PATH-INTEGRAL QUANTUM COSMOLOGY: SOME RECENT DEVELOPMENTS

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ABSTRACT: We review some recent advances in a path-integral approach to quantum cosmology developed over the last few years. We discuss the construction of the path integral for the wave function of the universe. Because the Euclidean Einstein-Hilbert action is not bounded from below, the integration is necessarily over a contour of complex metrics. Criteria for a suitable choice of contour are discussed, many of them motivated by studies of simple minisuperspace models. We show that if the contour is suitably chosen, the wave functions generated by the path integral will be solutions to the Wheeler-DeWitt equation and momentum constraints of the Dirac quantization procedure.

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1. Introduction

The goal of quantum cosmology is to understand the origin and evolution of the universe in quantum mechanical terms. The prime motivation for this endeavour concerns the issue of initial conditions in cosmology. Virtually all predictions of classical cosmology always depend to some extent, even though very weakly, on initial conditions. This dependence may not be so obvious for processes in the late universe, but in the very early universe, around the inflationary era, it is becomes apparent that the present universe could only have arisen from a certain class of initial conditions. What is the origin of these initial conditions? What sort of form does a theory of initial conditions take?

In looking for the point in time at which to impose initial conditions, it is natural to follow the evolution of the universe backwards. As we do this, the scale factor goes to zero, the energy density and curvature approach infinity, and the universe appears to have begun with an initial singularity. We know from the singularity theorems that these singularities are not artefacts of inaccurate modeling in cosmological models, but are generic. The existence of these regions of arbitrarily high curvature is not taken to imply that the real universe actually began with a singularity, but rather, from general dimensional reasoning, that it emerged from a regime in which quantum gravitational effects are important. This suggests that the issue of initial conditions in cosmology is most properly addressed from a framework in which gravity is quantized. Quantum cosmology aspires to be such a framework.

In brief, quantum cosmology is the application of quantum mechanics to the dynamical systems describing closed universes, generally involving a combination of canonical and path integral methods, based on the Einstein action [1,2]. The point of this talk is to some describe some recent developments in the use of path integral methods, much of it by the present author in collaboration with J.B.Hartle, and with J.Louko.

2. The Problem of Quantum Gravity

The first and perhaps most serious difficulty with quantum cosmology is the problem of quantum gravity. Einstein gravity is perturbatively non-renormalizable, and no-one to date has had any reasonable degree of success in making sense of it as a quantum field theory. It is claimed that string theory is, or at least contains a consistent well-defined quantum theory of gravity. But it

has not yet been cast in a manageable form allowing it to be directly applied to cosmology.

Given, on the one hand, the considerable interest in the issue of initial conditions, but on the other, the serious difficulties with quantum gravity, it seems reasonable at present in quantum cosmology to persevere with a (possibly sick) theory based on Einstein gravity coupled to matter (or possibly simple modifications of this theory, such as those discussed by Steinhardt in his lectures [3]). There are a number of reasons why this is a reasonable thing to do:

(i) There is a chance that it may be correct, that is, that the theory may make sense non-peturbatively. This possibility is currently being investigated by Ashtekar and collaborators [4], and also in numerical studies of DeWitt and collaborators [5].

(ii) The results obtained may be correct in the low-energy regime, well away from the Planck scale. This regime may in fact still contain some interesting physics. For example, in tunneling from nothing scenarios, the scale (of curvatures, energy densities *etc.*) is set by demanding that one match on to an inflationary model, and this sets the scale to be four or five orders of magnitude below the Planck scale [6].

(iii) A quantum gravity theory based on Einstein gravity possesses features and difficulties one might expect to appear in a full quantum theory of gravity. We can therefore have a "practice run" at the full theory. For example, we can try and learn how to quantize generally covariant theories, with the associated problem of time, *etc.* In fact, I would say that the main achievements of contemporary quantum cosmology have been in the investigation of *issues*, rather than detailed technical results.

These, then, are some of the reasons why one might want to study a quantum theory of cosmology based on the Einstein action. Let us turn now to the formalism of quantum cosmology.

