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Neutrinos in a Medium

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ABSTRACT

In these notes I describe recent work on the application of Finite Temperature Field Theory to the calculation of the properties of neutrinos that propagate through a medium. This method serves as a powerful framework to understand some of the unusual properties of the neutrinos in a medium and also provides very efficient techniques for practical calculations of relevant quantities, such as the neutrino index of refraction and the electromagnetic vertex. Both of these aspects are illustrated in the course of the lectures.

1 Introduction

A neutrino that propagates through a medium can have properties that differ from those in the vacuum in unexpected and important ways. For example, in the Standard Model of weak interactions, a neutrino in vacuum is guaranteed to be massless by the lepton number symmetry of the theory and the chiral nature of the neutrino field: the only term that is bilinear in the neutrino field and which also satisfies these two conditions, and therefore the only one that can appear in the effective action, is the kinetic energy term. Strictly speaking, this argument holds if we also require that the effective action be Lorentz invariant, as it should be in the vacuum. The situation in the medium is very different. Although the fundamental Lagrangian is Lorentz invariant, the effective action is not. The presence of a background medium introduces a special reference frame, namely the frame in which its center of mass is at rest. In this case, it is possible to have a term in the effective action that is bilinear in the neutrino field, respects the lepton number symmetry of the fundamental Lagrangian, and which is different from the kinetic energy term. The result of this additional term is that the effective equation for the neutrino field in the medium differs from the Weyl equation in such a way that the energy-momentum relation for a propagating neutrino is not the vacuum relation $E = p$. This can be interpreted in terms of an index of refraction, an effective potential or an effective mass for the neutrino.

Apart from the effect that the background can have on the properties of a single neutrino, it can also have great influence on the oscillations of neutrinos of one flavor to another as the neutrinos traverse a dense medium. Therefore, the oscillation of neutrinos in astrophysical contexts are very different from the oscillations in vacuum. For this reason,

Lectures delivered at the "Fourth Mexican School of Particles and Fields", Oaxtepec, Mexico; December 3-14, 1990.

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the problem of neutrino oscillations in matter and its implications in various astrophysical contexts, such as the solar neutrino problem, the observations of neutrinos from the supernova and the Early Universe, have been the subject of numerous articles and reviews[1]. The ramifications and implications of this problem have quickly become current areas of intense research.

Most of the work on the propagation of neutrinos in a medium has been focused in understanding the interesting and complicated phenomena contained in the evolution equation for neutrino oscillations in matter. These are the celebrated Mikheyev, Smirnoff, Wolfenstein equations[2], which take the typical form

$$-i\frac{d\psi}{dt} = (H_0 + V)\psi \ , \tag{1.1}$$

where ψ is a vector in the space of neutrino flavors and V is the effective potential for the neutrinos in the medium. For example, for electron and muon neutrinos propagating through normal matter,

$$V = \begin{pmatrix} \sqrt{2}G_F n_e & 0 \\ 0 & 0 \end{pmatrix} . \tag{1.2}$$

where G_F is the Fermi constant and n_e is the electron number density.

The main focus of these notes will not be the subject of neutrino oscillations in matter and its applications in the above-mentioned settings. As already mentioned, Ref.(1) contains an excellent review of the work carried out in these directions. Our purpose is to expose, in a pedagogical way, the approach that we have used to understand the properties of neutrinos that propagate through a medium. This includes understanding the origin of the terms in an equation such as Eq.(1.2), and learning efficient methods to calculate them in a variety of physical circumstances. The main ingredient in the approach that we have used and advocate is the calculation of the neutrino effective action using finite temperature field theory. In our experience, this approach has provided a convenient framework to, on one hand understand the intriguing aspects of this subject and, on the other, to carry out practical calculations. Our interest is to present the method and techniques of finite temperature field theory in a way that makes them useful to understand the existing results and extend the previous calculations of the properties of neutrinos in a medium.

The material contained in the next four sections is organized as follows. In Section 2 we will present the salient features of the formalism of real-time finite temperature field theory in the form that we have used it to calculate the properties of neutrinos in matter. The Feynman rules of this formalism, which are now well known[3], are somewhat complicated by the fact that they involve a duplication of the numbers of propagators and vertices of the original Lagrangian. We will present a simple derivation of the Feynman rules by following a straightforward generalization of the canonical quantization procedure in quantum field theory. This approach, which we call the canonical approach, provides a very intuitive understanding of the need to duplicate the vertices and propagators of the original Lagrangian in order to formulate the Feynman rules. In addition, since it involves familiar operator methods, the formalism is very suitable to study the implications of the symmetries of the Lagrangian and the background medium for the Green's function of the theory. In Section 3 we will derive the relation between the Green functions that are calculated with the Feynman rules and the macroscopic quantities of interest; e.g. the relation between the self-energy, which in this

formalism is a 2 x 2 matrix, and the equation of motion for the neutrino field in the medium. In Section 4 we will apply these techniques to the problem of neutrinos propagating in a medium. Specifically, we will consider in detail the case of a chiral neutrino of the Standard Model propagating through a background of electrons. We will learn to calculate the terms in an equation such as Eq.(1.2), show how to rederive previous results and how to extend them, and calculate some of the electromagnetic properties of the neutrino.

2 Canonical Approach

In this section we present a summary of the canonical quantization approach to the real-time formalism of finite temperature field theory[4]. We begin by considering in detail the case of a scalar boson and then summarize the analogous results for fermions afterwards. Our ultimate goal is to obtain the field equation satisfied by the effective field

$$\phi_{cl} = \langle \phi(x) \rangle \tag{2.1}$$

where the angle brackets indicate an average over the distribution of states of the system, given by

$$\langle O \rangle = \frac{TrZO}{TrZ}$$

Here Tr stands for the trace, and Z is the partition function defined by

$$Z = e^{-\beta P\cdot u + \sum_A \alpha_A Q_A} \tag{2.2}$$

where the Q_A are the (conserved) charges that commute with the Lagrangian, the α_A are the chemical potentials that parametrize the composition of the medium, P_μ is the momentum operator and u_μ represents the 4-velocity of the medium, which in its rest frame is simply the vector (1,0). As in ordinary quantum field theory, the field equation for the effective field is determined from the effective action, which is constructed from the knowledge of the Green's function. The latter are calculated by perturbation theory, and for this we need to know the expression for the propagators of the free fields.

2.1 Free-field scalar propagators

As will be seen later, the following propagators enter in the Feynman rules:

$$i\,\Delta_{F\,11}(x-y) = \langle T\phi(x)\phi^*(y)\rangle \tag{2.3a}$$

$$i\,\Delta_{F\,22}(x-y) = \langle \overline{T}\,\phi(x)\phi^*(y)\rangle \tag{2.3b}$$

$$i\,\Delta_{F\,12}(x-y) = \langle \phi^*(y)\phi(x)\rangle \tag{2.3c}$$

$$i\,\Delta_{F\,21}(x-y) = \langle \phi(x)\phi^*(y)\rangle \tag{2.3d}$$

where T stands for time-ordered product and $\overline{T}$ for the anti-time-ordered product. The explicit expression for the propagators in Eq.(2.3) are found straightforwardly by substituting the standard plane-wave expansion of canonical field theory,

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E}\left(a(\mathbf{k})e^{-ik\cdot x} + b^*(\mathbf{k})e^{ik\cdot x}\right)$$

and imitating the analogous steps of the vacuum case. The only difference is that in the present case we must use the relations

$$\langle a(\mathbf{k})a^*(\mathbf{k}')\rangle = (2\pi)^3 2E\delta(\mathbf{k}-\mathbf{k}')(f(\mathbf{k})+1)\,,$$

$$\langle b(\mathbf{k})b^*(\mathbf{k}')\rangle = (2\pi)^3 2E\delta(\mathbf{k}-\mathbf{k}')(\bar{f}(\mathbf{k})+1)\,,$$

where $f(\bar{f})$ is the Bose-Einstein distribution function for the particles (antiparticles), given by

$$f(\mathbf{k}) = \frac{1}{e^{\beta k\cdot u - \alpha} - 1} \tag{2.4a}$$

$$\bar{f}(\mathbf{k}) = \frac{1}{e^{\beta k\cdot u + \alpha} - 1} \tag{2.4b}$$

In Eq. (2.4) α is the value of the combination $\Sigma\, \alpha_A Q_A$ for the ϕ-particles. In this way, the propagators in momentum space are found to be given by

$$\Delta_{F\,11}(k) = \frac{1}{k^2 - m^2 + i\varepsilon} - 2\pi i\delta(k^2 - m^2)\eta(k\cdot u) \tag{2.5a}$$

$$\Delta_{F\,22}(k) = \frac{-1}{k^2 - m^2 - i\varepsilon} - 2\pi i\delta(k^2 - m^2)\eta(k\cdot u) \tag{2.5b}$$

$$\Delta_{F\,12}(k) = -\,2\pi i\delta(k^2 - m^2)\left(\eta(k\cdot u) + \theta(-k\cdot u)\right) \tag{2.5c}$$

$$\Delta_{F\,21}(k) = -\,2\pi i\delta(k^2 - m^2)\left(\eta(k\cdot u) + \theta(k\cdot u)\right), \tag{2.5d}$$

where

$$\eta(k\cdot u) = \theta(k\cdot u)n(x) + \theta(-k\cdot u)n(-x)\,, \tag{2.6}$$

with

$$n(x) = \frac{1}{e^x - 1}\,, \tag{2.7}$$

and the variable x is defined as

$$x = \beta k\cdot u - \alpha \tag{2.8}$$

Introducing the standard free propagator in vacuum,

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\varepsilon}\,, \tag{2.9}$$

the formulas in Eq.(2.5) can be written in the form

$$\Delta_{F\,11} = \Delta_F + (\Delta_F - \Delta_F^*)\eta \tag{2.10a}$$

$$\Delta_{F\,22} = -\Delta_F^* + (\Delta_F - \Delta_F^*)\eta \tag{2.10b}$$

$$\Delta_{F\,12} = (\Delta_F - \Delta_F^*)\left(\eta + \theta(-k\cdot u)\right) \tag{2.10c}$$

$$\Delta_{F\,21} = (\Delta_F - \Delta_F^*)\left(\eta + \theta(k\cdot u)\right) \tag{2.10d}$$

Furthermore, if the propagators Δ_{Fab} are collected together in a matrix $\tilde{\Delta}_F$, then the relations in Eqs.(2.5) and (2.10) can be summarized in the matrix equation

$$\tilde{\Delta}_F = U\begin{pmatrix} \Delta_F & 0 \\ 0 & -\Delta_F^* \end{pmatrix}U \tag{2.11}$$

where the elements of U are given by

$$U_{11} = U_{22} = \sqrt{1+\eta} \tag{2.12a}$$

$$U_{12} = \frac{\eta + \theta(-k\cdot u)}{\sqrt{1+\eta}} \tag{2.12b}$$

$$U_{21} = \frac{\eta + \theta(k\cdot u)}{\sqrt{1+\eta}} \tag{2.12c}$$

It is also useful to note that

$$\tilde{\Delta}_F^{-1} = \begin{pmatrix} k^2 - m^2 & 0 \\ 0 & -(k^2 - m^2) \end{pmatrix}\,. \tag{2.13}$$

