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The Gell-Mann–Oakes–Renner relation within the nuclear medium

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Abstract

We study the pion decay constant, associated with the time component of the axial current, in the nuclear medium in terms of the pion self-energy. The theoretical scheme exploits chiral properties of the s-wave πN amplitude as previously applied to the study of the medium renormalized pion mass. We show that the Gell-Mann–Oakes–Renner relation holds to a good approximation in the medium. The renormalizations of the pion decay constant and the pion mass are consistent with that predicted for the quark condensate in the medium.

1. INTRODUCTION . The problem of the modification of the weak coupling constants in the nuclear environment is especially interesting for the isovector axial current since it is linked to pion physics via its partial conservation (PCAC). In this work we will focus on the pion weak decay constant (denoted by f_π) which is of particular interest since it is related to the quark condensate via the well known Gell-Mann–Oakes–Renner (GOR) relation [1]. In free space the GOR relation reads:

$$f_\pi^2 m_\pi^2 = -2m_q < \text{vac} \mid \bar{q}q \mid \text{vac} > , \tag{1}$$

where the average of the up (u) and down (d) current quark masses has been denoted by $m_q = \frac{1}{2}(m_u + m_d)$, whereas $< \text{vac} \mid \bar{q}q \mid \text{vac} >$ stands for $\frac{1}{2} < \text{vac} \mid \bar{u}u + \bar{d}d \mid \text{vac} >$ and $f_\pi \approx 94 MeV$. We are concerned here with the renormalization of the decay constant associated with the time component of the axial current, the one which enters in the GOR relation. It is natural to expect that the GOR relation can be extended to the case of non–zero density as follows:

$$f_\pi^{*\,2} m_\pi^{*\,2} = -2m_q < 0 \mid \bar{q}q(\rho) \mid 0 > . \tag{2}$$

Here, the in-medium renormalized values have been labelled by the asterisk, $\mid 0 >$ denotes the ground state of dense matter, and $< 0 \mid \bar{q}q(\rho) \mid 0 >$ is the corresponding quark condensate. If this relation holds, then the study of the decay constant provides an information on the quark condensate in the dense medium. The evolution of this condensate has been shown to be governed by the Σ commutator and therefore linked to the soft πN amplitude [2], [3], [4]. On the other hand, the pion mass in the medium has recently been studied in terms of the chiral properties of the πN amplitude by Delorme et al. [5]. The main result of their investigation is that the pion mass remains almost unaltered as compared to the free space one.

In order to reach a consistent description for all the quantities entering the GOR relation, we will treat both the pion mass and the pion weak decay constant on equal footing, exploiting the approach of ref. [5].

2. THE PION PROPAGATOR AND THE PION DECAY CONSTANT IN THE MEDIUM. In order to keep to the essentials, we consider a piece of isospin symmetric homogeneous nuclear matter, ignoring all correlations . Con-

sider the pion propagator for a pion at rest ($\vec{q} = 0$) of energy q_0. It is defined according to

$$D_\pi(q_0, 0) \equiv D_\pi(q_0, \vec{q} = 0) = \int d^3x dt e^{iq_0 t} < 0 \mid T\, \Phi^i(\vec{x}, t), \Phi^i(0) \mid 0 >, \quad (3)$$

where T is the usual time ordered product and $i = 1, 2, 3$ is the isospin index of the pion field. In symmetric nuclear matter the pion propagator is obviously independent of the index i. In using closure over the states $\mid n >$ with energy E_n we obtain the spectral representation as

$$D_\pi(q_0) = \Sigma_n \frac{2E_n V \mid < n(\vec{p}_n = 0) \mid \Phi_i(0) \mid 0 > \mid^2}{q_0^2 - E_n^2 + i\epsilon}. \quad (4)$$

