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**Direct solution of the hard Pomeron problem for arbitrary conformal weight**

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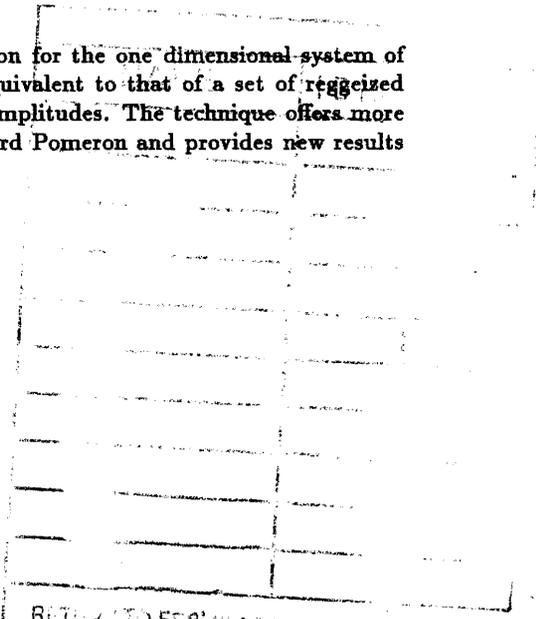
**Abstract**

A new method is applied to solve the Baxter equation for the one dimensional system of noncompact spins. Dynamics of such an ensemble is equivalent to that of a set of reggeized gluons exchanged in the high energy limit of the QCD amplitudes. The technique offers more insight into the old calculation of the intercept of the hard Pomeron and provides new results in the odderon channel.

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Quantitative description of the reggeization of QCD still remains a challenge for the Leading Logarithmic scheme and its extensions<sup>1,2</sup>. In the first approximation the problem separates into sectors with fixed number  $n$  of the reggeized gluons propagating in the  $t$  channel. The lowest nontrivial case,  $n = 2$ , was solved in the classical papers by Balitskii, Kuraev, Fadin and Lipatov<sup>3</sup> resulting in the simple expression for the intercept of the hard pomeron. The notable progress for arbitrary  $n$  was achieved by Lipatov and Faddeev and Korchemsky<sup>4,6</sup> who have established exact equivalence with the one dimensional chain of  $n$  noncompact spins. The success of this approach was confirmed by rederiving the Lipatov et al. result in the  $n = 2$  case<sup>6,7</sup>. However, the adopted procedure requires an analytic continuation from the integer values of the relevant conformal weight  $h$  (see later) because only for integer  $h$  they were able to diagonalize the two spin hamiltonian. The  $n = 3$  case, which gives the lowest contribution to the odderon exchange, was studied by Lipatov, Faddeev and Korchemsky,<sup>5,6,7</sup>. Again, the spectrum of the system for integer  $h$  can be found for any finite  $h = m$ . However, the general expression for arbitrary  $m$  is not known, and consequently the analytical continuation to  $h = 1/2$  is not available<sup>6</sup>.

We have developed a new approach which a) works for arbitrary values of the conformal weight  $h$ , providing explicitly above continuation, and b) gives the analytic solution of the  $n = 3$  problem for arbitrary  $h$  and  $g_3$ . Here we will apply the new method to the  $n = 2$  case rederiving directly the BFKL result without need of the analytical continuation. Our new results in the  $n = 3$  case<sup>8</sup> will be also shortly summarized.

The intercept of the Pomeron trajectory is given by

$$\alpha_P(0) = 1 + \frac{\alpha_s N_c}{4\pi} (\epsilon_2(h) + \bar{\epsilon}_2(\bar{h})), \quad (1)$$

where  $\epsilon_2$  and  $\bar{\epsilon}_2$  are respectively the largest eigenvalues of the  $n = 2$  reggeon hamiltonian and its antiholomorphic counterpart<sup>6,7</sup>. This system is equivalent to the misleadingly simple set of the two noncompact spins which for higher  $n$  generalizes to the one dimensional chain with nearest-neighbour

<sup>8</sup>The lowest state of the  $n = 3$  hamiltonian is believed to occur at  $h = 1/2$ .

