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Fourier transforms on groups and quantum relativistic kinematics. II. Description of vacuum

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Abstract

Assuming that vacuum has energy-momentum zero, we describe it in terms of particles. These particles are virtual ; some of them have negative energies ; their states are global states, which generalize those described in paper (I) because their statistical operators have infinite trace. Any system of vacuum particles is a part of a *keneme*, i.e. of a system of n particles which can, without violating the conservation laws, annihilate in the strict sense of the word (transform into nothing). The kenemes will first be studied for themselves. A keneme is a *homogeneous system*, i.e. its state is invariant by all transformations of the invariance group. But a homogeneous system is not necessarily a keneme. In the simple case of a spin system, where the invariance group is $SU(2)$, a homogeneous system is a system whose total spin is unpolarized ; a keneme is a system whose total spin is zero. The state of a homogeneous system is described by a statistical operator with infinite trace (von Neumann), to which corresponds a characteristic distribution. Then one studies vacuum. The characteristic distributions of the homogeneous systems of vacuum are defined ; their elementary properties are studied.

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I. Introduction

In the present article, the results of paper [1] are applied to the description of vacuum. (In the following that paper will be denoted by (I)).

We shall assume that vacuum contains particles, and that they belong to systems whose total energy-momentum (and other quantum numbers) are zero. Hence these systems contain particles with negative energy. The particles of vacuum are virtual particles ; their states are global states, which generalize those described in (I) because the statistical operators have infinite traces.

The central notion is that of *keneme*, i.e. of a system of particles which can annihilate in the strict sense of the word. Any subsystem of a keneme is a *homogeneous system*, i.e. its state is invariant by the transformations of the invariance group. Hence the density of the statistical operator which describes a particle belonging to a keneme is a multiple of the identity operator ; therefore it is not trace class.

This is why we have to take again the notion, due to von Neumann [2], of a relative statistical operator, which allows to compute relative probabilities. To give this notion a more precise meaning we use the analogy between quantum mechanics and probability theory. Rényi [3, 4] has generalized probability theory to make it able to define relative probabilities, which he calls *conditional probabilities* (in a sense which is a generalization of the usual one). Analogously, we define conditional statistical operators and we show that one may attach to them *characteristic distributions*, which are a generalization of the characteristic functions defined in (I). The characteristic distributions of homogeneous systems and of kenemes are studied, and the relations between those two types of systems are specified.

Then, in Section VI, the results achieved are applied to the study of the characteristic distributions which describe vacuum.

II. Vacuum as a state of zero energy-momentum

A. The energy-momentum of vacuum

In quantum field theory, one describes most often vacuum as a state of energy-

momentum zero [5]. When cosmological considerations bursted into particle theory this definition was challenged. Thus Zeldovich [6] remarked that the case of a vacuum with energy-momentum tensor $T_{\mu\nu}$ proportional to the metrical tensor $g_{\mu\nu}$ cannot be excluded *a priori* ; such a vacuum of course has infinite energy. Futhermore in quantum field theory, if one seeks to unify gravitation with other interactions between particles it becomes difficult to put equal to zero the energy of vacuum, because on a riemannian manifold one cannot define normal products of field operators [7]. On the other hand, if one takes literally the $(1/2)h\nu$ of the electromagnetic field oscillators one gets for the energy of vacuum a huge value, which should bring about gravitational effects ; the latter are not observed. To compensate these effects, one may introduce into the equation of general relativity a cosmological constant. But the value which should be given to this constant is some 120 orders of magnitude larger than that which is compatible with astronomical observations. Otherwise stated, to be in agreement with experimental and observational data the two terms of the effective cosmological constant should cancel to about 41 decimal places, or even much more (see Weinberg [8] ; for elementary discussions, see references [9] and [10]).

We shall leave these problems aside, and we shall consider that all the vacuum particles belong to systems whose energy-momentum, as well as other quantum numbers, are zero.

B. Negative energy particles

If one is willing to give a particle description of a vacuum with zero energy-momentum, one immediately meets the fact that some particles must have negative energy. But the energies of observed particles are always positive ; why, then, do particles with negative energies remain confined in vacuum ? Here we shall not answer to this question. It should be remarked, however, that even in quantum field theory it is not possible to have all energies positive : for a free scalar field, the energy P_0 is positive, but the component T_{00} of the tensor density of energy-momentum is not positive definite. The same result holds for any field which satisfies the usual postulates of local field theory [11].

