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TIME EVOLUTION IN QUANTUM MECHANICS ON THE QUANTUM LINE

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Abstract

We use the infinite product representation of the q-exponential function to derive some useful identities involving the product of q-exponentials of operators. We then use several of these results to formulate and compare some of the possible q-deformations of the time evolution of an operator in quantum mechanics and discuss some of their implications.

In the past few years quantum groups and quantum algebras [1] have had an increasing and broader range of applications in mathematics and physics. A deeper mathematical understanding of these algebraic constructions was achieved once they were identified as particular cases of Hopf algebras [2], while its relevance to physics can be traced back to the theory of integrable models and the role of the Yang-Baxter equation in two-dimensional statistical mechanics [3].

More recently, much effort has been directed in developing the representation theory of quantum groups and quantum Lie algebras [4], which at the same time has allowed the study of new physical situations which can be understood as q-deformed (generalized) versions of standard physical systems [5]. In particular, the formulation of a consistent q-deformation of quantum mechanics is essential to the development of a successful approach to a quantum field theory with quantum group symmetries. The formulation of noncommutative geometry [6], its relation to quantum groups [7] and the development of noncommutative differential calculus on the n-dimensional quantum space $\mathbf{R}_q^n$ in its bosonic [8] and fermionic formulations [9], in terms of the R-matrix of $Gl_q(n)$ and $SO_q(n)$ respectively, provides the basis for understanding the corresponding geometry and analysis. In particular, it has been shown [10] that differential operators acting on functions on $\mathbf{R}_q^n$ correspond to q-differential operators times a scaling operator acting on functions on $\mathbf{R}^n$, therefore bringing the theory of q-analysis to play the corresponding role in commutative geometry. The theory of differential q-analysis was originally formulated at the beginning of this century and posteriorly extended to include q-integration [11] such that the standard calculus can be recovered once one takes the limit q=1. Therefore, the parameter q can be seen as a deformation parameter of operators and functions. The study of the so called q-series dates back to the times of Euler, and it has played an important role in applications to the theory of partitions [12] and the theory of vector spaces over a finite field [13].

In particular, since the exponential function plays a prominent role in the q=1 (undeformed) case (e.g.partition functions,path integrals, coherent states), then the so called q-exponential function $exp_q(x)$ is probably one of the most interesting functions for physical applications in which involves a quantum group structure and therefore have a q-analog correspondence in commutative geometry. Some of its basic properties for q-commuting coordinates were discussed in [14] and more recently in [15].

In this letter our main interest resides in the formulation of the q-deformation of the time evolution equation in one-dimensional quantum mechanics. Since the natural generalization of this equation consists in replacing the exponential function by the q-exponential function we first discuss and generalize some of the relevant identities that the later satisfies. As we will shown, the $exp_q(x)$ richer algebraic structure, as compared with the standard exponential, does not allow a unique q-deformation. The letter is organized as follows. We first use the infinite product representation of the q-exponential [16] to derive some useful identities and then generalize to the case of the product of q-exponentials of arbitrary operators and discuss some interesting particular cases. Based on this we then give and compare in this context some of the possible deformations of the evolution equation of an operator in the Heisenberg picture and conclude with some remarks.

The q-exponential function $exp_q(\lambda x)$ is defined according to the rule

$$D_x\, exp_q(\lambda x) = \lambda\, exp_q(\lambda x) \quad , \tag{1}$$

where $|q| < 1$ and $D_x \equiv x^{-1}\,[x\partial_x]$ is the q-derivative and $[\alpha] \equiv \dfrac{1-q^{\alpha}}{1-q}$ with the requirement that

$$limit_{q \to 1}\left[exp_q(\lambda x)\right] = exp(\lambda x) \quad . \tag{2}$$

Eqs. (1) and (2) provide a representation in terms of the Eulerian series

$$exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \quad , \tag{3}$$

where the q-factorial is defined as

$$[n]! \equiv \prod_{m=0}^{n-1} [n-m] \quad . \tag{4}$$

An infinite product representation of eq. (3) can be written once we recall an important theorem due to Cauchy [12] which states that, for $|q| < 1$ and $|x| < 1$, the next identity follows

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \prod_{n=0}^{\infty} \frac{(1 - axq^n)}{(1 - xq^n)} \quad , \tag{5}$$

where we have used the standard compact notation $(a)_n = \prod_{m=0}^{n-1} (1 - q^m a)$. In particular, the $a = 0$ case gives that

$$exp_q\left(\frac{x}{1-q}\right) = \prod_{m=0}^{\infty} \frac{1}{1 - xq^n} \quad , \tag{6}$$

which clearly states that for two commuting variables $exp_q(x)\, exp_q(y) \neq exp_q(x+y)$ and therefore the function $exp_q(x+y)$ if it is represented by the Eulerian series in eq. (3) will not satisfy eq. (1), i.e. for $q \neq 1$ the function $exp_q(x)$ is not an additive character on the field **R**. From the product formula we can readily obtain some useful identities. In particular,