Here V denotes the normalization volume . The condition $\vec{p}_n = 0$ follows from space integration and will be implicitely understood in the following. To obtain the modification of this propagator in the medium we consider the s–wave pion self energy $\Pi^0(q_0) \equiv \Pi^0(q_0, \vec{q} = 0)$. For on shell pions, $\Pi^0(m_\pi)$ is simply related to the standard optical potential via

$$\Pi^0(m_\pi) = 2m_\pi V_{\rm opt}\,. \quad (5)$$

Now, the pion propagator in the medium $\widetilde{D}_\pi$ becomes

$$\widetilde{D}_\pi(q_0) = [q_0^2 - m_\pi^2 - \Pi^0(q_0)]^{-1}\,. \quad (6)$$

In dense matter we define a quasi-pion, that we denote by $\tilde{\pi}$, which is on shell in the medium. The energy of such a pion is nothing else than the effective pion mass m_π^*, which satisfies the following equation:

$$m_\pi^{*\,2} - m_\pi^2 - \Pi^0(m_\pi^*) = 0\,. \quad (7)$$

To find the effective mass we expand $\Pi^0(q_0^2)$ [1] around $q_0^2 = 0$, as in ref. [5], which leads to

$$\Pi^0(\,m_\pi^{*\,2}\,) = \Pi^0(0) + m_\pi^{*\,2}(\frac{\partial \Pi^0}{\partial q_0^2})_{q_0^2=0} + \dots\,. \quad (8)$$

In this way we obtain the effective pion mass as

$$m_\pi^{*\,2} = \frac{m_\pi^2[1 + \frac{\Pi^0(0)}{m_\pi^2}]}{1 - (\frac{\partial \Pi^0}{\partial q_0^2})_{q_0^2=0}}\,. \quad (9)$$

[1] In symmetric nuclear matter Π^0 is an even function of q_0.

In the same manner we perform an expansion of $\Pi^0(q_0^2)$ near $m_\pi^{*\,2}$ in the pion propagator to obtain

$$\begin{aligned}\widetilde{D}_\pi(q_0) &= \left(q_0^2 - m_\pi^2 - \Pi^0(m_\pi^{*\,2}) - (q_0^2 - m_\pi^{*\,2})(\frac{\partial \Pi}{\partial q_0^2})_{q_0^2=m_\pi^{*\,2}} + \ldots + i\epsilon\right)^{-1} \\ &= \left((1 - (\frac{\partial \Pi}{\partial q_0^2})_{q_0^2=m_\pi^{*\,2}})(q_0^2 - m_\pi^{*\,2} + \ldots + i\epsilon)\right)^{-1} . \end{aligned} \tag{10}$$

In this equation terms of order $(q_0^2 - m_\pi^{*\,2})^2$ have been ignored. Near the quasi-pion pole the propagator thus has the form

$$\widetilde{D}_\pi(q_0) \approx \frac{Z_\pi}{q_0^2 - m_\pi^{*\,2} + i\epsilon}, \quad \text{with} \quad Z_\pi = \frac{1}{1 - (\frac{\partial \Pi^0}{\partial q_0^2})_{q_0^2=m_\pi^{*\,2}}} . \tag{11}$$

On the other hand we extract the quasi-pion pole from the expression given in equ 4 for the pion propagator by isolating among the states $\mid n >$ the quasi-pion one, i. e. $\mid n >=\mid \tilde{\pi} >$ with $E_n = m_\pi^*$. As a result we find in the vicinity of $q_0^2 = m_\pi^{*2}$ the alternative expression for the in-medium pion propagator

$$\widetilde{D}_\pi(q_0) \approx \frac{2m_\pi^* V \mid < \tilde{\pi} \mid \Phi_i(0) \mid 0 > \mid^2}{q_0^2 - m_\pi^{*\,2} + i\epsilon} . \tag{12}$$

In identifying now eqs. (10) and (12) the matrix element of the pion field between the ground state of matter and the quasi-pion $\tilde{\pi}$ is obtained as

$$\mid < \tilde{\pi} \mid \Phi_i(0) \mid 0 > \mid^2 = \frac{Z_\pi}{2m_\pi^* V} . \tag{13}$$