2.2 Free-field fermion propagators

The free fermion propagators are defined by

$$iS_{F\,11}(x-y)_{\alpha\beta} \equiv \langle T\psi_\alpha(x)\overline{\psi}_\beta(y)\rangle \tag{2.14a}$$

$$iS_{F\,22}(x-y)_{\alpha\beta} \equiv \langle \overline{T}\psi_\alpha(x)\overline{\psi}_\beta(y)\rangle \tag{2.14b}$$

$$iS_{F\,12}(x-y)_{\alpha\beta} \equiv -\langle \overline{\psi}_\beta(y)\psi_\alpha(x)\rangle \tag{2.14c}$$

$$iS_{F\,21}(x-y)_{\alpha\beta} \equiv \langle \psi_\alpha(x)\overline{\psi}_\beta(y)\rangle \tag{2.14d}$$

Substituting the free field expansion

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2E}\sum_s \left(a_s(k)u_s(k)e^{-ik\cdot x} + b_s^*(k)v_s(k)e^{ik\cdot x}\right)$$

the following explicit formulas are obtained:

$$S_{F\,11}(k) = (\not{k}+m)\left\{\frac{1}{k^2-m^2+i\varepsilon} + 2\pi i\delta(k^2-m^2)\eta_F\right\} \tag{2.15a}$$

$$S_{F\,22}(k) = (\not{k}+m)\left\{\frac{-1}{k^2-m^2+i\varepsilon} + 2\pi i\delta(k^2-m^2)\eta_F\right\} \tag{2.15b}$$

$$S_{F\,12}(k) = (\not{k}+m)2\pi i\delta(k^2-m^2)\left(\eta_F - \theta(-k\cdot u)\right) \tag{2.15c}$$

$$S_{F\,21}(k) = (\not{k}+m)2\pi i\delta(k^2-m^2)\left(\eta_F - \theta(k\cdot u)\right) \tag{2.15d}$$

η_F is defined by

$$\eta_F = \theta(k\cdot u)n_F(x) + \theta(-k\cdot u)n_F(-x)\ , \tag{2.16}$$

where

$$n_F(x) = \frac{1}{e^x+1} \tag{2.17}$$

with x defined as in Eq.(2.8). The minus sign in Eq.(2.14c) is introduced so that the usual conventions for minus signs in diagrams involving fermion loops can be applied uniformly, independently of which propagators are involved in the diagram. The formulas in Eq.(2.10) and (2.11) for the scalar case have their counterpart here. In particular, if the fermion propagators given by Eq.(2.15) are collected in a 2 x 2 matrix, then we can write

$$\widetilde{S}_F = U_F\begin{pmatrix} S_F & 0 \\ 0 & -\overline{S}_F \end{pmatrix}U_F\ , \tag{2.18}$$

where

$$S_F = \frac{\not{k}+m}{k^2-m^2+i\varepsilon}\ , \tag{2.19}$$

and

$$\overline{S}_F \equiv \gamma^{0\dagger}S_F^\dagger\gamma^0 \tag{2.20}$$

U_F is given by a formula similar to Eq.(2.12), with the replacement $\eta \to -\eta_F$.

2.3 Feynman rules

As already mentioned, our final goal is to be able to determine the equations for the effective field from the knowledge of the Green's functions that we calculate in field theory. The precise relation between those two is the subject of the next section. Our immediate purpose now is to learn how to calculate the Green's functions. For illustrative purposes, let us begin by considering the following Green's functions in a scalar theory:

$$G(x_1,\ldots,x_N) = \langle T\phi(x_1)\ldots\phi(x_N)\rangle \tag{2.21}$$

Following the standard procedure in vacuum, we express the exact Heisenberg fields that appear in Eq.(2.13) in terms of fields ϕ_{in} that satisfy the free equations of motion. Thus we write

$$\phi = U^{-1}(t)\phi_{in}U(t)\ ,$$

where $U(t)$ is the evolution operator. Choosing a time τ satisfying

$$-\tau < t_1, t_2, \ldots, t_N < \tau\ ,$$

and introducing the notation $U(\tau,-\tau) = U(\tau)U^{-1}(-\tau)$, the identity

$$T\phi(x_1)\ldots\phi(x_N) = U^{-1}(\tau)\left\{T\phi_{in}(x_1)\ldots\phi_{in}(x_N)U(\tau,-\tau)\right\}U(-\tau) \tag{2.22}$$

holds. Substituting Eq.(2.22) into Eq.(2.21) yields

$$G(x_1,\ldots,x_2) = \langle U^{-1}(\tau)\left\{T\phi_{in}(x_1)\ldots\phi_{in}(x_N)U(\tau,-\tau)\right\}U(-\tau)\rangle\ . \tag{2.23}$$

In the case of the vacuum the result is similar, the only difference being that the average is with respect to the vacuum state. Formally, in the limit $\tau \to \infty$, $U(\infty) \to 1$, while $U(\infty)|0\rangle$ must be a phase times the vacuum state again because it is stable. Therefore, in that limit the $U^{-1}(\tau)$ can be factored out, giving the final formula for the vacuum Green function

$$G^{(0)}(x_1,\ldots,x_2) = \frac{\langle 0|\{T\phi_{in}(x_1)\ldots\phi_{in}(x_N)U(\infty)\}|0\rangle}{\langle 0|U(\infty)|0\rangle} \tag{2.24}$$

The perturbation expansion is obtained by expanding the formal formula

$$U(\infty) = Te^{i\int d^4x L_{int}(x)} \ , \tag{2.25}$$

and reducing the products of field operators through Wick's theorem, which yields the Feynman rules. In the present case, we can still use $U(-\infty) \to 1$, but the $U^{-1}(\tau)$ cannot be factored out, because it does not act on the vacuum state alone. Therefore, the final formula for the Green function is in this case

$$G(x_1,\ldots,x_N) = \langle U^{-1}(\infty)\{T\phi_{in}(x_1)\ldots\phi_{in}(x_N)U(\infty)\}\rangle \tag{2.26}$$

The perturbation expansion is obtained by expanding the formal formula for $U(\infty)$, but in this case we have to take into account the factor $U^{-1}(\infty)$, which is given by

$$U^{-1}(\infty) = \overline{T}e^{-i\int d^4x L_{int}(x)} \tag{2.27}$$

and expand it also. In Eq.(2.27), $\overline{T}$ indicates the anti time-ordered product.

From Eq.(2.26) it is not difficult to understand the need to introduce all the propagators defined in Eq.(2.3) and the corresponding doubling of vertices. When U and U^{-1} are expanded in Eq.(2.26), G consists of a sum of terms of the generic form

$$\langle \overline{T}(\overline{A}\,\overline{B}\ldots\overline{F})T(A,B,\ldots F)\rangle \tag{2.28}$$

where the bar over $A,B,\ldots$, is just to indicate that they are operators that appear under the $\overline{T}$ sign in Eq.(2.26). By a slight generalization of the standard Wick theorem for $\langle T(A,B,\ldots F)\rangle$ the product in Eq.(2.28) can be reduced to a sum of terms in which all the operators appear contracted in pairs. However, in the present case a contraction means any of the four propagators. Following previous authors, we will refer to the vertices that come from U and U^{-1} as type-1 and type-2 vertices, respectively. Then, a propagator that joins two vertices is one of the four given in Eq.(2.3) according to the following: if it joins two type-1 vertices it is Δ_{F11} ; if it joins two type-2 vertices it is Δ_{F22} ; if it joins a type-1 and a type-2 vertex, then it is Δ_{F12} if the particle propagates (i.e. the momentum flows) from the type-2 to the type-1 vertex, or Δ_{F21} if the converse is true.

We know sketch the procedure to reduce the products such as that in Eq.(2.28), which is a slight generalization of the standard Wick theorem, found in many textbooks[5]. Omitting the space dependence, which is not relevant for this analysis, the problem then reduces to consider a product such as

$$\langle \overline{T}(\overline{\alpha}_a(\bar{t}_a)\overline{\alpha}_b(\bar{t}_b)\ldots\overline{\alpha}_f(\bar{t}_f))T(\alpha_a(t_a)\alpha_b(t_b)\ldots\alpha_f(t_f))\rangle \ , \tag{2.29}$$

where the bar over the various quantities are only to indicate that they appear under the anti time-ordered product. Further, we can restrict the attention to the case in which the operators are labeled such that

$$\bar{t}_a < \bar{t}_b < \ldots < \bar{t}_f \tag{2.30a}$$

and

$$t_a > t_b > \ldots > t_f \ . \tag{2.30b}$$

If this is not the case, then we re-label the operators such that these conditions hold. The product in Eq.(2.29) then reduces to

$$\langle \overline{\alpha}_a(\bar{t}_a)\overline{\alpha}_b(\bar{t}_b)\ldots\overline{\alpha}_f(\bar{t}_f)\alpha_a(t_a)\alpha_b(t_b)\ldots\alpha_f(t_f)\rangle \ , \tag{2.31}$$

and from here on the procedure is identical to the standard one. The first operator, $\overline{\alpha}_a$ is commuted to the right until it reaches the end, and it is then brought back to its original place by using the cyclic property of the trace. To bring this operator to the right we express the product of two operators in the form

$$\overline{\alpha}_a\overline{\alpha}_b - \eta_{ab}\overline{\alpha}_b\overline{\alpha}_a = C_{ab}$$

where η_{ab} is taken as +1 if the two operators are fermions, or −1 if one or both are bosons. The quantity C_{ab} has the value 0, +1 or −1, which is easily obtained using the canonical commutation relations. The product in Eq.(2.31) is then equal to

$$C_{ab}\langle \overline{\alpha}_c(\bar{t}_c)\ldots\overline{\alpha}_f(\bar{t}_f)\alpha_a(t_a)\alpha_b(t_b)\ldots\alpha_f(t_f)\rangle \ + \ \eta_{ab}\langle \overline{\alpha}_b(\bar{t}_b)\overline{\alpha}_a(\bar{t}_a)\ldots\overline{\alpha}_f(\bar{t}_f)\alpha_a(t_a)\alpha_b(t_b)\ldots\alpha_f(t_f)\rangle$$