$$exp_q(x)\; exp_q(-x) = exp_{q^2}\left((1-q)x^2\right) \quad , \tag{7}$$

$$exp_q(x) = \prod_{n=0}^{m-1} \; exp_{q^m}(\, q^n x) \quad . \tag{8}$$

The inverse of $exp_q(x)$ can be found from the product formula

$$(exp_q(-x))^{-1} = \prod_{n=0}^{\infty} (1 + (1-q)\, xq^n) \quad , \tag{9}$$

such that with use of eq. (5) it can be shown to satisfy the useful identity

$$exp_q(x)\; exp_{q^{-1}}(-x) = 1 \quad . \tag{10}$$

The last equation indicates that more general identities can be found by investigating the product

$$exp_q(A)\, B\, exp_{q^{-1}}(-A) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{[n]!} \frac{1}{[m]_{q^{-1}}!} A^n B (-A)^m \quad . \tag{11}$$

With use of $[n]_{q^{-1}}! = q^{-n(n-1)/2}\, [n]!$, this product can be written in a very convenient form in terms of q-commutators as follows

$$exp_q(A)\ B\ exp_{q^{-1}}(-A) = B + [A, B] + \frac{1}{[2]!}[A,[A, B]_q] + \frac{1}{[3]!}[A,[A,[A, B]_{q^2}]_q] + \ldots$$

$$\ldots + \frac{1}{[n]!}[A, [A, \ldots [A,[A, B]_{q^{n-1}}]_{q^{n-2}} \ldots]_q] + \ldots \tag{12}$$

where $[A, B]_{q^n} = AB - q^n\ BA$. We can easily verify that

$$[A,[A, B]_{q^n}]_{q^m} = [A,[A, B]_{q^m}]_{q^n} \quad , \tag{13}$$

and therefore in eq. (12) the order of the q-commutators is irrelevant. This nice feature allows to derive simple identities. In what follows we display some particular cases:

1) If $[A,B]_q = 0$ one can check from eq. (12) that

$$exp_q(A)\ exp_q(B) = exp_q(A+B + [A,B]) \quad . \tag{14}$$

An example of this case corresponds to the coordinates of the quantum plane by defining $A = \hat{y}$ and $B = \hat{x}$, and therefore it would be useful to q-deformations of two-dimensional classical and quantum mechanics.

2) The case $[A,B]_q = 1$ is particularly interesting because it corresponds to the q-commutation relation satisfied by the q-derivative D_x and the coordinate x,and therefore $exp_q(iD_x)$ and $exp\ (ix)$ can be interpreted as the q-analog of the Weyl-Heisenberg operators. Similarly to the previous case, we also have

$$exp_q(A)\ B\ exp_{q^{-1}}(-A) = B + [A,B] \quad . \tag{15}$$

Since $[[A,B], B]_q = 0$, we finally obtain

$$exp_q(A)\ exp_q(B) = exp_q(B)\ \ exp_q([A,B])\ exp_q(A) \quad . \tag{16}$$

Now we use some of the previous identities to study different possible quantum deformations of the time evolution equation

$$\widehat{A}(t) = e^{i\widehat{H}t}\,\widehat{A}(0)\, e^{-i\widehat{H}t} \tag{17}$$

with $\widehat{H}$ a time-independent hermitian operator. Since the complex conjugate of the q-exponential does not correspond to its inverse; the deformation of (17) is far from being unique. Three possibilities that arise are

a) $$\widehat{A}(t) = e_q{}^{i\widehat{H}t}\,\widehat{A}(0)\; e_q{}^{-i\widehat{H}t} = T_1\,\widehat{A}(0)\; T_1^\dagger$$

b) $$\widehat{A}(t) = e_q{}^{i\widehat{H}t}\,\widehat{A}(0)\; e_{q^{-1}}{}^{-i\widehat{H}t} = T_2\,\widehat{A}(0)\,(T_2)^{-1} \tag{18}$$

c) $$\widehat{A}(t) = e_q{}^{i\widehat{H}t/2} e_{q^{-1}}{}^{i\widehat{H}t/2}\,\widehat{A}(0)\; e_q{}^{-i\widehat{H}t/2} e_{q^{-1}}{}^{-i\widehat{H}t/2} = U\,\widehat{A}(0)U^\dagger = U\,\widehat{A}(0)U^{-1}$$

Clearly the three cases are indistinguishable in the q=1 limit. The case a) can readily be discarded once we realize that it is not valid for the product of two operators. This problem does not arise in the other two cases. In particular, the operator U is unitary and the operator T_2 is not. We see that the case c) would be applicable if, as in standard quantum mechanics, the corresponding Hilbert space and its dual are related by the adjoint operation where in the case b) they should be related by an additional $q \rightarrow q^{-1}$ transformation with the condition that