C. Kenemes

In a system of particles with zero total energy-momentum, there are particles with both signs of energy. We shall use the notion of a *keneme* (from the Greek *kenos*, empty) : it is a system of particles which can transform into "nothing". If the particles $A_1, \cdots, A_n$ form a keneme, the two inverse processes

$$\emptyset \to A_1 + A_2 + \cdots + A_n \to \emptyset \tag{1}$$

are possible, i.e. they are compatible with the conservation laws. The symbol $\emptyset$ stands for "nothing".

From the notion of keneme emerges an intuitive one for crossing, independent of field theory. To cross a particle is to take it from one side to the other one in one of the equations (1), and to apply to its state the operation PCT. (Here T is not the anti-unitary operator defined by Wigner, but the naïve time reversal, represented by a unitary operator ; it reverses the sign of energy. This is why we stated ((I), Section II.A) that to add T to the invariance group does not set any special problem). The physical idea is that if vacuum contains the keneme (1), all reactions which derive from (1) by crossing are possible. Such is the case, for instance, with the process

$$\bar{A}_1 + \cdots + \bar{A}_p \to A_{p+1} + \cdots + A_n \quad , \tag{2}$$

where $\bar{A}_i$ is deduced from A_i by crossing.

III. The problem of the description of vacuum particles

By definition, the vacuum contains no real particle. Vacuum particles show themselves only by their interactions with real particles, which means that they are virtual particles.

How can one describe the state of a virtual particle ? The Landau-Lifchitz treatise [12] defines a virtual particle as an intermediate state of a process. They are concerned with a process whose graph can be cut into two disjoint parts by cutting a single line.

Let us consider for instance, in quantum electrodynamics, the two second order graphs which describe negaton-positon scattering. The Bhabha type graph describes a virtual annihilation (in the usual sense) of the pair. The state of the virtual photon is the global state of the incoming (or outgoing) negaton-positon pair. For the other graph, the photon has a spacelike energy-momentum vector ; if we cross, for instance, the outgoing negaton line, we get an ingoing positon line and again, the state of the virtual photon is the global state of the pair constituted by the ingoing negaton and the ingoing positon obtained by crossing. (To cross still means to apply PCT, see above).

We have seen ((I), Section V.B) that the global characteristic function is conserved in a transformation process. Hence if a virtual particle is the intermediate state of a process, its state is the global state of the ingoing (or outgoing) particles. We shall assume that the state of a virtual particle can always be characterized thus. Summarizing : *the vacuum particles are virtual particles, and their states are global states.*

Now if we describe vacuum with one particle statistical operators, two particle statistical operators, etc., in analogy with statistical mechanics, what has just been said implies that these statistical operators will describe global particles.

The one-particle state of vacuum will be described by an operator-valued measure of the form

$$W = \int_{\widehat{G}}^{\oplus} \rho_\chi \, d\chi \quad . \tag{3}$$

(see (I), equation (25)).

How does the operator W transform by an element g of the invariance group G ? It becomes the operator

$$W' = \int_{\widehat{G}}^{\oplus} \rho'_\chi \, d\chi \quad , \tag{4}$$

with

$$\rho'_\chi = U_{\chi,g} \, \rho_\chi \, U_{\chi,g^{-1}} \quad . \tag{5}$$

The invariance of vacuum by the elements of G now implies that for almost any χ one has

$$\rho'_\chi = \rho_\chi \quad . \tag{6}$$

Hence for almost any $\chi \in \widehat{G}$ and for any $g \in G$, ρ_χ commutes with $U_{\chi,g}$. This implies (by Schur's lemma) that ρ_χ is almost everywhere a multiple of the identity operator :

$$\rho_\chi = \sigma_\chi \, I_\chi \tag{7}$$

where I_χ denotes the identity operator on the space $\mathcal{H}_\chi$, and σ_χ is a scalar.

As for almost any χ the space $\mathcal{H}_\chi$ has infinite dimension, the operator-valued measure is not trace class (see (I), Section II.B), hence it has no inverse Fourier-Stieltjes transform, at least in the sense defined in (I).

We must therefore answer two questions :

1) Can one give a physical meaning to an operator-valued measure of the form (3), when the operator ρ_χ is not trace class ?

2) Can one define, in a more general sense than that given in (I), the inverse Fourier-Stieltjes transform of an operator-valued measure, when it is not trace class ?

IV. Conditional statistical operators

A. Conditional random variables and characteristic distributions

To answer the first question we shall draw our inspiration from Rényi's solution [3] of the analogous problem in classical theory of probabilities. Here the aim was for instance to be able to define a random variable uniformly distributed on the real line, which is impossible in the classical theory of probabilities.