This expression will be exploited later. We first define the pion weak decay constant within the medium which is naturally introduced via the axial current correlator for the time components

$$\Pi^A(\vec{q} = 0, q_0) = i \int \mathrm{d}^3x \mathrm{d}t e^{iq_0 t} < 0 \mid T A_0^i(x) A_0^i(0) \mid 0 > , \tag{14}$$

with the corresponding spectral representation being

$$\Pi^A(q_0) = \Sigma_n 2E_n V \frac{< 0 \mid A_0^i(0) \mid n(\vec{p}_n = 0) >< n(\vec{p}_n = 0) \mid A_0^i(0) \mid 0 >}{q_0^2 - E_n^2 + i\epsilon} . \tag{15}$$

The renormalized pion decay constant f_π^* is defined from the residue of the quasi pion pole at $q_0^2 = m_\pi^{*\,2}$ in the last equation to be

$$2m_\pi^* V \mid < \tilde{\pi} \mid A_0^i(0) \mid 0 > \mid^2 = f_\pi^{*\,2} m^{*\,2}_\pi . \tag{16}$$

From this definition the medium renormalization of the divergence of the axial current matrix element immediately follows as

$$\frac{|<\tilde{\pi} \mid \partial^0 A_0^i \mid 0>|^2}{|<\pi \mid \partial^0 A_0^i \mid \text{vac}>|^2} = \frac{m_\pi^{*\,3}}{m_\pi^3}\frac{f_\pi^{*\,2}}{f_\pi^2} \tag{17}$$

Next, in exploiting the PCAC hypothesis in the standard form $\partial^\mu A_\mu^i(x) = f_\pi m_\pi^2 \Phi^i(x)$, we assume the medium renormalization of the divergence of the axial current to be completely exhausted by the corresponding renormalization of the pion field, which amounts in the following equality:

$$\frac{|<\tilde{\pi} \mid \partial^0 A_0^i \mid 0>|^2}{|<\pi \mid \partial^0 A_0^i \mid \text{vac}>|^2} = \frac{|<\tilde{\pi} \mid \Phi^i \mid 0>|^2}{|<\pi \mid \Phi^i \mid \text{vac}>|^2} \tag{18}$$

Inserting the eqs. (13) and (17) in eq. (18) together with $<\pi \mid \Phi^i \mid \text{vac}>= 1/\sqrt{2m_\pi V}$ the following relation between f_π^* and f_π is obtained:

$$\frac{m_\pi^{*\,4} f_\pi^{*\,2}}{m_\pi^4 f_\pi^2} = Z_\pi = \frac{1}{1-(\frac{\partial \Pi^0}{\partial q_0^2})_{q_0^2=m_\pi^{*\,2}}}. \tag{19}$$

Replacing in the last equation the effective mass by its expression (9) we deduce the in-medium pion weak decay constant

$$\frac{f_\pi^{*\,2}}{f_\pi^2} = \frac{(1-(\frac{\partial \Pi^0}{\partial q_0^2})_{q_0^2=0})^2}{(1+\frac{\Pi^0(0)}{m_\pi^2})^2} Z_\pi \tag{20}$$

We further assume the derivatives of the pion self energy over q_0^2 at the points $q_0^2 = 0$ and $q_0^2 = m_\pi^{*\,2}$ to be the same and obtain the final result for the in-medium decay constant

$$\frac{f_\pi^{*\,2}}{f_\pi^2} = \frac{1-\frac{\partial \Pi^0}{\partial q_0^2}}{(1+\frac{\Pi^0(0)}{m_\pi^2})^2} \tag{21}$$

Up to this point we have expressed the renormalization effects on the pion mass and weak decay constant in terms of the pion self energy. We now express them in terms of the πN amplitude.