interactions. Applying Bethe ansatz one obtains in the  $n = 2$  case

$$\epsilon_2 = i \left( \frac{\dot{Q}_2(-i)}{Q_2(-i)} - \frac{\dot{Q}_2(i)}{Q_2(i)} \right) - 4, \quad (2)$$

where  $Q_2(\lambda)$  satisfies the following Baxter equation

$$(\lambda+i)^2 Q_2(\lambda+i) + (\lambda-i)^2 Q_2(\lambda-i) = (2\lambda^2 + g_2) Q_2(\lambda). \quad (3)$$

$g_2$  is the eigenvalue of the square of the total spin of the system  $\hat{g}_2$ . It commutes with the hamiltonian and its spectrum is known from the symmetry considerations

$$g_2 = h(1-h), \quad h = \frac{1}{2}(1+m) - i\nu, \quad m \in Z, \nu \in R. \quad (4)$$

In order to solve the Baxter equation, (3), the following integral representation is customarily used

$$Q_2(\lambda) = \int_{C_I} z^{-i\lambda-1} (1-z)^{i\lambda+1} Q(z) dz. \quad (5)$$

Then, if the boundary terms do not contribute, Eq.(3) is equivalent to the simple hypergeometric equation for  $Q(z)$

$$\left[ \frac{d}{dz} z(1-z) \frac{d}{dz} - g_2 \right] Q(z) = 0, \quad (6)$$

with the well known solutions. However, for arbitrary value of the conformal weight,  $h$  the singularity structure of the hypergeometric functions together with the nontrivial monodromy of the kernel  $K(z, \lambda) = z^{-i\lambda-1} (1-z)^{i\lambda+1}$  precludes existence of the contour such that the boundary contributions cancel. For integer  $h = m$ , however, the solution regular at  $z = 0$  does not have a cut and consequently the simple contour encircling both  $z = 0$  and  $z = 1$  points guarantees vanishing of the boundary terms. This observation was exploited in Refs.<sup>6,7</sup> leading to the elegant solution of the  $n = 2$  problem for integer conformal weight. The BFKL formula resulted after the analytic continuation in  $h$  to  $h = 1/2$ . However, the case of noninteger  $h$  requires further insight. In particular the boundary conditions for  $Q_2(\lambda)$  are not fully understood. For integer  $h$ , again, they can be deduced from the polynomial Bethe ansatz and are consistent with the above choice of the integration contour in Eq.(5). For arbitrary  $h$ , they are not

available. It would be very instructive to investigate the so called functional Bethe ansatz in this connection.

We will present here a different approach. It was observed in Ref.<sup>9</sup> that the double contour representation (c.f. Fig.1)

$$Q_2(\lambda) = \int_{C_I} z^{-i\lambda-1} (1-z)^{i\lambda+1} Q_I(z) dz \quad (7) \\ + \int_{C_{II}} z^{-i\lambda-1} (1-z)^{i\lambda+1} Q_{II}(z) dz,$$

together with simple boundary conditions on  $Q_{I/II}(z)$ , reproduced numerically the holomorphic energy in the half-integer case  $h = m + 1/2$ . Using the double contour representation we have subsequently derived the analytic expression for the holomorphic energy for arbitrary complex  $h$ . With the aid of the new formalism of the transition matrix this method was applied to the  $n = 3$  case and led to the analytic expression for the intercept of the odderon trajectory for arbitrary values of relevant parameters.

We begin with the general solutions of Eq.(6) and then show how the original freedom is restricted leading to the unique solution. To this

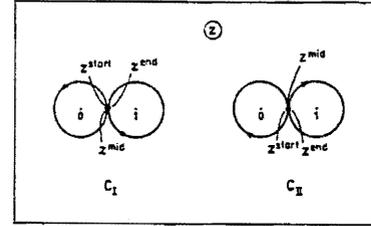


Figure 1: Integration contours used in Eq.(7). Start  $z^{\text{start}}$ , middle  $z^{\text{mid}}$ , and end  $z^{\text{end}}$  points coincide but they lie on the different sheets of the Riemann surface of the integrands.

end we write the two fundamental sets of two, linearly independent solutions of Eq.(6)

$$\vec{u}(z) = (u_1(z), u_2(z)), \quad (8) \\ \vec{v}(z) = (v_1(z), v_2(z)),$$

around  $z = 0$  and  $z = 1$  respectively.