A real random variable is a measurable mapping $X : \Omega \to \mathbf{R}$ of a probability space $(\Omega, \mathcal{G}, P)$ into $\mathbf{R}$. To the random variable X there corresponds a measure μ on $\mathbf{R}$, defined thus : for any Borel part B of $\mathbf{R}$, one has

$$\mu(B) = P\left[X^{-1}(B)\right] \quad .$$

Henceforth we shall forget the initial probability space ; our only concern will be the probability space $(\mathbf{R}, \mathcal{A}, \mu)$, where $\mathcal{A}$ stands for the σ-algebra of all Borel parts

of $\mathbf{R}$.

In the usual probability theory μ is a finite measure, which can be normalized if one puts $\mu(\mathbf{R}) = 1$. Rényi's generalization is concerned with the case where μ is not necessarily finite, but must be σ-finite (this means that $\mathbf{R}$ is a countable union of parts with finite measure).

Let $\mathcal{B}$ be the set of Borel parts of R whose measure μ is finite and non-zero. Such a part will be called an *admissible condition.*

Let B be an admissible condition. For any $A \varepsilon \mathcal{A}$, let us put

$$P(A|B) = \mu(AB)/\mu(B) \quad , \tag{8}$$

where AB stands for the intersection of A and B. $P(A|B)$ will be called the *probability of A, conditioned by B.* The triplet $S = (\mathbf{R}, \mathcal{A}, \mu)$, where μ is a σ-finite measure, will be called the *complete conditional probability space* defined by the measure μ. (See Rényi [4], definition 2.2.4). The measures μ and $c\mu$ $(c > 0)$ define the same space.

If μ is a finite measure, S is a usual probability space and $P(A|B)$ is the usual conditional probability.

Let B be an admissible condition. For any A, let us write

$$P_B(A) = P(A|B) \quad . \tag{9}$$

Then P_B is a usual probability on $\mathbf{R}$. The (usual) probability space $(\mathbf{R}, \mathcal{A}, P_B)$, denoted by $S|B$, is called the restriction of S to B. In $S|B$ the event B is certain, and the events incompatible with B are impossible.

Thus it turns out that a conditional probability space S can be considered as a family of usual probability spaces $S|B$; one gets the whole family when B runs through the set $\mathcal{B}$ of all admissible conditions. An important property of this family is that it is compatible, in the following sense. Let C be an admissible condition. For all spaces $S|B$ such as $C \subset B$, the probability of A conditioned by C, computed according to the usual formula (recall that P_B is a usual probability on $S|B$)

$$P_B(A|C) = P_B(AC)/P_B(C) \quad , \tag{10}$$

has the same value ; it is equal to $P(A|C)$, computed according to definition (8).

For a usual random variable one defines a characteristic function which depends only on the probability space $(\mathbf{R}, \mathcal{A}, \mu)$. Similarly, for a conditional random variable one defines a family of conditional characteristic functions. The expectation value of the conditional random variable X, conditioned by B, is by definition

$$< X|B >= \int_{-\infty}^{+\infty} x dP_B(x) \quad .$$

Then the conditional characteristic function of X, conditioned by B, is defined as the function ϕ_B :

$$\phi_B(t) =< e^{itX}|B > \quad . \tag{11}$$

Now for a conditional random variable one may also define a single *characteristic distribution.* Rényi (reference [3], chap. 26, Section 9) defines it directly from the measure μ, in analogy to the definition of a usual characteristic function :

$$F(t) = \mathcal{F}^{-1}(\mu)(t/2\pi) \tag{12}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. (It is known that the distribution thus defined exists provided μ is a measure with slow growth, i.e. the product of a finite measure by a polynomial ; see Schwartz [13]).

But the distribution F can also be defined as the limit of a sequence of functions. To do this, let us first notice that the conditional characteristic function can be written (using its definition (11) and the definition (8)-(9) of P_B) :

$$\phi_B(t) = [\mu(B)]^{-1} \int_B e^{itx} \, d\mu(x) \quad . \tag{13}$$

Let us now consider a family $\{B_n\}$ of admissible conditions such as

$$\text{if } n > k \quad , \qquad B_n \supset B_k \tag{14.a}$$

and

$$\sum_{n=1}^{+\infty} B_n = \mathbf{R} \quad . \tag{14.b}$$

One might believe that the distribution F is the limit of the sequence of functions ϕ_{B_n}. Such is not the case, however. Let us rather consider the "unnormalized characteristic functions" F_n :

$$F_n(t) = \mu(B_n)\phi_{B_n}(t) = \int_{B_n} e^{itx}\, d\mu(x) \quad . \tag{15}$$