3. Canonical Quantization

We begin by considering the canonical quantization of Einstein gravity coupled to matter for closed universes [7]. Our starting point is the action

$$S = \int d^4x \, \sqrt{-g} \, (R - 2\Lambda) \; + \; S_{matter} \tag{3.1}$$

where S_{matter} is the action for some matter field, a scalar field ϕ say, and we are using units in which $16\pi G = 1$. To obtain the Hamiltonian form of the action we first write the metric in $(3+1)$ form,

$$ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j \tag{3.2}$$

where N is the lapse, N^i is the shift and h_{ij} is the three-metric. The Hamiltonian form of the action may then be straightforwardly obtained, and takes the form

$$S = \int dt \; d^3x \left[\dot{h}_{ij}\pi^{ij} + \dot{\phi}\pi - N\mathcal{H} - N^i\mathcal{H}_i \right] \tag{3.3}$$

where π^{ij} and π_ϕ are the momenta conjuate to h_{ij} and ϕ respectively. The Hamiltonian is a sum of constraints,

$$\mathcal{H} = h^{-\frac{1}{2}} \left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^i_i \pi^j_j \right) - h^{\frac{1}{2}} \left({}^3R - 2\Lambda \right) + \mathcal{H}_{matter} = 0 \tag{3.4}$$

$$\mathcal{H}_i = -2\pi_{ij}^{\;\;|j} + \mathcal{H}_i^{matter} = 0 \tag{3.5}$$

with the lapse and shift playing the role of lagrange multipliers.

In the quantum theory, the state of the system is represented by a wave functional, $\Psi[h_{ij}, \phi]$, a functional of the field configurations on the three-surfaces. The wave function has no explicit dependence on time. There is therefore no Schrödinger equation, only the constraints of the Dirac quantization procedure, obtained by insisting that the state is annihilated by the operator version of the constraints:

$$\hat{\mathcal{H}}\Psi[h_{ij}, \phi] = 0, \qquad \hat{\mathcal{H}}_i\Psi[h_{ij}, \phi] = 0 \tag{3.6}$$

These relations are referred to, respectively, the Wheeler-DeWitt equation and the momentum constraints.

4. Path-Integral Quantization

The wave function for the universe may also be obtained from a path integral expression. Suppose we wish to obtain the wave function for a three-surface B on which the three-metric is h_{ij} and the matter field is ϕ. Regard B as part of the boundary of a four-manifold $\mathcal{M}$, on which the four-metric is $g_{\mu\nu}$ and the matter field is Φ, with these fields matching h_{ij} and ϕ on B. Then the

wave function is obtained by summing over all such metrics and matter fields on the manifold $\mathcal{M}$, and over some class of manifolds:

$$\Psi[h_{ij}, \phi, B] = \sum_{\mathcal{M}} \int \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi \; e^{-I} \tag{4.1}$$

Here, I is the Euclidean action of the gravity plus matter system. Perhaps the best known example of such a construction is the "no-boundary" proposal of Hartle and Hawking, in which the class of manifolds summed over is taken to be compact manifolds whose *only* boundary is B [8].

The path integral approach is potentially more powerful than the canonical approach in that it can in principle deal with multiple surface amplitudes, as might be involved in topology changing situations, or in the decoherence functional [2]. These more general amplitudes are likely to be the objects of greatest interest in future approaches to quantum cosmology. Here however, for simplicity, we will consider only single surface amplitudes.

It is not known how to define the sum over four-manifolds, so in practice what is done is to study the path integral for each candidate four-manifold separately. As with all such path integral expressions, a number of technical aspects of this construction then need to be specified before it can be regarded as properly and uniquely defined, such as gauge-fixing machinery, measure, regularization *etc.* After all of this has been done, two question naturally arise:

1) Does the wave function so generated satisfy the constraints of the Dirac quantization procedure, *i.e.*, the Wheeler-DeWitt equation and the momentum constraints?

2) The Euclidean action for gravity is not bounded from below. The integral will therefore not converge unless the fields are integrated along a *complex* contour. There are presumably many suitable contours. How is this complex contour chosen?

These two questions are the topic of the rest of this talk.

5. Derivation of the Wheeler-DeWitt Equation

We now consider the first question, that of the connection between the path integral and the constraints [9]. It is enlightening to cast this question in a more general setting, and ask, for more general systems with symmetry, whether or not the wave functions generated by the path integral satisfy the constraints of the Dirac quantization procedure. For definiteness, let us focus

on constrained Hamiltonian systems, although the following discussion is not restricted to this case.

A constrained Hamiltonian system [10] is one defined on a phase space (p_i, q^i), characterized by an action of the form

$$S[p_i, q^i, \lambda^\alpha] = \int_{t'}^{t''} dt \left[p_i \dot{q}^i - H_0 - \lambda^\alpha T_\alpha\right] \tag{5.1}$$

where λ^α are multipliers which when varied enforce the constraints $T_\alpha = 0$. The constraints satisfy the Poisson bracket algebra

$$\{T_\alpha, T_\beta\} = U^\gamma_{\alpha\beta} T_\gamma \tag{5.2}$$

H_0 is the physical Hamiltonian of the theory, and satisfies

$$\{H_0, T_\alpha\} = V^\beta_\alpha T_\beta \tag{5.3}$$

$U^\gamma_{\alpha\beta}$ and V^β_α may depend on p_i and q^i. H_0 vanishes identically when the physical time is included among the dynamical variables q^i (parametrized theories) and the action as a consequence is reparametrization invariant.