$$u_1(z) = F(h, 1-h, 1; z) = \sum_{n=0}^{\infty} f_n z^n,$$

$$u_2(z) = \frac{s(h)}{\pi i} \log z u_1(z) - \frac{s(h)}{\pi i} \sum_{n=0}^{\infty} g_n z^n, \quad (9)$$

$$g_n = f_n [2\psi(n+1) - \psi(n+h) - \psi(n+1-h)],$$

where  $F(a, b, c; z)$  is the hypergeometric function,  $\psi(z)$  denotes the digamma function and  $s(h) = \sin(\pi h)$ . The series in Eq.(9) are convergent in the unit circle  $K_0$  around  $z = 0$ . Similarly one can construct the  $\vec{v}(z)$  solutions in the unit circle  $K_1$  around  $z = 1$ . In fact, because of the symmetry of Eq.(6), we take

$$v_1(z) = iu_1(1-z), \quad v_2(z) = -iu_2(1-z). \quad (10)$$

Since any solution is a linear combination of the fundamental solutions, we have in general

$$Q_I(z) = au_1(z) + bu_2(z) \\ \equiv A \cdot \vec{u}(z) = A \cdot \Omega \vec{v}(z), \\ Q_{II}(z) = cu_1(z) + du_2(z) \\ \equiv B \cdot \vec{u}(z) = B \cdot \Omega \vec{v}(z), \quad (11)$$

with an obvious vector notation. The transition matrix  $\Omega$  is defined by

$$\vec{u}(z) = \Omega \vec{v}(z), \quad (12)$$

and provides the analytic continuation of our solutions  $Q(z)$  between  $K_0$  and  $K_1$ . It plays an important role for higher  $n$  and its direct calculation for  $n > 2$  is rather nontrivial. For the hypergeometric equation, and for the special choice of both bases, Eqs.(9,10),  $\Omega$  is very simple. Due to the identity  $u_2(z) = iu_1(1-z)$

$$\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (13)$$

Next we introduce the monodromy matrix  $M_\Omega$

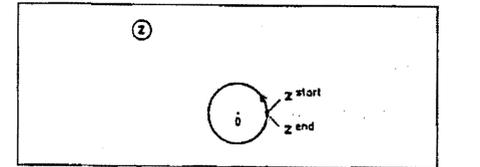


Figure 2: Closed contour used to define the monodromy matrix, Eq.(14).  $z^{\text{start}} = z^{\text{end}}$ , however they belong to the different sheets of the Riemann surface.

which describes the behaviour of the basis  $\vec{u}$  in the vicinity of the branch point  $z = 0$  (see Fig.2).

$$\vec{u}(z_{end}) = M_u \vec{u}(z_{start}), \quad M_u = \begin{pmatrix} 1 & 0 \\ 2s(h) & 1 \end{pmatrix}, \quad (14)$$

and similarly for the  $v$  basis. It is easy to see that  $M_v = M_u^{-1}$ .

We are now ready to write the condition for the cancellation of the boundary contributions in Eq.(7). With the choice of the contours  $C_I$  and  $C_{II}$  as shown in Fig.1, the boundary contributions cancel if

$$A^T M_I + B^T M_{II} = 0, \quad (15)$$

where the combined monodromy matrices for the corresponding contours read

$$M_I = \Omega M_v \Omega^{-1} - M_u^{-1}, \quad M_{II} = \Omega M_v^{-1} \Omega^{-1} - M_u. \quad (16)$$

In terms of the coefficients, condition (15) reads simply

$$a = c, \quad b = d. \quad (17)$$

Hence the original freedom of four coefficients in Eqs.(11) was reduced to the two free parameters. In fact the energy of the system, Eq.(2), is insensitive to the absolute normalization, hence only the ratio

$$\rho = a/b, \quad (18)$$

remains relevant. This variable parametrizes all possible boundary conditions which are consistent with the cancellation of the end-point contributions in the sum (7). The role of the remaining freedom is better seen when the explicit result for  $\epsilon_2$  is derived.