Then F turns out to be the limit, in the space $\mathcal{S}'$ of tempered ditributions, of the sequence F_n :

$$F = \lim_{(\mathcal{S}')} F_n \tag{16}$$

B. Conditional statistical operators

1) Let $\mathcal{H}$ be the Hilbert space attached to a quantum system, and let $\mathcal{A}$ be the set of all projectors on $\mathcal{H}$. A usual statistical operator is a positive trace class operator on $\mathcal{H}$. Let $\mathcal{L}(\mathcal{H})$ be the von Neumann algebra of all bounded operators on $\mathcal{H}$. As shown by the formula $\phi(T) = \mathrm{Tr}(WT)$, a statistical operator defines a positive linear form on $\mathcal{L}(\mathcal{H})$. According to Dixmier (reference [14], chap. I, section 4, theorem 1 and exercise 9), this form ϕ has the following property : for any countable family $\{E_i\}$ of pairwise orthogonal projectors , one has

$$\phi(\Sigma E_i) = \Sigma\phi(E_i) \quad . \tag{17}$$

This property, called complete additivity, is analogous to the σ-additivity of probability measures. The analogy is a precise one, since in quantum mechanics the projectors represent properties of the system and correspond to the events of probability theory.

2) In the simplest case a quantum-mechanical state is represented by a statistical operator W. Henceforth we shall not assume that our statistical operators are normalized. (Recall that in (I) we have represented the states by operator-valued measures on the dual of the invariance group ; such a measure gives statistical operators $W(K)$ which are not normalized). To any projector $E \in \mathcal{A}$, the state represented by W assigns the probability

$$P(E) = [\mathrm{Tr}W]^{-1}\ \mathrm{Tr}[WE] \quad . \tag{18}$$

We shall use a result of the theory of measurement in quantum mechanics [15, 16] : if the state of a quantum system is represented by the (unnormalized) statistical

operator W, and if a measurement performed on the system gives the result that the property represented by the projector B is true, then the state of the system after the measurement is represented by the statistical operator

$$W_B = BWB \quad . \tag{19}$$

The notation W_B stresses the analogy between measurement and conditioning : the operator W_B allows to compute probabilities related to the system of interest, conditioned by the fact that property B is true.

We may now carry over to quantum mechanics Rényi's generalization of probability theory. Let W be a bounded positive operator, whose trace is not necessarily finite. (The assumption analogous to the σ-finiteness of the measure μ in probability theory is automatically fulfilled, because in quantum mechanics one deals only with Hilbert spaces with countable basis). Let $\mathcal{B}$ be the set of all the projectors such that the trace of W_B be finite and non zero. We shall call them *admissible conditions.* For any property $A \in \mathcal{A}$, and for any admissible condition $B \in \mathcal{B}$, the operator W allows one to compute the probability of A conditioned by B :

$$P(A|B) = [\mathrm{Tr}(BWB)]^{-1}\ \mathrm{Tr}(BWBA) \quad . \tag{20}$$

This probability derives from the operator W_B, which will again be defined by equation (19) - now extended to the case where W has no longer necessarily a finite trace. Hence we may write instead of equation (20) :

$$P(A|B) = [\mathrm{Tr}(W_B)]^{-1}\ \mathrm{Tr}(W_B A) \quad . \tag{21}$$

Instead of giving the operator W, we might characterize the state defined by W by the family of all conditioned operators W_B ; one gets the whole family when B runs through the whole set $\mathcal{B}$ of admissible conditions.

Proceeding with our analogy, let us show that the family $\{W_B, B \in \mathcal{B}\}$ is compatible. Recall that there exists an order relation between projectors, denoted by $C \leq B$; it can be defined by the inclusion of images, $\mathrm{Im}C \subset \mathrm{Im}B$. This relation is equivalent to the property : B-C is a projector ; it means that property C implies property B. It is also equivalent to the property

$$BC = CB = C \quad . \tag{22}$$

Now let C be an admissible condition which implies the admissible condition B. Since W_B is a statistical operator, we may compute the conditional probability $P_B(A|C)$ according to equation (20) :

$$P_B(A|C) = [\mathrm{Tr}\,(CW_BC)]^{-1}\ \mathrm{Tr}\,(CW_BCA) \quad . \tag{23}$$

By using equation (22) we get

$$P_B(A|C) = P(A|C) \quad . \tag{24}$$

We see thus that the probability of A conditioned by C, computed according to equation (23) with all conditional operators W_B such that $C \leq B$, has always the same value, which is simply $P(A|C)$ computed according to equation (21).