For the case of general relativity the q^i represent the components of the three-metric, $q^i \sim h_{ij}(\mathbf{x})$, the λ^α represent the lapse and shift, $\lambda^\alpha \sim (N(\mathbf{x}), N^i(\mathbf{x}))$, and the T_α represent $(\mathcal{H}(\mathbf{x}), \mathcal{H}_i(\mathbf{x}))$. $H_0 = 0$ and the structure coefficients $U^\gamma_{\alpha\beta}$ depend on q^i, but not p_i. In any summation over indices, it is implicit that there is also an integration over spatial coordinates $\mathbf{x}$, but this will be omitted for notational simplicity

The Hamiltonian form of the action, (5.1), is invariant under canonical transformations generated by the constraints. To see this, consider the transformations

$$\delta p_i = \{p_i, \epsilon^\alpha T_\alpha\}, \quad \delta q^i = \{q^i, \epsilon^\alpha T_\alpha\} \tag{5.4}$$

where $\epsilon^\alpha(t)$ is a function of time. An elementary calculation shows that if

$$\delta\lambda^\alpha = \dot{\epsilon}^\alpha - U^\alpha_{\beta\gamma}\lambda^\beta\epsilon^\gamma - V^\alpha_\beta\epsilon^\beta \tag{5.5}$$

then

$$\delta S = \left[\epsilon^\alpha(t)\left(p_i \frac{\partial T_\alpha}{\partial p_i} - T_\alpha\right)\right]_{t'}^{t''} \tag{5.6}$$

The action is therefore invariant if $\epsilon^\alpha(t') = 0 = \epsilon^\alpha(t'')$, unless the constraints are linear in the momenta, in which case the terms in the square bracket vanish identically without restrictions on $\epsilon^\alpha(t)$.

Consider now the path integral representation of the wave functions of this theory. For convenience, introduce the notation $z^A = (p_i, q^i, \lambda^\alpha)$. Then the path integral representation of the wave functions has the form

$$\Psi(q^{i''}) = \int_{\mathcal{C}} \mathcal{D}z^A \; \delta[G^\alpha] \; \Delta_G \; e^{iS[z]} \; \delta(q^i(t'') - q^{i''}) \tag{5.7}$$

The sum is over a class $\mathcal{C}$ of paths, restricted by the explicit delta-function to end on the argument of the wave function. G^α are a set of gauge-fixing conditions, with associated weight factor Δ_G which may be constructed using the method of Batalin, Fradkin and Vilkovisky (BFV) [11]. For gauge theories it is the Faddeev-Popov determinant. $S[z]$ is the action. Here we have focussed on the Hamiltonian form of the path integral, but the form (5.7) is actually quite general, if z^A is suitably defined, *e.g.*, $z^A = (q^i, \lambda^\alpha)$ for the Lagrangian form of path integral, $z^A = (p_i, q^i, \lambda^\alpha, ghosts)$ for the BRST form [11].

We shall show that the constraints of the Dirac quantization procedure on $\Psi(q^i)$ follow very readily from this path-integral expression, assuming only certain invariance properties of the sum-over-histories. Let us spell out what we mean by an invariant path integral construction. Suppose the theory has an invariance under $z^A \to z^A + \delta z^A$, where δz^A depends linearly on some parameters $\epsilon^\alpha(t)$ and their derivatives, $\dot{\epsilon}^\alpha(t)$. We also need to assume that,

$$\delta q^i = \epsilon^\alpha f^i_\alpha(p_i, q^i) \tag{5.8}$$

(for Lagrangian theories, p_i denotes $\partial L/\partial \dot{q}^i$, where L is the Lagrangian). That is, the transformation of q^i does not depend on the Lagange multipliers or on $\dot{\epsilon}^\alpha$. This is clearly satisfied for constrained Hamiltonian systems, by virtue of (5.4). Then an invariant path integral construction would involve an action, measure and class of paths for which the following four properties hold under this transformation:

1) The action S changes at the most by a surface term of the form

$$\delta S = \left[\epsilon^\alpha F_\alpha(p_i, q^i)\right]_{t'}^{t''} \tag{5.9}$$

That is, it should not depend on the Lagrange multipliers, or on $\dot{\epsilon}^\alpha$.

2) The class of paths, $\mathcal{C}$, is invariant.

3) The path integral (5.7) is independent of the choice of gauge conditions G^α.