The same procedure is successively applied to $\overline{\alpha}_a$ and the operator that appears to its right until it appears at the end of the product. Then, using the cyclic property of the trace,

$$\begin{aligned}
&\langle \overline{\alpha}_b(\bar{t}_b)\ldots\overline{\alpha}_f(\bar{t}_f)\alpha_a(t_a)\alpha_b(t_b)\ldots\alpha_f(t_f)\overline{\alpha}_a(\bar{t}_a)\rangle \\
&\qquad = \frac{1}{\mathrm{Tr}Z}\mathrm{Tr}\overline{\alpha}_a(\bar{t}_a)Z\overline{\alpha}_b(\bar{t}_b)\ldots\overline{\alpha}_f(\bar{t}_f)\alpha_a(t_a)\alpha_b(t_b)\ldots\alpha_f(t_f) \\
&\qquad = e^{\lambda_a\varepsilon_a}\langle \overline{\alpha}_a(\bar{t}_a)\overline{\alpha}_b(\bar{t}_b)\ldots\overline{\alpha}_f(\bar{t}_f)\alpha_a(t_a)\alpha_b(t_b)\ldots\alpha_f(t_f)\rangle
\end{aligned} \tag{2.32}$$

where λ_a is +1 (−1) if $\overline{\alpha}_a$ is a creation (annihilation) operator and ε_a is the same quantity that appears in the argument of the exponential in Eq.(2.4a); namely, if the indices a,b,... label momentum states,

$$\varepsilon = \beta k\cdot u - \alpha$$

where α has the same meaning as in Eq.(2.4). The last step in Eq.(2.32) follows from the useful relation

$$Z^{-1}\overline{\alpha}_a Z = e^{\lambda_a\varepsilon_a}\overline{\alpha}_a$$

Putting everything together, we arrive at the formula

$$\langle \bar{\alpha}_a \bar{\alpha}_b \ldots \bar{\alpha}_f \alpha_a \alpha_b \ldots \alpha_f \rangle = \langle \bar{\alpha}_a^{\bullet} \bar{\alpha}_b^{\bullet} \ldots \bar{\alpha}_f \alpha_a \alpha_b \ldots \alpha_f \rangle + \ldots + \langle \bar{\alpha}_a^{\bullet} \bar{\alpha}_b \ldots \bar{\alpha}_f^{\bullet} \alpha_a \alpha_b \ldots \alpha_f \rangle$$

$$+ \langle \bar{\alpha}_a^{\bullet} \bar{\alpha}_b \ldots \bar{\alpha}_f \alpha_a^{\bullet} \alpha_b \ldots \alpha_f \rangle + \ldots + \langle \bar{\alpha}_a^{\bullet} \bar{\alpha}_b \ldots \bar{\alpha}_f \alpha_a \alpha_b \ldots \alpha_f^{\bullet} \rangle \tag{2.33}$$

with the understanding that, when fermions are involved, the usual bookkeeping of signs must be made to bring the two contracted operators together. In terms of the commutator (or anti-commutator for fermions) C_{ab} defined above, the contraction of any two operators β_1, β_2 is defined by

$$\beta_1^{\bullet} \beta_2^{\bullet} = \frac{C_{12}}{1 + \eta e^{\lambda_1 \beta_1}}$$

where η is +1 for fermions and −1 for bosons. On the other hand, by explicit evaluation the following result is easily verified:

$$\beta_1^{\bullet} \beta_2^{\bullet} = \langle \beta_1 \beta_2 \rangle$$

By using Eq.(2.33) recursively, we arrive at a formula for $\langle \bar{\alpha}_a \bar{\alpha}_b \ldots \bar{\alpha}_f \alpha_a \alpha_b \ldots \alpha_f \rangle$ in which all the operators are contracted in all possible ways. Further, the resulting formula also applies directly to the original product of Eq.(2.29) for arbitrary orderings of the time-arguments, provided that the contractions are reinterpreted to mean the following:

$$\alpha_a^{\bullet} \alpha_b^{\bullet} \equiv \langle T \alpha_a \alpha_b \rangle \tag{2.34a}$$

$$\bar{\alpha}_a^{\bullet} \bar{\alpha}_b^{\bullet} \equiv \langle \bar{T} \bar{\alpha}_a \bar{\alpha}_b \rangle \tag{2.34b}$$

$$\bar{\alpha}_a^{\bullet} \alpha_b^{\bullet} \equiv \langle \bar{\alpha}_a \alpha_b \rangle \ . \tag{2.34c}$$

2.4 Exact propagators

The knowledge of the exact propagators is important because they determine the quadratic part of the effective action, which in turn determines the effective field equation for the particle in the background. The precise connection between the propagators and the effective action will be elucidated in the next section. Here we will learn how to calculate the exact propagators. We consider first the scalar field.

The exact propagators are defined as follows

$$i\Delta'_{F\,11}(x-y) = \langle T\phi(x)\phi^{\bullet}(y) \rangle \tag{2.35a}$$

$$i\Delta'_{F\,22}(x-y) = \langle \bar{T}\phi(x)\phi^{\bullet}(y) \rangle \tag{2.35b}$$

$$i\Delta'_{F\,12}(x-y) = \langle \phi^{\bullet}(y)\phi(x) \rangle \tag{2.35c}$$

$$i\Delta'_{F\,21}(x-y) = \langle \phi(x)\phi^{\bullet}(y) \rangle \tag{2.35d}$$

Using the transformation in Eq.(2.22), we can express these propagators in the form

$$i\Delta'_{F\,11}(x-y) = \langle U^{-1}(\infty) \left[T\phi_{in}(x)\phi_{in}^{\bullet}(y) U(\infty) \right] \rangle \tag{2.36a}$$

$$i\Delta'_{F\,22}(x-y) = \langle \left[\bar{T}\phi_{in}(x)\phi_{in}^{\bullet}(y) U^{-1}(\infty) \right] U(\infty) \rangle \tag{2.36b}$$

$$i\Delta'_{F\,12}(x-y) = \langle \bar{T}\left[\phi_{in}^{\bullet}(y) U^{-1}(\infty) \right] \left[T\phi_{in}(x) U(\infty) \right] \rangle \tag{2.36c}$$

$$i\Delta'_{F\,21}(x-y) = \langle \left[\bar{T}\phi_{in}(x) U^{-1}(\infty) \right] T\left[\phi_{in}^{\bullet}(y) U(\infty) \right] \rangle \tag{2.36d}$$

In deriving these equations we have used identities such as (omitting the space dependence)

$$\phi(t_1)\phi^{\bullet}(t_2) = U^{-1}(-\tau) \left[\bar{T}\phi_{in}(t_1) U^{-1}(\tau,-\tau) \right] \left[T\phi_{in}^{\bullet}(t_2) U(\tau,-\tau) \right] U(-\tau) \tag{2.37a}$$

$$\bar{T}\phi(t_1)\phi^{\bullet}(t_2) = U^{-1}(-\tau) \left[\bar{T}\phi_{in}(t_1)\phi_{in}^{\bullet}(t_2) U^{-1}(\tau,-\tau) \right] U(\tau) \ , \tag{2.37b}$$

in the limit $\tau \to \infty$. From Eqs.(2.36) it follows that, if we collect the propagators in the matrix $\tilde{\Delta}'_F$ then

$$\tilde{\Delta}'_F(k) = \tilde{\Delta}_F(k) + \tilde{\Delta}_F(k)\tilde{\pi}(k)\tilde{\Delta}'_F(k) \tag{2.38}$$

where $\tilde{\Delta}_F$ is the matrix of free propagators, given in Eq.(2.20), and $\tilde{\pi}$ is the matrix of self-energies calculated from the diagrams represented in Fig. 1.

From the definitions in Eq.(2.36), the following identities are easily derived:

$$\left(i\Delta'_{F\,11}(k) \right)^{*} = i\Delta'_{F\,22}(k) \tag{2.39a}$$

$$\Delta'_{F\,11} + \Delta'_{F\,22} = \Delta'_{F\,12} + \Delta'_{F\,21} \tag{2.39b}$$

$$\Delta'_{F\,21}(k) = e^{x}\Delta'_{F\,12}(k) \ , \tag{2.39c}$$

where x is defined in Eq.(2.8). These relations imply that only one component of $\tilde{\Delta}'_F$ is independent. A convenient parametrization that automatically satisfies Eqs.(2.39) is

$$\tilde{\Delta}'_F = \mathrm{U} \begin{pmatrix} \Delta'_F & 0 \\ 0 & -\Delta'^{*}_F \end{pmatrix} \mathrm{U} \ , \tag{2.40}$$

where U is the matrix given in Eq.(2.12). Eqs.(2.39) in turn imply that only one component of the self-energy matrix is independent. Defining π by writing

$$\Delta'_F \equiv \frac{1}{k^2 - m^2 - \pi} \ , \tag{2.41}$$

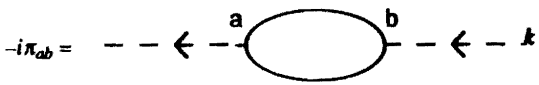

Fig. 1. Self-energy diagram for a scalar.

and comparing with Eq.(2.38) it follows that

$$\tilde{\pi} = U^{-1}\begin{pmatrix} \pi & 0 \\ 0 & -\pi^* \end{pmatrix} U^{-1} \tag{2.42}$$

or, explicitly,

$$\pi_{11} = \pi + (\pi - \pi^*)\eta \tag{2.43a}$$

$$\pi_{22} = -\pi^* + (\pi - \pi^*)\eta \tag{2.43b}$$

$$\pi_{12} = -(\pi - \pi^*)\big(\eta + \theta(-k\cdot u)\big) \tag{2.43c}$$

In addition, it is useful to observe that

$$(\tilde{\Delta}_F')^{-1} = (k^2 - m^2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \tilde{\pi} \tag{2.44}$$

which follows either from Eq.(2.38) or by inverting (2.40)[6]. Eqs.(2.43) are very useful because they show explicitly how to obtain all the components of $\tilde{\pi}$ from one of them. For example, if π_{11} is calculated, then from Eq.(2.43a)

$$\mathrm{Re}\,\pi = \mathrm{Re}\,\pi_{11} \tag{2.45a}$$

$$\mathrm{Im}\,\pi = \frac{\mathrm{Im}\,\pi_{11}}{1+2\eta} = \tanh(\tfrac{1}{2}x)\varepsilon(k\cdot u)\mathrm{Im}\,\pi_{11}\,, \tag{2.45b}$$

so that all the components of $\tilde{\pi}$ are expressed in terms of π_{11}. In practice, instead of using Eq.(2.45b), it is simpler to use[7]

$$\mathrm{Im}\,\pi = \frac{i\pi_{12}}{2\big(\eta + \theta(-k\cdot u)\big)} = \frac{i\pi_{12}\varepsilon(k\cdot u)}{2n(x)}\,, \tag{2.45c}$$

In Eqs.(2.36b) and (2.36c), $\varepsilon(k\cdot u) = \theta(k\cdot u) - \theta(-k\cdot u)$.