In a homogeneous medium the pion self energy is related to the (isospin symmetric) πN amplitude D_N^+ by $\Pi^0 = D_N^+ \rho$. The Born term of D_N^+ vanishes in the static limit and we will neglect it in a first step. The non–Born part is smoothly varying and can be expanded near the soft point as a power serie in ν^2,

t, q^2 and q'^2. At the soft point $D_N^+(0,0,0,0) = -\Sigma_N/f_\pi^2$ where $\Sigma_n \approx 45$ MeV is the πN sigma commutator defined in the standard way as

$$\delta_{ij}\Sigma_N =< N \mid [Q_i^5,[Q_j^5,H]] \mid N > . \tag{22}$$

Here Q_i^5 stands for the total axial charge. The expansion of the amplitude limited to the s-wave term writes:

$$D_N^+(0,0,0,0) = -\frac{\Sigma_N}{f_\pi^2}(1-\frac{q^2+q'^2}{m_\pi^2}) + \beta\nu^2 + \dots . \tag{23}$$

In expanding the optical potential around the soft pion point we put $\vec{q} = \vec{q}' \equiv 0$ and $q_0 = q_0' = \nu$ in eq. (24) after which we are led to the following expansion:

$$\Pi^0(q_0^2) = \frac{\Sigma_N\rho}{f_\pi^2} + q_0^2\,\rho\,(-\frac{2\Sigma_N}{f_\pi^2 m_\pi^2} - \beta) + \dots . \tag{24}$$

Thus, $\Pi^0(0)$ is related to the sigma term by $\Pi^0(0) = \Sigma_N\rho/f_\pi^2$ On the other hand, the value of the πN scattering length is experimentally found to be very small, which implies that

$$D_N^+(\nu^2 = m_\pi^2, t = 0, q^2 = q'^2 = m_\pi^2) = \frac{\Sigma_N}{f_\pi^2} + \beta m_\pi^2 \approx 0 . \tag{25}$$

The derivative of the self energy is then

$$\left(\frac{\partial\Pi^0}{\partial q_0^2}\right)_{q_0^2=0} = \rho\left(-\frac{2\Sigma_N}{f_\pi^2 m_\pi^2} - \beta\right) \approx -\frac{\rho\Sigma_N}{f_\pi^2 m_\pi^2} , \tag{26}$$

which results in an almost independence of the pion mass on density, as appears from the expression [2]

$$m_\pi^{*\,2} = m_\pi^2 \frac{[1+\frac{\Sigma_N\rho}{f_\pi^2 m_\pi^2}]}{1-(\frac{\partial\Pi^0}{\partial q_0^2})_{q_0^2=0}} \approx m_\pi^2 . \tag{27}$$

The pion decay constant instead is affected according to

$$\frac{f_\pi^{*\,2}}{f_\pi^2} \approx \frac{1}{1+\frac{\Sigma_N\rho}{f_\pi^2 m_\pi^2}} . \tag{28}$$

[2]Our formal expression for $m_\pi^{*\,2}$ differs from that of Delorme et al. [5] although the conclusion about the stability of the mass is the same. The reason for this difference is that, in order to make the link with the chiral lagrangian approach Delorme et al. [5] considered a correlated medium, as explained in their work.

This represents for f_π a quenching of $\approx 15\%$ at normal density.

If we include the nuclear correlations and release the static approximation, the self energy is no longer proportional to the πN amplitude and cannot be represented by the simple expression (25). According to π mesic data the pion self energy at $q_0^2 \approx m_\pi^2$ is determined as $\Pi^0(m_\pi^2) = 4\pi b_{\rm eff}\,\rho$ with $b_{\rm eff} \approx -.024\, m_\pi^{-1}$. Taking for the derivative

$$\frac{\partial \Pi^0}{\partial q_0^2} = \frac{\Pi^0(m_\pi^2) - \Pi^0(0)}{m_\pi^2} = \frac{4\pi b_{\rm eff}\rho}{m_\pi^2} - \frac{\Sigma_N \rho}{f_\pi^2 m_\pi^2} \approx -0.18\frac{\rho}{\rho_0}\,, \tag{29}$$

we find that the pion mass squared is now increased by $\approx 13\%$ at normal density while the quenching of f_π ($\approx$19% instead of 15%) is somewhat more pronounced.