4) The combination of the measure and the gauge-fixing weight factor transform under a symmetry transformation according to

$$\mathcal{D}z^A \Delta_G[z^A] \to \mathcal{D}z^A \Delta_{G_\epsilon}[z^A] \quad . \tag{5.10}$$

where $G_\epsilon[z^A] = G[z^A + \delta_\epsilon z^A]$.

In addition to these four properties characterizing the invariance of the path integral (5.7) we will also need to assume the following:

5) Integrals of the form (5.7) weighted by functions of p_i and q^i on the final surface are equal to corresponding, appropriately ordered operators acting on $\Psi(q^{i''})$. That is, for given $\mathcal{F}(p_i, q^i)$,

$$\int_{\mathcal{C}} \mathcal{D}z^A \; \mathcal{F}\left(p_i(t''), q^i(t'')\right) \; \delta\left(q^i(t'') - q^{i''}\right) \; \Delta_G[z^A] \; \delta[G^\alpha(z^A)] \; \exp\left(iS[z^A]\right)$$

$$= \mathcal{F}\left(-i\frac{\partial}{\partial q^{i''}}, q^{i''}\right) \Psi(q^{i''}) \tag{5.11}$$

for some deducible operator acting on the right-hand side.

Criterion (1) is clearly satisfied for constrained Hamiltonian systems. Perhaps the most important condition is (2), which concerns the domains of integration of the integral. From (5.5), it implies that the Lagrange multipliers, and in particular the lapse function, must be integrated over an infinite range. Criteria (3) and (4) may be shown to be properties of the BFV path integral construction [9,11], and (5) may be shown to be true in a time-slicing implementation of the path integral. We will not devote any further space to discussing whether or not particular implementations of the sum-over-histories satisfy these criteria. This is discussed in detail in Ref.[9]. It suffices to state that they may be shown to be satisfied by standard implementations. The important point is that the above set of criteria is the minimal set that any implementation must satisfy if it is to satisfy the constraints.

Turn now to the derivation of the constraints from (5.7). Translate the integration variables in (5.7) by a symmetry transformation for which $\epsilon^\alpha(t)$ is nonvanishing only in a neighborhood of t''. The overall integral is unchanged because we are merely changing the variables of integration. The class of paths is unchanged because it is invariant. The action changes according to (5.9) with only the surface term at $t = t''$ contributing because $\epsilon^\alpha(t') = 0$. The change in measure and gauge-fixing machinery consists of no more than a change of gauge conditions. The integral with translated integrand therefore takes the form

$$\Psi(q^{i''}) = \int_{\mathcal{C}} \mathcal{D}z^A \; \delta\left(q^i(t'') + \delta q^i(t'') - q^{i''}\right) \; \Delta_{G_\epsilon}[z^A] \; \delta[G^\alpha_\epsilon(z^A)]$$

$$\times \exp\left(i(S[z^A] + \delta S[z^A])\right) \quad . \tag{5.12}$$

Using the assumed independence of gauge-fixing condition, G^α_ϵ may be replaced by G^α in (5.12). Then subtracting (5.7) from (5.12), using (5.8) and (5.9), and expanding to first order in ϵ^α, we have

$$0 = \int_{\mathcal{C}} \mathcal{D}z^A \; \epsilon^\alpha(t'') \; \delta\left(q^i(t'') - q^{i''}\right) \; \Delta_G[z^A] \; \delta[G^\alpha(z^A)] \; \exp\left(iS[z^A]\right)$$

$$\times \; \epsilon^\alpha(t'') \left[iF_\alpha(p_i(t''), q^i(t'')) - f^i_\alpha(p_i(t''), q^i(t'')) \frac{\partial}{\partial q^{i''}} \right] \tag{5.13}$$

And using (5.11), we may pull the term in square brackets outside the intergal, to deduce that

$$\left[F_\alpha(\hat{p}_i, \hat{q}^i) - f^i_\alpha(\hat{p}_i, \hat{q}^i)\hat{p}_i \right] \Psi(q^i) = 0 \tag{5.14}$$

where

$$\hat{p}_i = -i\frac{\partial}{\partial q^i}, \qquad \hat{q}^i = q^i \tag{5.15}$$

and we have dropped the double primes for convenience. This is the main result – a differential operator identity for $\Psi(q^i)$ satisfied as a consequence of a path integral satisfying invariance properties (1)-(5). Finally, computing (5.14) for the case of constrained Hamiltonian systems, by comparing (5.8) and (5.9) with (5.4) and (5.6), we readily derive the constraints of the Dirac quantization procedure:

$$T_\alpha(\hat{p}_i, \hat{q}^i)\Psi(q^i) = 0 \tag{5.16}$$

In particular, for the case of general relativity for closed cosmologies, these constraints are the Wheeler-DeWitt equation and the momentum constraints.

