert space where α_1^* denotes the hermitean conjugate of α_1, the eigenvalues of $A_1^+ A_1$ are real and non-negative. Moreover, by the relation (2.6) the operator $A_1^+ A_1$ is bounded and the eigenvalues of this operator are in the interval $[0,1)$. Thus one can take

$$\alpha_2 = (1 - A_1^+ A_1)^{-1/2} A_2 . \tag{3.20b}$$

We require that the relations which we will obtain, after the defining the transformations, are the same with the relations (3.16) and (3.15). Thereby we will take

$$\alpha_1 = (1 - q^2)^{1/2} A_1, \tag{3.21a}$$

$$\alpha_2 = (1 - q^2)^{1/2} X^{-1} A_2, \tag{3.21b}$$

where

$$X^2 = A_1 A_1^+ - A_1^+ A_1. \tag{3.22}$$

Note that

$$\begin{aligned} &A_1 X^2 = q^2 X^2 A_1, \quad A_1^+ X^2 = q^{-2} X^2 A_1^+ \\ &A_1 X^{\pm 1} = q^{\pm 1} X^{\pm 1} A_1, \quad X^{\pm 1} A_1^+ = q^{\pm 1} X^{\pm 1} A_1^+ \\ &[A_2, X^2] = 0 = [A_2^+, X^2]. \end{aligned} \tag{3.23}$$

Now,using the commutation relations (3.19) with (3.23) it is easy to check that the relations (3.16) and (3.15) are invariant under the transformations (3.21). Indeed, for example,

$$\begin{aligned} \alpha_1 \alpha_2 &= (1 - q^2) A_1 X^{-1} A_2 \\ &= (1 - q^2) X^{-1} A_2 A_1 = \alpha_2 \alpha_1, \end{aligned}$$

and

$$\begin{aligned} \alpha_2 \alpha_2^* - q^2 \alpha_2^* \alpha_2 &= (1 - q^2)(X^{-1} A_2 A_2^+ X^{-1} - q^2) A_1^+ X^{-2} A_2) \\ &= (1 - q^2)(A_2 A_2^+ - q^2 A_2^+ A_2) X^{-2} \\ &= 1 - q^2. \end{aligned}$$

Consequently, we have shown that there is a one to one

correspondence between the matrix elements of the matrix representation of $CP_q(n)$ and covariant Pusz-Woronowicz oscillators.

In the general case, one can take

$$\alpha_1 = (1 - q^2)^{1/2} A_1$$

and for $k = 2,3,\ldots,n$

$$\alpha_k = (1 - q^2)^{1/2} \left[\prod_{i=1}^{k-1} X_i^{-1} \right] A_k \tag{3.24}$$

where

$$X_i^2 = (A_i A_i^+ - A_i^+ A_i) \tag{3.25}$$

and a straightforward shows that the relations (3.19) are satisfied .

Finally,array the oscillators in Jordan-Schwinger [21] bilinears:

$$e_{ij} = \alpha_i^* \alpha_j \ , \quad i,j = 1,2 \tag{3.26}$$

where the q-oscillators α_i have to satisfy the relations (3.15) and (3.14) and

$$e_{ij}^+ = e_{ji}.$$

Then, after some algebra, one obtains

$$[e_{11},e_{22}] = 0,\ [e_{12},e_{21}] = (1-q^2)(e_{11}-e_{22}),$$

$$[e_{11},e_{12}]_{q^2} = (1-q^2)e_{12},$$

$$[e_{21},e_{11}]_{q^2} = (1-q^2)e_{21}, \tag{3.27}$$

$$[e_{22},e_{12}]_{q^{-2}} = (1-q^{-2})e_{12},$$

$$[e_{21},e_{22}]_{q^{-2}} = (1-q^{-2})e_{21},$$

where

$$[a,b]_{q^2} = ab - q^2 ba, \tag{3.28}$$

or by the transformations

$$E_{ij} = (1-q^2)^{-1}e_{ij}\ ,\quad i,j = 1,2 \tag{3.29}$$

one obtains

$$[E_{11},E_{22}] = 0,\ [E_{12},E_{21}] = E_{11} - E_{22},$$

$$[E_{11},E_{12}]_{q^2} = E_{12},\ [E_{21},E_{11}]_{q^2} = E_{21}, \tag{3.30}$$

$$[E_{22},E_{12}]_{q^{-2}} = -q^{-2}E_{12},\ [E_{21},E_{22}]_{q^{-2}} = -q^{-2}E_{21}.$$

These relations are same with the ones appearing in Ref.10 for the single quantum parameter case.