The formalism for fermions is similar. The exact propagators are defined by

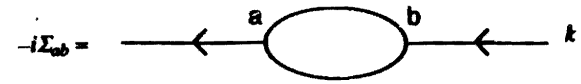

Fig.2 Self-energy diagram for fermions.

$$iS_{F\,11}(x-y)_{\alpha\beta} \equiv \langle T\psi_\alpha(x)\bar{\psi}_\beta(y)\rangle \tag{2.46a}$$

$$iS_{F\,22}(x-y)_{\alpha\beta} \equiv \langle \bar{T}\psi_\alpha(x)\bar{\psi}_\beta(y)\rangle \tag{2.46b}$$

$$iS_{F\,12}(x-y)_{\alpha\beta} \equiv -\langle \bar{\psi}_\beta(y)\psi_\alpha(x)\rangle \tag{2.46c}$$

$$iS_{F\,21}(x-y)_{\alpha\beta} \equiv \langle \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle \tag{2.46d}$$

If the free and exact propagators are assembled in the matrices $\tilde{S}_F$ and $\tilde{S}'_F$, respectively, then

$$\tilde{S}'_F = \tilde{S}_F + \tilde{S}_F\tilde{\Sigma}\tilde{S}'_F\,, \tag{2.47}$$

where $\tilde{\Sigma}$ is the matrix of self-energies calculated from the diagrams represented in Fig. 2. Similarly to the scalar case, the exact propagators are not independent, but satisfy the following relations that can be derived from their definition:

$$S'_{F11}(k) + S'_{F22}(k) = S'_{F21}(k) + S'_{F12}(k) \tag{2.48a}$$

$$\overline{S'_{F22}(k)} = -S'_{F11}(k) \tag{2.48b}$$

$$S'_{F21}(k) = -e^x S'_{F12}(k) \tag{2.48c}$$

These relations imply that the propagator can be parametrized in the form

$$\tilde{S}'_F = U_F\begin{pmatrix} S'_F & 0 \\ 0 & -\overline{S'_F} \end{pmatrix} U_F\,, \tag{2.49}$$

where

$$\overline{S'_F} \equiv \gamma^{0\dagger} S'^{\dagger}_F \gamma^0 \tag{2.50}$$

and U_F is defined as in Eq.(2.20). If we write

$$S'_F = \frac{1}{\not{k} - m - \Sigma}\,, \tag{2.51}$$

then the self-energy matrix can be expressed in the form

$$\tilde{\Sigma} = U_F^{-1}\begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} U_F^{-1}$$

or, explicitly,

$$\Sigma_{11} = \Sigma - \left(\Sigma - \bar{\Sigma}\right)\eta_F \tag{2.52a}$$

$$\Sigma_{22} = -\bar{\Sigma} - \left(\Sigma - \bar{\Sigma}\right)\eta_F \tag{2.52b}$$

$$\Sigma_{12} = \left(\Sigma - \bar{\Sigma}\right)\left(\eta_F - \theta(-k\cdot u)\right) \tag{2.52c}$$

$$\Sigma_{21} = \left(\Sigma - \bar{\Sigma}\right)\left(\eta_F - \theta(k\cdot u)\right) , \tag{2.52d}$$

where the bar over Σ has the same meaning as in Eq.(2.50). From Eqs.(2.52), it follows for example,

$$\mathrm{Re}\ \Sigma = \mathrm{Re}\ \Sigma_{11} \tag{2.53a}$$

$$\mathrm{Im}\ \Sigma = \frac{\Sigma_{12}}{2i\left(\eta_F - \theta(-k\cdot u)\right)} = \frac{\varepsilon(k\cdot u)\Sigma_{12}}{2in_F(x)} , \tag{2.53b}$$

where the symbols Re and Im here should be interpreted as the dispersive and absorptive part, respectively. For future reference we note that, in analogy with Eq.(2.44), the inverse fermion propagator can be expressed as

$$\tilde{S}_F^{-1} = (\not{k} - m)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \tilde{\Sigma} \tag{2.54}$$

In summary, although the propagators and the self-energy are 2 x 2 matrices in this formalism, there is only one independent component. In practice we only have to calculate Σ_{11} for example, and then we can determine the other components using Eqs.(2.45) and (2.43). The same occurs for fermions. However, there is still the question as to what is the relation between the propagators and self-energy matrices that we have introduced, with the effective action or, equivalently, the effective field equation for the particle. This is the subject to which we now turn our attention.

3 Effective Action

As before, we begin by considering the case of a scalar field and then generalize the results to include he case of fermion fields.

1. Scalar field

In the presence of a source, j, the classical field is defined as

$$\phi_{cl} = \langle\phi(x)\rangle$$

Using the transformation in Eq.(2.22) we have the two equivalent formulas

$$\phi_{cl}(x) = \langle U_j^{-1}(\infty)T\left(U_j(\infty)\phi_{in}(x)\right)\rangle \tag{3.1a}$$

$$\phi_{cl}(x) = \langle \bar{T}\left(U_j^{-1}(\infty)\phi_{in}(x)\right)U_j(\infty)\rangle \tag{3.1b}$$

where the subscript j on $U(\infty)$ indicates that it is the evolution operator in the presence of the external source. In order to be able to obtain these as functional derivatives, we define

$$e^{iW[j_1,j_2]} = Z[j_1,j_2] = \langle U_{j_2}^{-1}(\infty)U_{j_1}(\infty)\rangle = \langle \bar{T}e^{-i\left[S_{in} + (j_2\cdot\phi + H.c.)\right]}Te^{i\left[S_{in} + (j_1\cdot\phi + H.c.)\right]}\rangle \tag{3.2}$$

Corresponding to each source, we introduce a classical field by

$$\phi_1 = \frac{\delta W}{\delta j_1} \tag{3.3a}$$

$$\phi_2 = \frac{\delta W}{\delta j_2} \tag{3.3b}$$

which, using

$$Z[j_1=j_2] = 1 ,$$

are such that

$$-\phi_2(j_2 = j_1) = \phi_1(j_2 = j_1) = \phi_{cl} \tag{3.4}$$

In similar fashion, we define

$$\tilde{\phi}_a = \frac{\delta W}{\delta j_a^*} \tag{3.5}$$

For $j_2 \neq j_1$, $\tilde{\phi}_a$ is not the Hermitean conjugate of ϕ_a, but

$$\tilde{\phi}_a(j_2 = j_1) = \phi_a^*(j_2 = j_1) \tag{3.6}$$

Introducing Γ by (here we omit indices as well as summation and integration symbols)

$$\Gamma\left[\phi, \tilde{\phi}\right] = W - j\cdot\phi - j^*\cdot\tilde{\phi} \ , \tag{3.7}$$

where the currents are taken to be functionals of the fields, then

$$\frac{\delta\Gamma}{\delta\phi_a} = -j_a \ ; \ \frac{\delta\Gamma}{\delta\tilde{\phi}_a} = -j_a^* \ . \tag{3.8}$$

If we now take the *physical limit* $j_2 = j_1 = j$, then using Eq.(3.6) it follows that

$$\left(\frac{\delta\Gamma}{\delta\phi_a}\right)_{\phi_1 = -\phi_2 = \phi_c} = -j \ , \tag{3.9a}$$

$$\left(\frac{\delta\Gamma}{\delta\tilde{\phi}_a}\right)_{\phi_1 = -\phi_2 = \phi_c} = -j^* \ . \tag{3.9b}$$

Eqs.(3.9) are the effective equations of motion for the classical field. What remains to be done is to find the expression for Γ in terms of the Green functions calculated through the Feynman diagrams. Below we will do precisely this, but only for the contribution to Γ coming from the propagators. This contribution gives rise to the linear part of the equations of motion and, in general, is the one that determines how the particle propagates through the medium.

Before we go into the details we give the results: either of Eqs.(3.9) imply that the classical field, in momentum space, satisfies the equation

$$\left(k^2 - m^2 - \pi_{eff}\right)\phi_{cl}(k) = -j^*(k) \ , \tag{3.10}$$

where

$$\pi_{eff} = \pi\theta(k\cdot u) + \pi^*\theta(-k\cdot u) \tag{3.11}$$

ϕ_{cl}^* satisfies the conjugate equation.

To derive these results we proceed as follows. We assume that the theory is such that if $j_a = 0$, then $\phi_a = 0$. Eqs.(3.8) then imply

$$\left(\frac{\delta\Gamma}{\delta\phi_a}\right)_{\phi,\tilde{\phi}=0} = 0 \ ; \ \left(\frac{\delta\Gamma}{\delta\tilde{\phi}_a}\right)_{\phi,\tilde{\phi}=0} = 0 \ ,$$

which means that there is no symmetry-breaking. Γ then has the expansion

$$\Gamma = \int (dk)\tilde{\phi}_a(k)\Gamma_{ab}^{(2)}(k)\phi_b(k) + \dots \ , \tag{3.12}$$

where the dots indicate higher order terms in the fields, which we will not consider. From this point on we omit the argument k and the integration symbols. Taking a derivative with respect to ϕ_a on both sides of the following equation

$$\phi_a = \frac{\delta W}{\delta j_a} \ ,$$

we arrive at

$$\delta_{ab} = \left(\frac{\delta^2 W}{\delta j_a \delta j_c^*}\right)\left(\frac{-\delta^2\Gamma}{\delta\phi_c^*\delta\phi_b}\right)$$

Evaluating this for $\phi_a = 0$ which, as we have assumed, implies that $j_a = 0$, we obtain

$$\delta_{ab} = \left(\frac{\delta^2 W}{\delta j_a \delta j_c^*}\right)_{j=0}\left(-\Gamma_{cb}^{(2)}\right) \tag{3.13}$$

The second derivative of W is related to the propagators. In fact, from Eq.(3.2) it follows that

$$\left(\frac{\delta^2 W}{\delta j_a \delta j_c^*}\right)_{j=0} = -\left(\tau_3\tilde{\Delta}'_F\tau_3\right)_{ac} \ ,$$

where τ_3 is the third isospin matrix. Substituting this in Eq.(3.13) and inverting, then

$$\Gamma^{(2)} = \tau_3(\tilde{\Delta}_F^{-1} + \tilde{\pi})\tau_3 \tag{3.14}$$

where we have used Eq.(2.44). $\tilde{\pi}$ is the self-energy matrix given in Eqs.(2.42) and (2.43), and

$$\tilde{\Delta}_F^{-1} = (k^2 - m^2)\tau_3$$

The equations of motion follow from Eqs.(3.9) and (3.14). For example, neglecting the higher order terms,

$$\frac{\delta\Gamma}{\delta\phi_1^*} = \Gamma_{11}^{(2)}\phi_1 + \Gamma_{12}^{(2)}\phi_2 \ .$$

Eq.(3.9b) then becomes

$$\left(\Gamma_{11}^{(2)} - \Gamma_{12}^{(2)}\right)\phi_{cl} = -j^* \ ,$$

which, using Eqs.(3.14) and (2.43), reduces to Eq.(3.10).

There is an important observation we would like to make. We would not get the correct equations of motion by taking the limit $-\phi_2 = \phi_1 = \phi_{cl}$ in Eq.(3.12) and then take a derivative with respect to ϕ_{cl}. The effective action Γ gives the equations of motion for the fields ϕ_1, ϕ_2, and it is in those equations that we can take the physical limit $-\phi_2 = \phi_1 = \phi_{cl}$.