We have seen, therefore, that the path integral expression for the wave function for the universe will satisfy the Wheeler-DeWitt equation and the momentum constraints, provided that the integral is constructed in an invariant manner.

It is important to be precise about which invariance it is that leads to the constraints. It is sometimes claimed that the Wheeler-DeWitt equation is the quantum mechanical expression of diffeomorphism invariance, in the same way that the Gauss law constraint in canonically quantized gauge theories is the quantum expression of gauge invariance. This is not quite true. The symmetry used to derive the contraints was the symmetry of the Hamiltonian form of the action, (5.4), (5.5). On the lapse function N this symmetry transformation has the form, $\delta_\epsilon N = \dot{\epsilon} + \cdots$. Compare this to four-dimensional diffeomorphisms, $\delta_\xi x^\mu = -\xi^\mu$, which on the lapse function have the form $\delta_\xi N = \dot{\xi} N + \cdots$. The transformations coincide if one makes the identification of parameters, $\epsilon = N\xi$. To derive the constraints, it is important that the symmetry transformation is parametrized in such a way that (5.8) and (5.9) do not depend on the Lagrange

multpliers (otherwise we could not make the step from (5.13) to (5.14)). This is true of the parametrization in terms of ϵ, but is not true of the parametrization in terms of ξ. The symmetry associated with the Wheeler-DeWitt equation, therefore, is not four-dimensional diffeomorphism invariance, but is the closely related symmetry of the Hamiltonian form of the action, in which ϵ and not ξ is the freely specifiable parameter.

This subtle distinction has the following consequence. One might have thought that one could obtain a solution to the Wheeler-DeWitt equation by integrating the lapse function over a half-infinite range, $N > 0$. This is, after all, a diffeomorphism-invariant range. However, as we have just pointed out, the relevant symmetry is not diffeomorphisms, but is the symmetry associated with the Hamiltonian form of the action. $N > 0$ is not an invariant range under this symmetry, since $\delta_\epsilon N = \dot{\epsilon} + \cdots$ may map positive lapse to negative lapse. This symmetry is slightly larger (and therefore more restrictive) than the diffeomorphisms, and the only invariant range is $-\infty < N < +\infty$.

The above derivation can also be carried out for a path integral of Euclidean signature. One finds that the Wheeler-DeWitt equation is the same, irrespective of whether one takes a Euclidean or Lorentzian theory as the starting point [9].

For the special case of minisuperspace models, a derivation of the Wheeler-DeWitt equation paying particular attention to the measure and operator ordering may be carried out [12]. Other discussions of the connection between the path integral and the constraints may be found in Ref.[13]. The case of global symmetries of field theory (*e.g.* Lorentz or de Sitter invariance) and their associated constraints on the wave functions was discussed in Ref.[14].

6. Complex Contours

Now we turn to the second question, that of choosing suitable complex contours. For definiteness, we will concentrate on the path integral for the no-boundary wave function, although much of what follows is quite general. As stated earlier, the Euclidean Einstein action is not bounded from below, so the path integral will not converge if one integrates over real Euclidean metrics. To obtain convergence it is necessary to integrate along some complex contour in the space of complex four-metrics and matter fields. It is, therefore, rather misleading to refer to the expression (4.1) as a "Euclidean" path integral. One should from the very beginning think of it as a *complex* integral representation of the wave function.

How do we choose a suitable complex contour? Numerous proposals for the contour have been made over the years. Perhaps the best known is the conformal rotation of Gibbons, Hawking and Perry [15]. This involves writing the four-metric as $g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}$, where $\hat{g}_{\mu\nu}$ is a fiducial metric fixed by some suitable condition, such as $R(\hat{g}) = 0$ (for the asymptotically flat case) or $R(\hat{g}) = 4\Lambda$ (for the closed case). The integral over Ω is then rotated to lie parallel to the imaginary axis, and one can argue that the resulting integral will converge. The main problem with this proposal is that not all metrics can be conformally transformed to one satisfying the fiducial condition, so certain metrics are being missed out. And in general, there appears to be no entirely satisfactory proposal for defining a suitable complex contour, for the case of closed cosmologies considered here. For this reason, it seems sensible to embark on a general search for suitable complex contours with which to define the no-boundary wave function.