The renormalization of f_π has previously been calculated by Akhmedov et al. [6] , [7]. These authors considered the in medium PCAC relation and combined a mean field approach with Migdal's finite Fermi system theory with the result

$$\frac{f_\pi^{*\,2}}{f_\pi^2} = (1 - \frac{M\rho}{f_\pi^2 m_\sigma^2})^2 (1 - \frac{\partial \Pi^0}{\partial q_0^2})^2\,. \tag{30}$$

Here m_σ denotes the mass of the effective σ mean field. This expression (equ 30), although superficially different from our equ 21 is actually similar to our result. In order to compare the two expressions, we first note that in the linear σ model the sigma term Σ_N is $\Sigma_N = M m_\pi^2/m_\sigma^2$. Second we expand our denominator in equ 21: $(1 + \Sigma_N \rho/f_\pi^2 m_\pi^2)^{-2} \approx (1 - \Sigma_N \rho/f_\pi^2 m_\pi^2)^2$ (dilute gas approximation). Hence, the only difference between our result (equ 21) and Akhmedov one's (equ 30) lies in the different powers of the term $1 - \frac{\partial \Pi^0}{\partial q_0^2}$. This is simply a matter of definition of f_π^*. Our choice of definition is consistent with the in-medium GOR as written in equ 2. We stress on the other hand that our approach is based on a unique and consistent framework which relies entirely on the properties of the pion propagator in the medium.

3. THE GELL-MANN-OAKES-RENNER RELATION AT FINITE DENSITIES AND THE Σ TERM IN THE MEDIUM.

To derive the generalization of the GOR relation at finite density we start from the underlying QCD theory and write the axial current in terms of the fundamental quark fields q as

$$A_\mu^i(x) = \bar{q}(x)\gamma_\mu\gamma_5\frac{\tau^i}{2}q(x) \quad \text{with} \quad q(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}. \tag{31}$$

Neglecting possible strange content of the pion we define a pion-like operator in terms of the quark fields as

$$\Phi_5^i(x) = i\bar{q}(x)\gamma_5\frac{\tau_i}{2}q(x)\,. \tag{32}$$

From canonical commutation relations one obtains the equal time commutator of the axial charge with the pion operator as

$$[Q_5^i(x_0), \Phi_5^j(x)] = -\frac{i}{2}\delta^{ij}\left(\bar{u}(x)u(x) + \bar{d}(x)d(x)\right). \tag{33}$$

On the other hand the divergence of the axial current is linked to the operator Φ by

$$\partial^\mu A_\mu^j = 2\,m_q\Phi_5^j(x)\,. \tag{34}$$

To obtain the GOR relation we sandwich the commutator (33) taken at $x = 0$ between the ground state of nuclear matter and saturate it with a complete set of states $\mid n >$. Making use of the link (34) between the operator Φ and the divergence of the axial current we get

$$\begin{aligned}\int \mathrm{d}^3y \qquad & (< 0 \mid A_0^i(\vec{y},0)) \mid n >< n \mid (\frac{1}{2m_q})\partial^\mu A_\mu^i(0) \mid 0 > \\ - \qquad & < 0 \mid (\frac{1}{2m_q})\partial^\mu A_\mu^i(0) \mid n >< n \mid A_0^i(\vec{y},0) \mid 0 >) \\ = \quad & -\frac{i}{2} < 0 \mid \bar{u}u(\rho) + \bar{d}d(\rho) \mid 0 > \,.\end{aligned} \tag{35}$$