$H(q^{-1}) = H(q)$. Now, to find the equation of motion we take the q-derivative of (18) such that with use of the infinite product representation of $exp_{q^{-1}}(x)$ and the q-analog of the Leibniz rule we obtain for the case b) that

$$D_t \hat{A}(t) = i\, exp_q(i\hat{H}t)\, [\hat{H},\hat{A}(0)]\, exp_{q^{-1}}(-\,qi\hat{H}t)$$

$$D_t \hat{A}(t) = i\ [\hat{H},\hat{A}(t)]\, \frac{1}{1 - i(1-q)\,\hat{H}\,t} \tag{19}$$

For the case c) the corresponding difference equation is easy to obtain but it is much more complicate to solve. Applying the definition of D_t we obtain

$$D_t \hat{A}(t) = \frac{t^{-1}}{1-q}\left[\hat{A}\,,\frac{\lambda_-}{\lambda_+}\right]\frac{\lambda_+}{\lambda_-} \tag{20}$$

where $\lambda_{\pm} = 1 \pm \frac{i\,(1-q)\,H}{2}\,t$.

The simplest example to consider is, of course , a "free quantum particle" $\hat{H}_0 = \frac{\hat{p}^2}{2m}$. Then, in b) the coordinate operator satisfies the following difference equation

$$D_t \hat{x}(t) = \frac{i}{2m}\,[\hat{p}^2, \hat{x}\,]\ \frac{1}{1 - i\,(1-q)\,\frac{\hat{p}^2}{2m}\,t} \tag{21}$$

Since we want to find its solution on the quantum line we consider a quantum deformation of the phase space

$$\hat{p}\,\hat{x} - q\,\hat{x}\,\hat{p} = -i\,f(q) \tag{22}$$

where $f(q)$ is an arbitrary function such that $f(q=1) = 1$. A very simple calculation gives

$$D_t \hat{x}(t) = \left[\frac{\hat{p}\,(1+q)}{2m\,q^2} f(q) + \frac{i}{2m} \frac{(q^2-1)}{q^2} \hat{p}^2 \hat{x} \right] \frac{1}{1 - i\,(1-q)\,t\,\hat{p}^2/2m} \tag{23}$$

which indicates that for $q \neq 1$: $D_t^2\, x(t) \neq 0$ although $D_t \hat{p} = 0$. In fact, the corresponding difference equation is given by

$$D_t^2\, \hat{x}(t) = -\frac{i\,q\,(1-q)}{2m} [D_t \hat{x}(t)]\; \hat{p}^2 \frac{1}{1 - i\,q(1-q)\,t\,\hat{p}^2/2m} \tag{24}$$

which vanishes, as expected, for q=1. The solution of this equation is rather simple, denoting $D_t \hat{x}(t) = \hat{u}(t)$ we obtain

$$\hat{u}(t) = \hat{u}_0\;\; exp_q\!\left(iq^2 \hat{H}_0\, t\right)\; exp_{q^{-1}}\!\left(-\,iq\hat{H}_0\, t\right)$$

$$= \hat{u}_0 \left(1 - iq\,(1-q)\hat{H}_0\, t\right) \tag{25}$$

where $\hat{u}_0$ satisfies $[\hat{p},\hat{u}_0]_q = 0$. We see that $\hat{u}$ becomes a time-independent operator in the q=1 limit. Now, the coordinate operator can be obtained by applying the well known q-integral operator with the result

$$\hat{x}(t) = \hat{u}_0 \left(t - i \frac{q\,(1-q)\,\hat{H}_0\, t^2}{[2]} \right) + \hat{x}_0 \tag{26}$$

For the case c) the difference equation of motion for the coordinate operator is given by

$$D_t\,\hat{x}(t) = \left[-\,i\,(1-q^2)\,\hat{x}\,\hat{H} + \frac{f(q)\,(1+q)\hat{p}}{2m} \right] \frac{1}{\lambda_-\,\lambda_+'} \qquad (27)$$

with $\lambda_+' = 1 + \dfrac{i\,q^2(1-q)\,\hat{H}}{2}\,t$.

If we had chosen instead of eq. (22) the usual commutations relations for $\hat{p}$ and $\hat{x}$ we will of course obtain different difference equations but still leading to complex solutions.

In this letter, we have formulated some q-deformations of time evolution in quantum mechanics. The two deformations we have analyzed are not equivalent. In fact, one can check that the operators T_2' and U satisfy different difference equations, and therefore they would correspond to two different deformations of the Schrödinger equation. Our starting point consisted in the replacement of $exp(x)$ by $exp_q(x)$ and therefore of the standard derivative by the q-derivative D_t. Since D_t is a difference operator, the q-parameter introduces a discretization of time, and hence the deformations discussed here could be of relevance to approach problems of quantization of space-time. Some fundamental aspects in this direction has been discussed in ref. [17]. The approach in this paper and other alternative methods based exclusively on the q-deformation of the commutator algebra [18] could be seen as different options, and therefore giving different predictions, to the q-deformation of a quantum mechanical system.

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