It is sometimes suggested that the indefiniteness of the Euclidean action for gravity means that the theory is sick, and that in integrating along a complex contour one is pretending to cure the incurable. This is not necessarily the case. The indefiniteness of the gravitational action is intimately related to the fact that general relativity is an example of a parametrized theory – a theory in which the physical time is included amongst the basic dynamical variables. By way of example, consider a much simpler parametrized system, namely the relativistic point particle. It's Lorentzian action may be written

$$S[X^\mu] = \int d\tau \left[-\frac{(\dot{X}^0)^2}{N} + \frac{(\dot{X}^i)^2}{N} - Nm^2 \right] \tag{6.1}$$

Here τ is the parameter time, the physical time is X^0, N is a Lagrange multiplier, and a dot denotes a derivative with respect to τ. The object commonly called the Euclidean action, I, is obtained by letting $\tau = -i\tilde{\tau}$ and $I = -iS$. One thus obtains,

$$I[X^\mu] = \int d\tilde{\tau} \left[-\frac{(\dot{X}^0)^2}{N} + \frac{(\dot{X}^i)^2}{N} + Nm^2 \right] \tag{6.2}$$

(where with a slight abuse of notation, a dot now denotes a derivative with respect to $\tilde{\tau}$). This action is indefinite, but it is clear that one needs only to let $X^0 \to iX^0$ to obtain a positive definite action. It is therefore necessary to perform *two* rotations to achieve a postive definite action, one of the unphysical parameter time τ, one of the physical time X^0.

In fact, generally for parametrized theories, one would expect that a bounded Euclidean action could be obtained by identifying the internal time

parameter and then rotating it. The Euclidean action for gravity is analagous to the action (6.2), and is also unbounded – in the direction of rapidly varying conformal factors. And what the conformal rotation is doing is regarding the conformal factor as the physical time, and rotating it. But the conformal factor is not really the true internal time of general relativity. Indeed, the problem of identifying this internal time – the so-called problem of time – remains one of the oustanding problems of canonical general relativity [7]. It is because we do not know exactly what to rotate (but we know we need to rotate something) that is seems reasonable to look for *general* complex contours. Of course, we have not proved anything here. We have merely stressed that the indefiniteness of the action is not a sickness, and that the need for complex contours is not artifical. These things are just aspects of parametrized theories generally.

It is also worth pointing out that the contour would still be an issue even if the action for gravity were bounded from below. The contour issue is not really about convergence, which is in fact quite easy to achieve by choosing suitable complex contours. Rather, the point is that there are *many* complex contours leading to solutions to the Wheeler-DeWitt equation, each of which may be regarded, mathematically at least, as candidate wave functions for the universe. The real issue is to identify those contours which lead to physically interesting wave functions.

The situation may again be compared to that of the relativistic point particle. It has many two-point functions one may be interested in, *e.g.* Wightman, Feynman, advanced, retarded. All such two-point functions may be given by the same integral representation,

$$G(X'', X') = \int d^4p \; \frac{e^{ip\cdot(X''-X')}}{p^2 + m^2} \tag{6.3}$$

The various types of two-point function are obtained by appropriate choice of the contour in the p_0 plane. This is fixed by the boundary conditions of the problem. The integral (6.3) is analogous to the integral (4.1) for quantum cosmology. But none of the boundary condition proposals that have been made so far in quantum cosmology seem to fix a particular contour. It therefore seems reasonable to ask what restrictions one should put on the contour in order to obtain physically interesting wave functions.

7. Five Criteria

We now consider five criteria which naturally suggest themselves as reasonable restrictions on a contour defining the wave function of the universe, and how they might be satisfied [16]. They are:

(1) The integral along the contour should converge.

As we have stated, this requires that the contour be complex. It is generally quite easy to find complex contours along which the integral converges. For example, in a variety of simple minisuperspace models, the path integral readily reduces to a single integration over the lapse, and one may integrate along the contours of steepest descent [17].

(2) The wave function generated should satisfy the Wheeler-DeWitt equation and the momentum constraints, (3.6).

This criterion imposes restrictions on the end-points of the contour of integration. It implies, in particular, that the contour for the lapse function N must be invariant, in the sense discussed above. Invariance may be achieved by taking a contour which is infinite in length, and may lie anywhere in the complex plane, consistent with convergence. Another possibility for the lapse is to have a closed contour. This is not strictly invariant under the Hamiltonian symmetry discussed above, in that it is shifted in the complex plane; but it is invariant in the sense that the shifted contour is equivalent to the original one by Cauchy distortion.

These first two criteria are mathematical, and any contour satisfying them defines an admissable wave function for the universe. But we need further criteria to single out wave functions making sensible physical predictions. We therefore impose the following three requirements.

(3) The wave function should predict classical spacetime on scales large compared to the Planck length.