This leads to the sum rule

$$\begin{aligned}\Sigma_n\, 2\,E_nV \mid< n \mid A_0^i(0) \mid 0 >\mid^2 \quad &= \quad -m_q(< 0 \mid \bar{u}u(\rho) + \bar{d}d(\rho) \mid 0 > \\ &\equiv \quad -2m_q < 0 \mid \bar{q}q(\rho) \mid 0 > \,.\end{aligned} \tag{36}$$

with the implicit condition $\vec{p}_n = 0$. A priori all the unnatural parity pion–like states contribute to the saturation. Among them there are low lying particle hole states. However the contribution of the ph-sector is expected to be small (a few percent) since the axial charge of the nucleon, $\sim \vec{\sigma}\cdot\vec{p_N}/M$, vanishes in the static limit. Hence, we can only retain only the contribution of the quasi-pion pole. With this assumption and using the definition of f_π^* from eq. (16) we immediately get

$$f_\pi^{*\,2}m_\pi^{*\,2} = -2m_q < 0 \mid \bar{q}q(\rho) \mid 0 >\,, \tag{37}$$

the in-medium GOR relation. Its validity relies on the neglect of all intermediate nuclear states but the quasi-pion, which should be a good approximation. Inserting in the last equation the expressions for m_π^* and f_π^* given in eqs. (2.9) and (21) we find that in the medium the quark condensate is renormalized according to:

$$\begin{aligned} \frac{< 0 \mid \bar{q}q(\rho) \mid 0 >}{< \text{vac} \mid \bar{q}q \mid \text{vac} >} &= \frac{f_\pi^{*\,2} m_\pi^{*\,2}}{f_\pi^2 m_\pi^2} \\ &= \frac{1}{1 + \frac{\Pi(0)}{m_\pi^2}} \end{aligned} \tag{38}$$

The modification of the condensate is found to depend only on the soft pion self energy, the derivative disappearing in the product $f_\pi^{*\,2} m_\pi^{*\,2}$, as it should. Indeed the evolution with density of the condensate is governed by the nuclear Σ commutator, i.e. by the soft pion–nucleus scattering amplitude which depends only on the soft pion potential.

For a homogeneous medium where $\Pi(0) = \Sigma_N \rho / f_\pi^2$ equ 38 becomes

$$\frac{< 0 \mid \bar{q}q(\rho) \mid 0 >}{< \text{vac} \mid \bar{q}q \mid \text{vac} >} = \frac{1}{1 + \frac{\rho \Sigma_N}{f_\pi^2 m_\pi^2}} \tag{39}$$

At low densities the last expression (39) can be approximated by

$$\frac{< 0 \mid \bar{q}q(\rho) \mid 0 >}{< \text{vac} \mid \bar{q}q \mid \text{vac} >} \approx (1 - \frac{\rho \Sigma_N}{f_\pi^2 m_\pi^2}), \tag{40}$$

a result already obtained by several authors [2], [3] and [4]. The full expression (39) has been derived by M. Ericson [8] in a multiple scattering scheme for soft pions. In this approach the replacement of the linear density dependence of equ (40) into the hyperbolic one of equ (39) is an effect of the distortion of the soft pion wave. In the present work the hyperbolic expression naturally apppears as a consequence of the GOR relation.

In summary, we have studied the in-medium renormalizations of the pion decay constant and the pion mass in a consistent approach based on the chiral properties of the πN amplitude. We have shown that the decay constant is quenched by about 15–20% at normal density, while the pion mass remains stable (as in ref. [5]). Using the GOR relation, which we expect to be valid to a good approximation in nuclei, we have deduced the quark condensate at finite density. At low density our result coincides with that previously obtained by other authors

as quoted above. At higher densities on the other hand and for a homogeneous medium we find by a different technique the expression which recently has been given by one of us[8], which implies a reaction of the nuclear medium against chiral symmetry restoration.

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