2. *Fermion field*

Following steps analogous to those leading to Eq.(3.13) we obtain in this case

$$\left[-\left(\tau_3 \tilde{S'}_F \tau_3\right)_{ab}\right]\left(-\Gamma^{(2)}_{bc}\right) = \delta_{ac} \tag{3.15}$$

so that

$$\Gamma^{(2)} = \tau_3 \tilde{S'}_F^{-1} \tau_3$$

Substituting Eq.(2.54)

$$\Gamma^{(2)} = (\not{k} - m)\tau_3 - \tau_3 \tilde{\Sigma} \tau_3 \tag{3.16}$$

The field equations

$$\left(\Gamma^{(2)}_{11} - \Gamma^{(2)}_{12}\right)\psi = -j \quad ,$$

then become

$$\left(\not{k} - m - \Sigma_{eff}\right)\psi(k) = -j(k) \ , \tag{3.17}$$

where

$$\Sigma_{eff} = \Sigma\theta(k\cdot u) + \bar{\Sigma}\theta(-k\cdot u) \tag{3.18}$$

4 Chiral Neutrinos in a medium

In this section we will apply the techniques developed in the previous sections to study the propagation of a massless, chiral neutrino of the Standard Model, through a medium. The presentation is based on our work on the subject[8,9], which is in turn based on an idea due to Weldon[10]. In addition, we will discuss some aspects of the electromagnetic interactions of the neutrinos in a background of electrons[11]. In all these cited references, the attention was restricted to the real part of the self-energy. Although the explicit calculations that we reproduce here are also restricted to the real part, the general discussion that we present in Sections 4.1 and 4.2 is not.

4.1 Self-energy

Chirality implies that the self-energy of a left-handed neutrino of momentum k is of the form

$$\Sigma_{eff} = RML \tag{4.1}$$

where $L,R = \frac{1}{2}(1 \pm \gamma_5)$ and M is a combination of gamma matrices. In vacuum, the most general form of Σ is

$$\Sigma_{eff} = a\not{k}L$$

where a is some function of k^2. In this case, although the residue of the propagator is modified, the pole remains at $k^2 = 0$, which implies that the particle remains massless.

The situation in the medium is different. The effects of the medium will introduce in Σ an additional dependence on the velocity four-vector u^μ of the background. The self-energy is still given by Eq.(4.1), but the general form of Σ is now

$$\Sigma_{eff} = (a\not{k} + b\not{u})L \tag{4.2}$$

where a and b are functions of the Lorentz-invariant variables

$$\omega = k\cdot u \ , \ K = \left(\omega^2 - k^2\right)^{1/2} \tag{4.3}$$

We will indicate explicitly this dependence when needed.

Some general properties of a and b follow from the discrete symmetries CP and CPT, when they apply. To discuss these properties let us recall that the generating function that incorporates the effects of the medium is given by Eq.(2.2). A background is said to be CP symmetric if the chemical potentials of charges that are CP odd are zero. If, in addition, the Lagrangian (and hence H) is CP invariant, then Z satisfies

$$(CP)^{-1}Z(u)(CP) = Z(A^{-1}u) \tag{4.4}$$

where we have indicated explicitly the dependence of Z on u_μ, and $A = \mathrm{diag}(1,-1,-1,-1)$. Similar considerations hold for CPT with the distinction that the Lagrangian is always assumed to be CPT invariant.

If the Lagrangian is CP invariant and the background is CP symmetric (so that Eq.(4.4) holds) then the self-energy must satisfy the following relation (see Appendix):

$$\bar{\Sigma}^{CP}_{eff}(-k) = \Sigma_{eff}(k) \tag{4.5}$$

where Σ^{CP}_{eff} is obtained from Σ_{eff} by making the substitutions

$$\left(\gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu\gamma_5, \gamma_5\right) \to -\left(\gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu\gamma_5, \gamma_5\right)$$

$$\left(1, \sigma_{\mu\nu}\gamma_5\right) \to \left(1, \sigma_{\mu\nu}\gamma_5\right)$$

Thus, CP invariance of the Lagrangian and the background implies that

$$a^*(-\omega,K) = a(\omega,K) \tag{4.6a}$$

$$b^*(-\omega,K) = -b(\omega,K) \tag{4.6b}$$

On the other hand, if the background is CPT symmetric (we are assuming that the Lagrangian is CPT invariant) then the self-energy satisfies

$$\gamma_5\bar{\Sigma}_{eff}(-k)\gamma_5 = \Sigma_{eff}(k) \tag{4.7}$$

which implies the same relations in Eq.(4.6) for the coefficients a and b. We will show below that whenever Eq.(4,6) holds, the neutrino and antineutrino have the same dispersion relation. Therefore, if the background is CPT symmetric then, independently of whether or not the Lagrangian is CP invariant, the neutrino and antineutrino have the same dispersion relation. Equivalently, the neutrino and antineutrino have different dispersion relations only if the background has non-zero chemical potentials of charges that are CPT odd.

4.2 Dispersion relation and wave function

The effective field equation,

$$(\not{k} - \Sigma_{eff})\psi_L(k) = 0$$

can be written in the form

$$\not{V}\psi_L(k) = 0 \tag{4.8}$$

where

$$V_\mu \equiv (1-a)k_\mu - bu_\mu \tag{4.9}$$

Eq.(4.8) has solutions only if ω and K are such that $V^2 = 0$. Thus, the energy-momentum relation is given by the solution of the equation

$$(1-a)\omega - b = \pm(1-a)K \tag{4.10}$$

which is in general complex. We denote by ω_K (ω_K') the solution with the positive (negative) sign. It is useful to define

$$\bar{\omega}_K = -\omega_K'^* \tag{4.11}$$

which, in analogy with the situation in the vacuum, we identify as the dispersion relation for the antiparticle.

Notice that $\bar{\omega}_K$ is given by the solution of

$$\left[1 - a(-\bar{\omega}_K^*,K)^*\right]\bar{\omega}_K + b(-\bar{\omega}_K^*,K)^* = \left[1 - a(-\bar{\omega}_K^*,K)^*\right]K$$

or equivalently

$$\left[1 - a^*(-\bar{\omega}_K,K)\right]\bar{\omega}_K + b^*(-\bar{\omega}_K,K) = \left[1 - a^*(-\bar{\omega}_K,K)\right]K \tag{4.12}$$

Therefore, in general,

$$\bar{\omega}_K \neq \omega_K$$

so that the neutrino and antineutrino have different dispersion relations. However, if the Lagrangian is CP conserving and the background is CP symmetric, then $\bar{\omega}_K = \omega_K$, since Eq.(4.6) implies that ω_K and $\bar{\omega}_K$ satisfy the same equation. In this case the neutrino and antineutrino have the same energy-momentum relation and the same lifetime. As remarked below Eq.(4.6) the same conclusion holds under the less restrictive case that the background is CPT symmetric, independently of whether or not the Lagrangian and the background are CP symmetric.

In order to determine the wave function, we choose to work in the rest frame of the medium, which is defined by setting

$$u^\mu = (1,0) \tag{4.13}$$

The wave function in an arbitrary frame can then be obtained by a Lorentz transformation. We consider first the wave function for the neutrino, which we denote by $u_L(\mathbf{k})$. Putting $k = (\omega_K, \mathbf{k})$ in Eq.(4.8) and remembering that ω_K satisfies Eq.(4.10) (with the + sign), the equation for the spinor becomes

$$\not{n}u_L = 0 \ , \tag{4.14}$$

where

$$n^\mu = \left(1, \frac{\mathbf{k}}{|\mathbf{k}|}\right) \tag{4.15}$$

Equation (4.14) is of the same form as the equation for a massless spinor in vacuum. Its solution can be given explicitly once a particular representation of the gamma matrices is chosen. However, for most purposes, the knowledge of the projection operator is sufficient. Apart from a normalization factor, this projection operator is proportional to $L\not{n}$, so that we write

$$u_L\bar{u}_L = \frac{\omega_K L\not{n}}{N_K} \tag{4.16}$$

where N_K is a normalization constant that is determined following the same procedure as in the case of the vacuum. Firstly, the imaginary part of the self-energy is ignored. Then the field equation can be obtained from the effective Lagrangian

$$\begin{aligned} L_{eff} &= \bar{\psi}_L(k)(\not{k} - \Sigma_{eff})\psi_L(k) \\ &= \bar{\psi}_L(k)\not{V}\psi_L(k) \end{aligned} \tag{4.17}$$

The classical theory defined by Eq.(4.17) is now quantized, so that ψ_L is regarded as a quantum field for the neutrino in the medium. In analogy with the case of the vacuum, the field has the plane wave expansion

$$\psi_L(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(\frac{1}{\omega_K}a(\mathbf{k})u_L(\mathbf{k})e^{-ik\cdot x} + \frac{1}{\bar{\omega}_K}b^*(\mathbf{k})v_L(\mathbf{k})e^{i\bar{k}\cdot x}\right) \tag{4.18}$$

where

$$k^\mu = (\omega_K, \mathbf{k}) \ , \ \bar{k}^\mu = (\bar{\omega}_K, \mathbf{k}) \tag{4.19}$$

We denote by $|0\rangle$ the new vacuum state and by $|k\rangle$ the one-neutrino state obtained as

$$|k\rangle = a^*(\mathbf{k})|0\rangle$$

We adopt the normalization of the one-neutrino state to be

$$\langle k'|k\rangle = (2\pi)^3 2\omega_K \delta^{(3)}(\mathbf{k} - \mathbf{k}') \ ,$$

so that the anticommutation relations of the creation and annihilation operators is fixed to be

$$\left[a(\mathbf{k}), a^*(\mathbf{k}')\right]_+ = (2\pi)^3 2\omega_K \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

Therefore, for example,

$$\langle 0|\psi_L(x)|k\rangle = u_L(\mathbf{k})e^{-ik\cdot x}$$

The normalization of the spinor is then determined by imposing on the field the canonical anticommutation relations implied by the Lagrangian in Eq.(4.17) or, equivalently, by requiring that the one-neutrino contribution to the propagator S_F defined by

$$\langle 0|T\psi_L(x)\overline{\psi_L}(y)|0\rangle \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} iS_F(k) \ ,$$

coincides with the singular term of the inverse kinetic energy operator

$$L\frac{\not{V}}{V^2}R \tag{4.20}$$

in the limit $k_0 \to \omega_k$. The one-neutrino contribution to the propagator is given by

$$S_F(k)_{1\ \text{particle}} = \frac{u_L\overline{u_L}}{2\omega_K(k_0 - \omega_K)}$$

$$= \frac{L\not{k}}{2N_K(k_0 - \omega_K)} \tag{4.21}$$

where Eq.(4.16) has been used in the second line. On the other hand, remembering that

$$[(1-a)\not{k} - b\not{k}]_{k_0 \to \omega_K} = (1-a)K\not{k}$$

we can take the limit in Eq.(4.20) and comparing with (4.21) we obtain

$$N_K = \left(\frac{\partial f}{\partial \omega}\right)_{\omega = \omega_K} \ , \tag{4.22}$$

where

$$f \equiv ((1-a)(\omega - K) - b \tag{4.23}$$

Incidentally, notice that the equation for ω_K is equivalent to $f(\omega_K) = 0$ while, in terms of $\bar{f}$ defined below, the equation for $\bar{\omega}_K$ is $\bar{f}(-\bar{\omega}_K) = 0$.