Classical spacetime is a manifest property of the late universe, but a property of only very special quantum states. There are a number of conditions that need to be satisfied before one can say classical spacetime is a prediction. For the purposes of the present analysis, the most important one is that the histories are highly correlated according to classical laws. There are, of course, many wave functions which lead to classical correlations (*e.g.* Ref.[18]). However, classical correlations are most commonly signaled in quantum cosmology when the wave function is well-approximated by a certain type of semi-classical approximation. A typical form corresponding to classical spacetime and quantum matter is:

$$\Psi\left[h_{ij}, \phi\right] \approx \sum_p \Delta_p[h_{ij}] \exp\left(iS_p\left[h_{ij}\right]\right) \psi_p\left[\phi, \hat{g}_p\right] \tag{7.1}$$

Here, S_p is a classical action obeying the Lorentzian Hamiltonian-Jacobi equation for gravity coupled to the expectation value of matter fields. For this approximation to be valid, S_p must vary much more rapidly with h_{ij} than the prefactor Δ_p or ψ_p. The wave function (7.1) is then strongly peaked about the ensemble of classical spacetimes $\hat{g}_p$ defined by the integral curves of S_p [18]. The $\psi_p\left[\phi, \hat{g}_p\right]$ are the associated states of the matter field in these spacetime backgrounds. They should be normalizable in the variables $\phi(\mathbf{x})$ on a spacelike surface of g_p [19]. Mathematically, a semiclassical approximation to Ψ_0 of the form (7.1) arises when, in a steepest-descents approximation to the functional integral the dominating saddle-points are *complex*. The integral will frequently have many saddle-points, and for that reason we have included the sum over the discrete label p. The number of saddle-points supplying the *dominant* contribution to the integral, however, is typically very small so p runs over only a very small number of discrete values (one or two in simple examples) [20]. For each p, the prefactor Δ_p is of the form $e^{-I_R}\Delta_{WKB}$ where Δ_{WKB} is the usual WKB prefactor. Because the dominating saddle-points are generally complex, the prefactor also includes the factor e^{-I_R}, where I_R is the real part of the complex action.

The question of whether a no-boundary proposal predicts classical spacetime is the question of whether the contour passes through the appropriate complex saddle-points to make an approximation like (7.1) valid when the universe is large. No general statement is available concerning the explicit restrictions this criterion places on the contour, but it turns out to be straightforward to apply it in simple models.

(4) A closely related requirement is the reproduction of familiar quantum field theory for matter when spacetime is approximately classical.

This is connected with the matter fluctuations in the steepest-descents approximation about the complex saddle-points. More precisely, it is the question of whether the wave functions $\psi_p\left[\phi, \hat{g}_p\right]$ in (7.1) correctly describe quantum field theory in the spacetime $\hat{g}_p$. The way in which this could fail to be the case is as follows. Because we are allowing all possible complex metrics in the path integral, it is possible to end up with saddle-points for which $\sqrt{g} < 0$ (as well as "usual" ones with $\sqrt{g} > 0$). The action for matter fluctuations about such saddle-points will be negative definite. In the semi-classical approximation the matter wave functions are of the form, $\psi_p \approx e^{-I_m^{cl}}$, where I_m^{cl} is the matter action of the classical solution satisfying the appropriate boundary conditions. Because $I_m^{cl} < 0$ at saddle-points with $\sqrt{g} < 0$, the matter wave functions will not be normalizable, and not belong to the Hilbert space of matter states. This implies that the contour should not be dominated by saddle-points with $\sqrt{g} < 0$.

(5) To the extent that wormholes makes the cosmological constant dependent on initial conditions, the wave function should predict its vanishing.

Recent work by Hawking [21], Coleman [22], Giddings and Strominger [23] and others has shown that if non-trivial "wormhole" topologies are included in the sum-over-histories (4.1) then the coupling constants of the effective low energy theory accessible to us may depend as much on the initial conditions of the universe as they do on the form of the fundamental Lagrangian. In particular wormholes may provide a mechanism for making the cosmological constant vanish. A reasonable restriction on a no boundary proposal is that it predict this. We measure the cosmological constant only through the evolution of the universe itself, that is, through the dynamics of the classical spacetime of the late universe. A wave function of the universe predicts a distribution of cosmological constants when it predicts a family of classical spacetimes, each obeying Einstein's equation, but with *different* possible cosmological constant terms, Λ. That is, a semiclassical approximation of the form

$$\Psi[h_{ij}, \phi] = \int d\Lambda \sum_p \Delta_p[\Lambda, h_{ij}] \exp\left(iS_p[\Lambda, h_{ij}]\right) \psi_p\left[\phi, \hat{g}_{\Lambda p}\right] \tag{7.2}$$

Here, S_p is the classical action for gravity with cosmological constant Λ coupled to expectation values of matter fields in the state ψ_p. Roughly speaking, in the effective theory, the integral over metrics in Eq.(4.1) contributes the few terms in (7.2) labeled by different values of p, while the sum over manifolds becomes the sum over Λ. The distribution of the cosmological constant implied by $\Delta_p[\Lambda, h_{ij}]$ is thus closely connected with the semiclassical approximation and therefore with the contour of integration of a no-boundary proposal. In particular, the prediction of vanishing cosmological constant depends on the contour being dominated by saddle-points for which I_R, the real part of the complex action, is negative, and of the form $I_R \sim -\frac{1}{\Lambda}$. For then one gets weight factors of the form $\exp\left(\frac{1}{\Lambda}\right)$ in Δ_p. However, this will not be the case if the contour is dominated by the saddle-points for which $\sqrt{g} < 0$, for then $I_R \sim +\frac{1}{\Lambda}$. Again, therefore, we deduce that the contour must not be dominated by a saddle-point for which $\sqrt{g} < 0$.