To this end we substitute Eq.(11) with (17) in (7) and integrate resulting expression term by term expanding  $Q_{I/II}(z)$  in the  $u$  basis on  $C_I$ , and in the  $v$  basis on  $C_{II}$ . Since the involved series are absolutely convergent in corresponding domains, the final result for  $\epsilon_2(h)$  is the analytic function of  $h$ . Consistent choice of the branches of the kernel  $K(z, \lambda)$  and of  $Q(z)$  must be made. After some calculations we obtain

$$\epsilon_2(h) = 4\psi(1) - 2\psi(h) - 2\psi(1-h) - \frac{i\pi}{s}(\rho - \rho^{-1}). \quad (19)$$

It is instructive to compare this result with the original hamiltonian of the two spins<sup>7</sup>

$$\hat{\mathcal{H}}_2 = 4\psi(1) - 2\psi(-\hat{J}_{12}) - 2\psi(1 + \hat{J}_{12}). \quad (20)$$

where the eigenvalues of  $\hat{J}_{12}$  are equal to  $-h$  c.f. Eq.(4). It is now evident that the choice

$$\rho = \pm 1, \quad (21)$$

gives the correct spectrum of energies. We emphasize, however, that the additional information was required to fix the remaining freedom. This is different in the  $n = 3$  case (see below). It is important to note that the above choice is independent of  $h$  which *a priori* is not guaranteed.

Substituting Eq.(19), with (21), in Eq.(1), and setting  $h = \bar{h} = 1/2$ , we reproduce the BFKL formula

$$\alpha_P(0) = 1 + \frac{\alpha_s N}{\pi} 4 \log 2. \quad (22)$$

This was also obtained in Ref.<sup>7</sup> after analytic continuation of their result from integer values of  $h$ . The difference between both approaches is best seen by comparing Eq.(19) with Eq.(6.31) of Ref.<sup>7</sup>. It follows from the form of the hamiltonian, Eq.(20), that the complete holomorphic eigenenergy  $\epsilon_2(h)$  is singular also at positive integer  $h$ . This is true for our result, Eq.(19). On the other hand, as seen from Eq.(1), in order to calculate the physical intercept only the *real* part of  $\epsilon_2$  is required. It is finite for positive integer  $h$  and was correctly reproduced by the method of Faddeev and Korchemsky, c.f. Eq.(6.31) in Ref.<sup>7</sup>.

One of the ingredients of the calculation presented in Ref.<sup>7</sup> is the prescription how to fix an overall constant term in the two spin Hamiltonian, Eq.(20). In the present formalism the result (19) also has a freedom which is parametrized by  $\rho$ . It would be interesting to see if the arbitrariness seen in both methods had the same origin.

Our method can be extended to higher  $n$ . For  $n = 3$  we have carried out this procedure explicitly<sup>8</sup>. The complete set of linearly independent solutions of the corresponding third order differential equation was constructed. The transition matrix between the  $\vec{u}$  and  $\vec{v}$  bases was also obtained. Since in this case there is no simple identity connecting linearly independent solutions, the  $\Omega$  matrix is nontrivial. Remarkably it turns out that the condition for cancellation of the end-point contributions in the double integral representation determines *uniquely* the final solution of the

Baxter equation. Existing arbitrariness in both transforms  $Q_{I/II}(z)$  is irrelevant. Consequently we have obtained the holomorphic (and antiholomorphic) energies as the analytic function of the two relevant parameters  $h$  and  $q_3$ . The new variable  $q_3$  is the eigenvalue of the second, commuting with hamiltonian, observable  $\hat{q}_3$  which is known but unfortunately was not diagonalized in spite of many attempts<sup>10,11,12</sup>. We have therefore mapped numerically the analytic structure of  $\epsilon_3(1/2, q_3)$  in the complex  $q_3$  plane. Result is sketched in Fig.3. The holomorphic energy has a series of poles at imaginary  $q_3$ <sup>b</sup>. The intercept of the odderon trajectory is smaller than one for almost all values of  $q_3$  including all  $q_3 \in \mathbb{R}$ . However in the vicinities of the poles it can be arbitrarily large. Therefore any further conclusion about the numerical value of the  $\alpha_0(0)$  depends crucially on the spectrum of  $q_3$ .

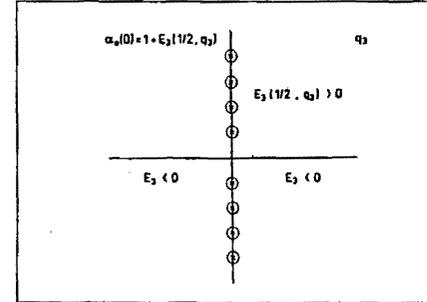


Figure 3: Schematic map of the analyticity structure of  $\epsilon_3(1/2, q_3)$  in the complex  $q_3$  plane.  $E_3$  is positive only in the vicinity of the poles.

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<sup>b</sup>Our definition of  $q_3$  is the same as in Ref.<sup>7</sup>

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