For the antineutrinos, the spinor $v_L(\mathbf{k})$ that appears in Eq.(4.18) is the solution to Eq.(4.8) for

$$k = (-\bar{\omega}_K, -\mathbf{k}) \equiv -\bar{k}$$

Using Eq.(4.12), the equation for $v_L(\mathbf{k})$ is then

$$\not{k}v_L(\mathbf{k}) = 0$$

In analogy with Eq.(4.16) we write

$$v_L\bar{v}_L = \frac{\bar{\omega}_K L\not{k}}{\bar{N}_K} \tag{4.24}$$

and following the same steps that led to Eq.(4.22) we obtain

$$\bar{N}_K = \left(\frac{\partial \bar{f}}{\partial \omega}\right)_{\omega = -\bar{\omega}_K} \ , \tag{4.25}$$

where

$$\bar{f} \equiv (1-a)(\omega + K) - b \tag{4.26}$$

Equations (4.16) and (4.24) give the projection operators that must be used for neutrinos and antineutrinos that appear as external particles in a physical process.

This completes the summary of the formalism that we have used to study the properties of neutrinos in a medium. The rest of this section is devoted to the application of the formalism to several specific calculations.

4.3 Index of refraction in matter

As our first example, we consider a neutrino propagating through a background of electrons. The diagrams that enter in the calculation of the self-energy are shown in Fig. 3. Diagram (a) contributes only to the electron neutrino. Diagram (b) gives an identical contribution to all neutrino species and therefore it can be ignored in the analysis of neutrino oscillations in matter. Diagram (c) contributes only if there are neutrinos in the background, as it is the case in, for example, the early universe.

We assume that the temperature is such that there is no appreciable number of W's and Z's in the background. Thus, their propagators remain the same as in the vacuum. The diagrams give, in addition to the temperature-dependent part, the standard vacuum contribution to the wave function renormalization which, to the order that we are

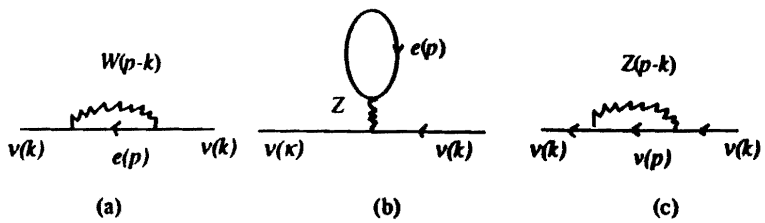

Fig. 3 Diagrams for the neutrino self-energy in a background of electrons.

calculating, can be factored out and has no effect on the background-dependent terms. It is not difficult to deduce that for temperatures and neutrino energies small compared to the W and Z masses, the 1-2 component of the self-energy is proportional to the number density of electrons evaluated at energies of the order of the W and Z masses, which is negligible. Therefore, we restrict ourselves to the calculation of the real part of the self-energy. In this case, according to Eq.(2.53) we need to calculate only the real part of the 1-1 element.

We consider first Diagram (a). Its contribution to the background-dependent part of the self-energy is then

$$-i\Sigma_{eff}^{(a)} = \left(\frac{-ig}{\sqrt{2}}\right)^2 \gamma^\mu L \int \frac{d^4p}{(2\pi)^4} \frac{-i}{(p-k)^2 - m_W^2} iS'_F(p)\gamma_\mu L$$

where S'_F is the temperature-dependent part of the 1-1 element of the electron propagator, which is given by

$$S'_F(p) = (\not{p} + m)2\pi i\delta(p^2 - m^2)\eta_F(p\cdot u)$$

Terms of order $(m_e/m_W)^2$, which in the Feynman-'t Hooft gauge arise from the exchange of unphysical Higgs scalars, have been neglected. To leading order in the inverse W mass, we replace in the W propagator

$$(p-k)^2 - m_W^2 \to -m_W^2$$

Then, using the identity $\gamma^\mu \not{p} \gamma_\mu = -2\not{p}$, we obtain

$$\Sigma_{eff}^{(a)} = \frac{2(g/\sqrt{2})^2}{m_W^2} \int \frac{d^4p}{(2\pi)^3} \not{p} L \delta(p^2 - m^2)\eta_F(p\cdot u)$$

To do the integral, notice that it can depend only on the vector u. Therefore, it must be of the form

$$u^\mu I$$

where

$$I = \int \frac{d^4p}{(2\pi)^3} p\cdot u\,\delta(p^2 - m^2)\eta_F(p\cdot u)$$

We now evaluate this integral in the rest frame of the medium, where $p\cdot u = p_0$. Then,

$$\int dp_0 \delta(p^2 - m^2)\eta_F(p_0) = \frac{1}{2}(f(p) - \bar{f}(p)) ,$$

where $f(\mathbf{p})$ and $\bar{f}(\mathbf{p})$ are the momentum distributions of electrons and positrons respectively, given by

$$f(\mathbf{p}) = \frac{1}{e^{\beta E(\mathbf{p}) - \alpha} + 1} , \bar{f}(\mathbf{p}) = \frac{1}{e^{\beta E(\mathbf{p}) + \alpha} + 1}$$

Thus, we obtain

$$\Sigma_{eff}^{(a)} = \frac{g^2}{m_W^2} \frac{(n_e - \bar{n}_e)}{4} \not{u} L , \tag{4.27}$$

where

$$n_e = 2\int \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p}) \tag{4.28a}$$

$$\bar{n}_e = 2\int \frac{d^3\mathbf{p}}{(2\pi)^3} \bar{f}(\mathbf{p}) \tag{4.28b}$$

give the number densities of electrons and positrons.

In order to calculate the contribution from Diagram (b), we write the piece of the Lagrangian that describes the interaction of the Z with fermions in the form

$$L_Z = -g_Z Z^\mu \left[\sum_{e,\mu,\tau,\ quarks} \bar{f}\gamma_\mu(A + B\gamma_5)f + \bar{\nu}_L \gamma_\mu \nu_L \right]$$

where

$$g_Z = \frac{g}{2\cos\theta_W} ,$$

$$A = I_3 - 2Q\sin^2\theta_W ,$$

$$B = -I_3$$

Therefore, for the electron,

$$A_e = \frac{-1}{2} + 2\sin^2\theta_W ,$$

$$B_e = \frac{1}{2}$$

The contribution to the self-energy from Diagram (b) is

$$-i\Sigma_{eff}^{(b)} = (-ig_Z)^2\gamma^\mu L\int\frac{d^4p}{(2\pi)^4}\frac{-i}{-m_Z^2}(-1)Tr\left[iS'_F(p)\gamma_\mu(A_e + B_e\gamma_5)\right]$$

$$\Sigma_{eff}^{(b)} = \frac{4g_Z^2A_e}{m_Z^2}\int\frac{d^4p}{(2\pi)^4}\not{p}L\delta(p^2-m^2)\eta_F(p\cdot u)$$

$$= \frac{g^2A_e}{4m_W^2}(n_e-\bar{n}_e)\not{u}L \tag{4.29}$$

As already mentioned, Diagram (c) is relevant only if there are neutrinos in the background. In that case its contribution can be obtained from Eq.(4.27) by making obvious substitutions.

For the electron neutrino the self-energy is given by the sum of Eqs.(4.27) and (4.29), while for the other species it is given by Eq.(4.29) alone. In terms of the coefficients a and b introduced in Eq.(4.2)

$$a = 0 \text{ (for all species)},$$

$$b = \frac{g^2}{4m_W^2}(1+A_e)(n_e-\bar{n}_e) \text{ (for } \nu_e) \ ,$$

$$b = \frac{g^2}{4m_W^2}A_e(n_e-\bar{n}_e) \text{ (for } \nu_\mu, \nu_\tau) \ , \tag{4.30}$$

The dispersion relations are simply

$$\omega = K \pm b \tag{4.31}$$

with the + (−) sign holding for the neutrino (antineutrino). In the context of neutrino oscillations in matter, the term proportional to A_e gives a common phase in the solution to the evolution equations, and it is permissible to discard it. If this is done, and we introduce an index of refraction by

$$n = \frac{K}{\omega} \ ,$$

Eq.(4.31) yields, for the electron neutrino,

$$n \cong 1 \quad \frac{b}{\omega} = 1 - \frac{\sqrt{2}G_F}{\omega}(n_e-\bar{n}_e) \ , \tag{4.32}$$

which reproduces the well-known results[12].

This calculation also shows that, to leading order in the inverse W mass, the wave function renormalization factor N_K , given by Eq.(4.22), is unity. Equation (4.31) also illustrates that in a CPT-asymmetric background the dispersion relations for the antineutrino

and antineutrino are different. On the other hand, if the chemical potentials are zero, then $n_e = \bar{n}_e$ and the dispersion relations are the same. Finally it should be noticed that Eq.(4.30) holds for a relativistic or non-relativistic, and a degenerate or non-degenerate electron gas.

4.4 Electromagnetic properties

The effects of the medium can have great influence on the electromagnetic properties of neutrinos. For example, in the vacuum, the electric- and magnetic- dipole operators are odd under CPT. Therefore, the dipole moments of a Dirac neutrino are opposite to those of its antiparticle while the dipole moments of a Majorana neutrino vanish because this particle is its own antiparticle[13]. In a medium the situation can be drastically different[14]. In general, the effect of the medium gives rise to CPT asymmetries in the effective electromagnetic interactions of the neutrinos. These asymmetries manifest themselves as new terms in the effective action which are identical for a particle and its antiparticle, and therefore can be non-zero even for a Majorana neutrino. The conditions under which this can happen were analyzed in Ref.[14]. Apart from qualitative differences such as the one just mentioned, there can be important quantitative differences as well. For example, consider the radiative neutrino decay, which has been studied extensively in the context of various astrophysical and cosmological problems. It is well known that in the vacuum the rate is suppressed by the Glashow-Iliopoulos-Maiani (GIM) mechanism. However, a medium that contains electrons but not muons or taons is not flavor-symmetric. Therefore, in such a medium the background-dependent part of this amplitude is not GIM-suppressed. As a result, the decay rate could be much larger than in the vacuum.