These then, are the conditions it seems reasonable to impose on the contour, and how they might be satisfied. Many interesting results have come out of a study of these criteria, but it is perhaps useful to focus some of these results into concise statements:

a) Contours yielding physically interesting wave functions are readily found in a variety of simple (minisuperspace) models [17].

b) The no-boundary proposal, as it currently stands, does not define a *unique* wave function of the universe. The proposal to sum over a given class of manifolds does not fix the contour. Even after applying the above criteria there is residual ambiguity, and so there are a number of wave functions laying rightful claim to be called the no-boundary wave function.

c) Any boundary condition proposal in quantum cosmology, in path integral form, must specify a choice of contour.

8. Lorentzian Contours

The indefiniteness of the Euclidean action has prompted a number of authors to suggest that one should never go to the Euclidean version of the theory in the first place, but always remain in the Lorentzian sector. That is, one should find a way of defining the Lorentzian path integral over real Lorentzian fields [24]. In connection with such proposals, we offer the following remarks.

(i) From the perspective of the present work, Lorentzian contours are possible choices amongst the set of all possible complex contours, and one must ask whether or not they satisfy the five criteria set forth above. Convergence is not a difficulty because a real Lorentzian contour can always be pushed off the real Lorentzian axis into the space of complex metrics in such a way that exponentially rapid convergence is obtained. Similarly, satisfying the constraints is straightforward. Criteria (3)-(5) however, concerning the nature of the dominating saddle-points, are not obviously satisfied, and it would be necessary to check these in particular models.

(ii) The description "Lorentzian" does not fix the contour uniquely. The action, and hence the integand of the path integral has a singularity at $\sqrt{g} = 0$, which lies on the real Lorentzian axis. Contours passing either side of the singularity could each be regarded as Lorentzian yet will be inequivalent. The above criteria, however, may resolve this ambiguity.

(iii) Lorentzian contours are sometimes proposed on the grounds that the classical spacetime we live in is a real Lorentzian geometry. However, this fact is not directly related to the nature of the contour, but rather, depends on the nature of the dominating saddle-points. As discussed above, classical spacetime is predicted when (amongst other conditions), the chosen contour may be distorted into a steepest descent contour dominated by a complex saddle-point for which the imaginary part of the action varies much more rapidly than the real part. Clearly there could be many complex, non-Lorentzian contours for which this is true. On the other hand, fixing the contour to be Lorentzian does not guarantee that it will be dominated by the correct saddle-points. One could say that a Lorentzian contour is

neither a *necessary* nor *sufficient* condition for the prediction of classical spacetime.

Finally, we remark that there exist alternative suggestions for dealing with the indefiniteness of the Euclidean action, which do not involve a general study of complex contours. There is, for example, the observation of Deser and Nicolai, in the context of supergravity, that it is formally possible to transform the action to quadratic form, and so explicitly identify what it is that one needs to rotate to obtain a positive action [25]. Another intriguing suggestion is that of Greensite and Halpern, which involves replacing the factor of e^{-I} with the solution to a certain Fokker-Planck equation [26].

9. Summary

We have reviewed recent work on the definition of the path integral in quantum cosmology, and its connections with the Wheeler-DeWitt equation. This work is part of a continuing effort to make path-integral quantum cosmology more precise. There is much more that can be done in this program.

The derivation of the constaints considered here was formal, in that little or no attention was paid to the details of the measure, regularization, operator ordering *etc.* For quantum mechanical-type systems (*e.g.* minisuperspace models) these issues are readily dealt with [12]. For field theories, however, they are likely to be highly non-trivial. It would be interesting, for example, to understand how the derivation of the constraints is affected by the presence of anomalies. It would also be interesting and useful to construct not just single-surface amplitudes like wave functions, but more complicated amplitudes such as the decoherence functional[2]. Finally, as stated at the beginning of this talk, the difficulties of quantum gravity cast a dark shadow over much of quantum cosmology. It would therefore be particularly interesting to somehow incorporate string theory into quantum cosmology in a useful way.

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