These result can be generalize as follows:

$$\alpha_i\alpha_i^* - q^2\alpha_i^*\alpha_i = 1 - q^2$$

$$[\alpha_i,\alpha_j] = 0 = [\alpha_i^*,\alpha_j^*]\ ,\ i \neq j$$

$$e_{ij} = \alpha_i^*\alpha_j\ ,\ i,j = 1,2,\ldots,n-1$$

$$[e_{ij},e_{jk}]_{q^2} = (1-q^2)e_{ik}\ ,\ i \neq k \tag{3.31}$$

$$[e_{ij},e_{kl}] = 0\ ,(i<l \text{ and } j\neq k) \text{ or } (i\neq l \text{ and } j<k)$$

$$[e_{ij},e_{ji}] = e_{ii} - e_{jj}\ ,\ i < j.$$

IV. DISCUSSION

In this work,we haven taken the matrix elements of the unitary quantum group as the independent q-oscillators and have shown that the entries of the matrix representations of the quantum projective space $CP_q(n)$ are the annihilation operators of Pusz-Woronowicz oscillators. Moreover, we give an explicit construction of Jordan-Schwinger bilinears in terms of q-oscillators which are the matrix elements of the matrices $M_{i,i+1}$ appearing in the matrix representation of $CP_q(n)$.These are commutative and one of them satisfy the q-oscillator relation (2.7).

The quantum projective spacehas two different $q \longrightarrow 1$ limits. In the first case it reduces to the ordinary complex projective space CP(n), in the other case it reduces to the n-dimensional ordinary oscillator.

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REFERENCES

[1] E.K.Sklyanin, Func.Anal.Appl. 16, 262(1982).

[2] P.Kulish and N.Y.Reshetikhin, J.Sov.Math. 23,24(1983).

[3] L. D. Faddeev, N. Y. Reshetikhin and L. A. Takhtajan, "Quantization of Lie groups and Lie algebras",Leningrad preprint LOMI (1987).

[4] S. L. Woronowicz, Comm. Math. Phys. 111, 613 (1987).

[5] Yu.I.Manin "Quantum groups and noncommutative geometry" Montreal Univ. preprint CRM-1561 (1988).

[6] A.J.Macfarlane, J.Phys.A 22, 4581 (1989).

[7] L.C.Biedenharn, J.Phys.A 22, L873 (1989).

[8] M.Chaichian and P.P.Kulish, Preprint CERN-TH. 5969/90 (1990).

[9] P.P.Kulish and E.V.Damaskinsky,J.Phys.A 23,L415(1990).

[10] S.P.Vokos,J.Math.Phys. 32(11),2979(1991); D.B.Fairlie and C.K.Zachos, Phys. Lett. B 256, 43(1991).

[11] M.Arik, Z.Phys.C 51, 627(1991).

[12] M. Chaichian, P. Kulish and J. Lukierski,Phys.Lett. B 237, 401(1990) and 262, 43(1991).

[13] C.Daskaloyannis, J. Phys. A 23, L789(1990).

[14] D.Bonatsos, C.Daskaloyannis and K.Kokkotas, J.Phys. A 23, L795(1990).

[15] M. Arik and M. Mungan, Phys. Lett. B 282, 101(1992).

[16] W.Pusz and S.L.Woronowicz, Reports Math.Phys. 27, 231 (1989).

[17] J.Wess and B.Zumino, Nucl. Phys. B (Proc.Suppl.) 18B, 302 (1990).

[18] M.Arik and S.Celik, "Unitary quantum groups, quantum projective spaces and q-oscillators," Bogazici Univ. preprint BUFB-92-02 (1992) (to be published in Z.Phys.C Particles and Fields).

[19] M. Arik, F. Aydin, S. Celik and E. Hizel, "Quantum Grassmannian manifolds,"Bogazici Univ. Preprint BUFB-92-04(1992).

[20] P.van Nieuwenhuizen,"General Theory of Coset Manifolds and Antisymmetric Tensors Applied to Kaluza-Klein Supergravity'84," edited by B.de Wit, P.Fayet and P.van Nieuwenhuizen,Proceedings of the Trieste Spring School, 1984.

[21] P. Jordan, Z.Phys.C 94, 531(1935); J. Schwinger, "On Angular Momentum," in Quantum Theory of Angular Momentum, edited by L.C.Biedenharn and H.van Dam (Academic, New York, 1965).