In this section we present a summary of the calculation of the background-dependent part of the $\nu\nu\gamma$ vertex for the case in which the medium consists of a gas of electrons[15]. As usual, the electron gas is assumed to be embedded in a uniform positive-ion background. However, since the background-dependent terms turn out to be proportional to the inverse of the electron mass in the classical and non-relativistic limit, the effect of the ions is negligible in most circumstances.

The off-shell vertex function is defined in such a way that

$$\langle\nu(k')|j_\mu^{em}(0)|\nu(k')\rangle = \bar{u}(k')\Gamma_\mu u(k)$$

The diagrams that enter in the calculation are shown in Fig. 4. We assume as before that the temperature is such that there are no W or Z bosons in the background. Therefore, only the electron propagator has a background-dependent term. The calculation of the background-dependent part of Γ_μ, which we denote by Γ'_μ , is simplified by making the following observations. Since the integrals involved are cut-off by the electron-positron distributions, the diagram marked as (W') gives a contribution which is suppressed by an extra power of $1/m_W^2$ relative to the other two diagrams. Therefore, to leading order in the inverse W mass, only the diagrams marked as (W) and (Z) need to be calculated. In the 't Hooft-Feynman gauge, the diagrams in which the W's are replaced by the unphysical Higgs fields are also suppressed by extra powers of $1/m_W^2$ and therefore do not contribute to Γ'_μ, to leading order in $1/m_W^2$. In addition, this also implies that the leading terms (in $1/m_W^2$) that arise from the diagrams marked (W) and (Z) are gauge invariant. This contrasts with the situation in the vacuum, where the leading term cancels by the GIM mechanism and the next term receives contributions from every diagram.

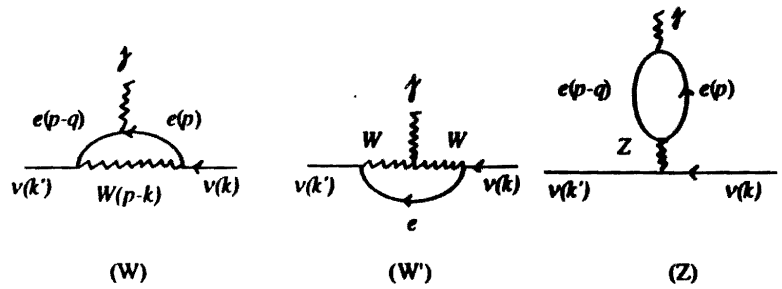

Fig. 4 Diagrams for the neutrino-neutrino photon vertex.

With these preliminaries, our task is reduced to the calculation of the following quantities:

$$-i\Gamma_\mu^{(W)} = \frac{eg^2}{2}\int \frac{d^4p}{(2\pi)^4}\,\gamma^\alpha L i S_F(p-q)\gamma_\mu i S_F(p)\gamma_\alpha L \frac{1}{(k-p)^2 - m_W^2} \tag{4.33a}$$

$$-i\Gamma_\mu^{(Z)} = \frac{eg_Z^2}{q^2 - m_Z^2}\gamma^\alpha L \int \frac{d^4p}{(2\pi)^4}\, Tr\left[iS_F(p-q)\gamma_\mu i S_F(p)\gamma_\alpha(A_e + B_e\gamma_5)\right] \tag{4.33b}$$

Notice that the coupling of the electron neutrino to photons is given by the sum of these two terms, while only the second one contributes to the other flavors of neutrinos. In view of what was argued above, we retain only the leading term in powers of $1/m_W{}^2$. Therefore, we will neglect the momentum dependence of the W and Z propagators.

We will calculate only the real part of the amplitude; thus, the electron propagator to be used is the 1-1 element. When it is substituted in Eq. (4.33), several terms are produced. The terms that are independent of the electron and positron distributions contribute only to the standard vacuum part and we drop them. The rest of the terms contain either one or two factors of the distribution functions. However, those involving two factors of the distribution functions contribute only to the imaginary part of the amplitude, and we also drop them. Therefore, we retain only those terms involving one factor containing the distribution functions. In addition, the integrals are interpreted as their principal-value part.

The expression in Eq. (4.33a) can be rewritten using the Fierz-type identity

$$\gamma^\alpha L A \gamma_\alpha L = -Tr(A\gamma_\alpha L)\gamma^\alpha L \tag{4.34}$$

which is valid for any 4 x 4 matrix (see Appendix B). We can then write both expressions in Eq. (4.33) in the form

$$\Gamma_\mu^{(W,Z)} = T_{\mu\nu}^{(W,Z)}\gamma^\nu L\ , \tag{4.35}$$

where

$$T_{\mu\nu}^{(Z)} = \frac{eg_Z^2}{m_Z^2}\int \frac{d^4p}{(2\pi)^3} Tr\left[(\not p - \not q + m)\gamma_\nu(\not p + m)\gamma_\mu(A_e + B_e\gamma_5)\right]$$

$$x\left[\frac{\delta[(p-q)^2 - m^2]\pi[(p-q)\cdot u]}{p^2 - m^2} + \frac{\delta(p^2 - m^2)\pi(p\cdot u)}{(p-q)^2 - m^2}\right] \tag{4.36}$$

and $T_{\mu\nu}^{(W)}$ is given by an expression that is obtained from this by making the following replacements:

$$\frac{g_Z^2}{m_Z^2} \to \frac{g^2}{2m_W^2},\ A_e \to \frac{1}{2},\ B_e \to -\frac{1}{2} \tag{4.37}$$

We henceforth concentrate on the evaluation of $\Gamma_\mu^{(Z)}$ only. The results for $\Gamma_\mu^{(W)}$ can be recovered at any stage by using the substitutions given in Eq. (4.37).

Making the change of variable $p \to p + q$ in the first integral of Eq. (4.36) and carrying out the traces, we then obtain

$$T_{\mu\nu}^{(Z)} = \frac{4eg_Z^2}{m_Z^2}\int \frac{d^3\mathbf{p}}{(2\pi)^3 2E}\left\{A_e(f + \bar f)\left[\frac{2p_\mu p_\nu + (p_\mu q_\nu + q_\mu p_\nu) - g_{\mu\nu}p\cdot q}{q^2 + 2p\cdot q} + (q \to -q)\right]\right.$$

$$\left. -B_e(f - \bar f)i\epsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta\left[\frac{1}{q^2 + 2p\cdot q} + (q \to -q)\right]\right\}\ , \tag{4.38}$$

where

$$p^\mu = (E, \mathbf{p})\ ,\ E = \sqrt{\mathbf{p}^2 + m^2}\ .$$

The remainder of the calculation can be conveniently organized by making use of the following observations. Electromagnetic gauge invariance implies

$$q^\mu T_{\mu\nu} = 0$$

From Eq. (4.38) it is easily verified that both $T_{\mu\nu}^{(W)}$ and $T_{\mu\nu}^{(Z)}$ separately satisfy this equation. However, from Eq. (4.38) we also see that both of these tensors have the additional property that they are symmetric under the simultaneous interchange

$$\mu \leftrightarrow \nu,\ q \to -q$$

Therefore, $T^{\mu\nu}$ also satisfies

$$q^\nu T_{\mu\nu} = 0$$

Moreover, from Eq. (4.36) or (4.38) we can see that with the approximation of keeping only the leading terms in $1/m_W^2$, $T_{\mu\nu}$ does not depend on k and k' separately, but is a function only of q. The most general form of $T_{\mu\nu}$ that is consistent with all these properties is

$$T_{\mu\nu} = T_T R_{\mu\nu} + T_L Q_{\mu\nu} + T_P P_{\mu\nu} \tag{4.39}$$

where

$$R_{\mu\nu} = \tilde{g}_{\mu\nu} - Q_{\mu\nu} ,$$

$$Q_{\mu\nu} = \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2} ,$$

$$P_{\mu\nu} = \frac{i}{Q} \varepsilon_{\mu\nu\alpha\beta} q^\alpha u^\beta . \tag{4.40}$$

We have defined

$$\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$$

and

$$\tilde{u}_\mu \equiv \tilde{g}_{\mu\nu} u^\nu$$

Further, the coefficients T_T, T_L, T_P, are functions of the variables

$$\Omega \equiv q \cdot u ,$$

$$Q \equiv \sqrt{\Omega^2 - q^2} , \tag{4.41}$$

the latter of which also appears in the definition of $P_{\mu\nu}$ in Eq. (4.40).

The tensors $R_{\mu\nu}$, $Q_{\mu\nu}$, $P_{\mu\nu}$, have the useful property that when one of them is contracted with the others the result is zero, while

$$R^{\mu\nu} R_{\mu\nu} = 2 ,$$

$$Q^{\mu\nu} Q_{\mu\nu} = 1 ,$$

$$P^{\mu\nu} P_{\mu\nu} = -2 . \tag{4.42}$$

The form factors $T_{T,L,P}$ can thus be obtained by projecting as follows:

$$T_T = \frac{1}{2} R_{\mu\nu} T^{\mu\nu} = \frac{1}{2}\left[T^\mu{}_\mu - \frac{u^\mu u^\nu}{\tilde{u}^2} T_{\mu\nu} \right] ,$$

$$T_L = Q_{\mu\nu} T^{\mu\nu} = \frac{u^\mu u^\nu}{\tilde{u}^2} T_{\mu\nu} ,$$

$$T_L = \frac{-1}{2} P_{\mu\nu} T^{\mu\nu} . \tag{4.43}$$

Applying this to the expression for $T_{\mu\nu}$ given in Eq. (4.38), we obtain

$$T_T^{(Z)} = \frac{2 e g_Z^2 A_e}{m_Z^2} \left[A - \frac{B}{\tilde{u}^2} \right] ,$$

$$T_L^{(Z)} = \frac{4 e g_Z^2 A_e}{m_Z^2} \frac{B}{\tilde{u}^2} ,$$

$$T_P^{(Z)} = \frac{-4 e g_Z^2 B_e}{m_Z^2} QC , \tag{4.44}$$

where

$$A = \int \frac{d^3p}{(2\pi)^3 2E} (f + \bar{f}) \left[\frac{2m^2 - 2p \cdot q}{q^2 + 2p \cdot q} + (q \to -q) \right] ,$$

$$B = \int \frac{d^3p}{(2\pi)^3 2E} (f + \bar{f}) \left[\frac{2(p \cdot u)^2 + 2(p \cdot u)(q \cdot u) - p \cdot q}{q^2 + 2p \cdot q} + (q \to -q) \right] ,$$

$$C = \int \frac{d^3p}{(2\pi)^3 2E} (f - \bar{f}) \left(\frac{p \cdot \tilde{u}}{\tilde{u}^2} \right) \left[\frac{1}{q^2 + 2p \cdot q} + (q \to -q) \right] . \tag{4.45}$$

The expressions for $T_{T,L,P}^{(W)}$ can be obtained from Eq. (4.44) by making the substitutions indicated in Eq. (4.37).

The integrals in Eq. (4.45) must be interpreted as their principal value, as mentioned before. The convenience of expressing the form factors in this form lies in the fact that A, B, and C are scalars, so that the integrations can be performed in any reference frame, in particular in the rest frame of the medium. In an arbitrary frame the distribution functions are expressed in the form

$$f = \frac{1}{e^{\beta p \cdot u - \alpha} + 1} , \bar{f} = \frac{1}{e^{\beta p \cdot u + \alpha} + 1} \tag{4.46}$$

If we denote the components of p^μ in the rest frame of the medium by

$$p^\mu = (\mathcal{E}, \mathcal{P}) ,$$

then the expressions for f and $\bar{f}$ become simply

$$f = \frac{1}{e^{\beta\mathcal{E}-\alpha}+1}, \bar{f} = \frac{1}{e^{\beta\mathcal{E}+\alpha}+1} \tag{4.47}$$

In this fashion it is straightforward to carry out the integrals for A, B, C. This has been done in Ref. (15) in considerable detail. In addition, Ref. (15) contains several applications of these calculations, such as as a very simple rederivation of the known formulas for the plasmon decay rate into neutrinos. Among other things, those examples serve to show the way to calculate other, similar decay processes in a medium. One particularly interesting case is the neutrino radiative decay, to which we now turn our attention.

4.5 Neutrino radiative decay

In this section we calculate the rate for the radiative decay of a heavier neutrino to a lighter one[16]:

$$\nu(k) \to \nu'(k') + \gamma(q)$$

We begin the calculation by assuming that the dispersion relations of the particles are not significantly affected by the medium, so that we can use the relations valid in the vacuum. In order to be self-consistent, we should then take the wave-function renormalization factors associated with the external lines equal to unity. In effect these assumptions imply that we consider only the transverse polarization states of the photon. In addition, we assume that the mass of the final neutrino vanishes, or at least that it is so small compared to the mass of the decaying neutrino, that it can be neglected.

The matrix element for the process can be written as

$$iM = -i\bar{u}(k')\Gamma'_\alpha u(k)e^{\alpha *}(q),$$

where $u(k)$ denotes the Dirac spinor with momentum k and e^α denotes the polarization vector of the photon. We have used the primes on M and Γ to indicate that we will consider their background-dependent part only. Following the notation of Eq. (4.35) we can write

$$\Gamma'_\alpha = U^*_{e\nu}U^*_{e\nu}T_{\alpha\beta}\gamma^\beta L,$$

where U is the lepton mixing matrix. As already mentioned, we assume that the background contains electrons but not muons or taons, nor any W bosons for that matter. As discussed in the previous section, to the lowest order in $1/m_W^2$ the most general form of $T_{\alpha\beta}$ is

$$T_{\mu\nu} = T_T R_{\mu\nu} + T_L Q_{\mu\nu} + T_P P_{\mu\nu}$$

The quantities $T_{T,L,P}$ are given in terms of the integrals A, B, C defined in Eq. (4.45). In the present case we are interested in the value of these functions for

$$q^2 = 0;\ \Omega = Q$$

Since

$$\tilde{u}^2 = -\frac{\mathbf{q}^2}{q^2},$$

it is easy to see from Eq. (4.45) that $C = 0$ when $q^2 = 0$. The expression for B in Eq. (4.45) shows that, while it is not zero when $q^2 = 0$, it has a finite value. Since B appears in the formulas for $T_{T,L,P}$ in the combination $B/\tilde{u}^2$, it does not contribute in Eq. (4.44). Therefore, the non-zero form factor in the present case is

$$T_T = -\frac{eg^2}{2m_W^2}\int\frac{d^3p}{(2\pi)^3}[f(p) + \bar{f}(p)] \tag{4.48}$$

where

$$p^\mu = (E,\mathbf{p}) \text{ with } E = \sqrt{\mathbf{p}^2 + m^2},$$

and $f, \bar{f}$ are given in Eq. (4.46) and (4.47). Using

$$g^2/8m_W^2 = G_F/\sqrt{2}$$

we thus obtain

$$T_T^{(NR)} \cong -\sqrt{2}eG_F n_e/m_e \tag{4.49}$$

in the case of a non-relativistic background of electrons, and

$$T_T^{(ER)} \cong -eG_F T^2/3\sqrt{2} \tag{4.50}$$

in the extremely relativistic case, when the temperature is much greater than the electron mass, so that positrons are also present. The square of the matrix element can be evaluated easily using the polarization sum

$$\sum_{\lambda = \text{transverse modes}} e^*_\alpha(q\lambda)e_\beta(q\lambda) = -R_{\alpha\beta}$$

Thus,

$$|M'|^2 = m^2|U^*_{e\nu}U_{e\nu}|\,|T_T|^2\left[\frac{(k+k')\cdot u}{\Omega} - \frac{m^2}{2\Omega^2}\right], \tag{4.51}$$

where m is the mass of the decaying neutrino. The differential decay rate in any frame is given by

$$d\Gamma' = \frac{1}{2k_0}(2\pi)^4\delta^4(k-k'-q)|M'|^2\frac{d^3k'}{(2\pi)^3 2k'_0}\frac{d^3q}{(2\pi)^3 2q_0}$$

If the background contains photons we should put a factor of

$$1 + \frac{1}{e^{\beta q\cdot u} - 1}$$

on the right-hand side of this formula to take into account the stimulated emission of photons. Since this factor is always larger than unity, this will increase the rate further. Similarly, if the background contains neutrinos, we should also include the Pauli-blocking factor on the right-hand side. In the presence of both, neutrinos and photons, the inverse process must also be taken into account. In our estimates we have neglected all these effects.

The quantity of interest is the decay rate in the rest frame of the medium, where $u_\mu = (1,0)$. The integration can be carried out in terms of the momentum variables of that frame, in which we denote by v the three-velocity of the decaying neutrino, or in terms of the variables in the rest frame of the decaying neutrino, in which case we have to boost the velocity 4-vector of the medium. The result is of the integration is

$$\Gamma' = \frac{m}{16\pi}\left|U_{e\nu}^{*}U_{e\nu'}\right|\left|T_T\right|^2 F(v) \tag{4.52}$$

where the function F is given by

$$F(v) = (1 - v^2)^{1/2}\left[\frac{2}{v}\ln\left(\frac{1+v}{1-v}\right) - 3\right] . \tag{4.53}$$

The crucial difference between Eq. (4.52) and the formula for the decay rate in the vacuum[17] is that the latter contains the GIM suppression factor

$$\left|\sum_{l=e,\mu,\tau}\frac{m_l^2}{m_W^2}U_{l\nu}^{*}U_{l\nu'}\right|^2$$

while Eq. (4.52) is free from that suppression. This is just a consequence of the fact that the GIM suppression mechanism is not operative in ordinary matter. A more detailed comparison of the two rates is given in Ref. (16).

Conclusions

In these lectures I presented the approach that we have used to understand some of the properties of particles as they propagate through a background medium, with focus on the case of neutrinos. The application of Finite Temperature Field Theory to this class of problem provides a well defined setting to study general questions and gives efficient methods for carrying out practical calculations.

I wish to thank Arnulfo Zepeda, José Luis Lucio and the organizers of the school for inviting me to give these lectures, and my friends Palash Pal and Juan Carlos D'Olivo for the collaborations that have led to most of the material that I have presented here.

Appendix A

For illustrative purposes let us consider the consequences of CP invariance for the self-energy of a scalar field. Denoting the CP operator by θ, the transformation of a scalar field is

$$\theta^{-1}\phi(x)\theta = \eta\phi^{*}(\Lambda^{-1}x)$$

where $\Lambda = \mathrm{diag}(1,-1,-1,-1)$. We assume that the chemical potentials are zero, so that Eq. (4.4) holds. Then by taking the trace of the identity

$$Z\phi^{*}(y)\phi(x) = \theta^{-1}Z(\Lambda u)\phi(\Lambda y)\phi^{*}(\Lambda x)\theta$$

we obtain

$$\Delta_{12}(x-y;\,u) = \Delta_{21}(\Lambda y - \Lambda x;\,\Lambda u)\,,$$

where we have made explicit the dependence on the vector u_μ, and we have omitted the prime and the subscript F in the exact propagator. Since

$$\Delta_{11}(x-y) = \Delta_{21}(x-y)\theta\left[(x-y)\cdot u\right] + \Delta_{12}(x-y)\theta\left[(y-x)\cdot u\right],$$

it follows that

$$\Delta_{11}(x-y;\,u) = \Delta_{11}(\Lambda y - \Lambda x;\,\Lambda u)\,,$$

and a similar relation also holds for Δ_{22}. Therefore, CP invariance implies that the propagator matrix satisfies

$$\Delta_{ab}(x-y;\,u) = \Delta_{ba}(\Lambda y - \Lambda x;\,\Lambda u)$$

or, in momentum space,

$$\Delta_{ab}(k) = \Delta_{ba}(-k)$$

The inverse propagator must satisfy a similar relation and, from Eq. (2.44), also the self-energy matrix. Thus,

$$\pi_{ab}(-k) = \pi_{ba}(k)$$

which in turn implies

$$\pi(-k) = \pi(k)$$

From the relation between π and π_{eff} given in Eq. (3.11) we finally obtain that, if CP invariance holds, then

$$\pi_{eff}^{*}(-k) = \pi_{eff}(k)$$

Similar considerations for the fermion case lead to Eq. (4.5).

Appendix B

Let ψ_1 and ψ_2 be two spinors, and let M be any 4 x 4 matrix. Consider the matrix element

$$A \equiv \bar{\psi}_2 \gamma_\alpha L M \gamma^\alpha L \psi_1$$

If we denote by e_i a set of basis spinors, then

$$A = \sum_{i=1,4} (\bar{\psi}_2 \gamma_\alpha L M e_i)(e_i^T \gamma^\alpha L \psi_1)$$

$$= -\sum_{i=1,4} (\bar{\psi}_2 \gamma_\alpha L \psi_1)(e_i^T \gamma^\alpha L M e_i)$$

In the second line we have used the standard Fierz identity

$$(\bar{\psi}_2 \gamma_\alpha L \phi)(\bar{\chi} \gamma^\alpha L \psi_1) = -(\bar{\psi}_2 \gamma_\alpha L \psi_1)(\bar{\chi} \gamma^\alpha L \phi),$$

with

$$\phi = M e_i, \; \bar{\chi} = e_i^T$$

Now A can be written in the form

$$A = -(\bar{\psi}_2 \gamma_\alpha L \psi_1) \sum_{i=1,4} Tr(e_i e_i^T \gamma^\alpha L M)$$

$$= -(\bar{\psi}_2 \gamma_\alpha L \psi_1) Tr(\gamma^\alpha L M)$$

Since this is valid for any two spinors $\psi_{1,2}$, we have the matrix relation given in Eq. (4.34).

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16. We follow the discussion given in the third paper cited in Ref. (11).
17. See e.g., P. B. Pal and L. Wolfenstein, Phys. Rev. D **25** (1982) 766, and references therein.