3) All what has just been said - about passing from usual statistical operators (trace class operators on $\mathcal{H}$) to statistical operators which are not necessarily trace class - could be repeated word for word about passing from statistical operators of the form

$$W = \int_{\widehat{G}}^{\oplus} \rho_\chi \, d\chi \tag{25}$$

(see (I), equation (25)), where ρ_χ is almost everywhere trace class, to operators of the same form where ρ_χ is not necessarily trace class. One has just to replace everywhere the trace by the integral of the trace over the dual, as in equation (24) of (I).

Summarizing, we have given a sense to operator-valued measures on the dual $\widehat{G}$ (Dirac measures or measures with a density), in the case where the operators are still positive but not necessarily trace class.

C. Characteristic distributions

Let us now define the characteristic distribution of such an operator-valued measure. We have first the general result of Bonnet (reference [17], proposition (3.3)) : any positive operator-valued measure has an inverse Fourier-Stieltjes transform, provided it is a measure with slow growth. As the latter condition does not look very restrictive, we may say in advance that all operator-valued measures that we shall deal with have inverse Fourier-Stieltjes transforms.

The characteristic distribution may also be defined as the limit of a sequence of functions. Let us consider a family $\{B_n\}$ of admissible conditions, such that

$$\text{if } n > k \quad , \qquad B_n \geq B_k \tag{26.a}$$

and

$$\lim_{n \to +\infty} B_n = I \tag{26.b}$$

where I stands for the identical operator. Then W is the limit of the sequence of (unnormalized) statistical operators

$$W_n = B_n \; W \; B_n \quad . \tag{27}$$

The characteristic distribution of W is the limit of the sequence of the characteristic functions of the (trace class) operators W_n.

V. Kenemes and homogeneous systems

A. Characteristic distributions : transformations, elementary properties

We have seen in Section III that if one describes the state of a vacuum particle by an operator-valued measure of the form (3), this measure cannot be trace class. That such a description is possible has been shown in detail in Section IV. Let us now explicitly define the characteristic distribution which corresponds to such an operator-valued measure. As the latter has a density, the characteristic distribution is simply the inverse Fourier transform of the operator field $\chi \to \rho_\chi$.

Here we follow Bonnet ([17], definition (3.4)). Let θ be a test function belonging to the space $\mathcal{D}(G)$. Let us define the function $\widehat{\theta}$ by

$$\widehat{\theta}_g = \theta_{g^{-1}} \tag{28}$$

and let us denote by $\chi \to \mathcal{T}(\theta)_\chi$ the operator field, Fourier transform of the test function θ :

$$\mathcal{T}(\theta)_\chi = \int_G U_{\chi,g}\ \theta_g\ dg \quad . \tag{29}$$

If T is a distribution, the number obtained by smearing it with the test function θ will be denoted by $< T, \theta >$. This being said, the characteristic distribution of the operator-valued measure (3) (i.e., its inverse Fourier transform) is defined by

$$< T, \theta >= \int_G \mathrm{Tr}\left[\mathcal{T}(\widehat{\theta})_\chi\ \rho_\chi\right] d\chi \quad . \tag{30}$$

The extension of these definitions to the n-particle case is straightforward. We have to consider an operator-valued measure of the form

$$\int^{\oplus} \rho_{\chi_1 \cdots \chi_n}\ d\chi_1 \cdots d\chi_n \quad . \tag{31}$$

If we put for a moment $G = \mathbf{P}^n$, this measure can again be written in the form (3). The definitions (28) and (29) still hold, and the characteristic distribution is still defined by equation (30).

If ϕ is a function over G, we shall call *left translate* and *right translate* of ϕ by an element $\gamma \in G$ the functions, denoted respectively by $(\gamma)\phi$ and $\phi(\gamma)$:

$$(\gamma)\phi_g = \phi_{\gamma^{-1}g}$$
$$\phi(\gamma)_g = \phi_{g\gamma} \quad .$$

If T is a distribution over G, its left and right translates, denoted respectively by $(\gamma)T$ et $T(\gamma)$, are defined by

$$< (\gamma)T, \phi >=< T, (\gamma^{-1})\phi >$$
$$< T(\gamma), \phi >=< T, \phi(\gamma^{-1}) > \quad .$$

These definitions will be used in the case where G is equal to $\mathbf{P}^n$, while γ belongs to the diagonal subgroup G_d of G.

With these notations, the transformation law of a characteristic function (see (I), equation (18)) reads

$$\gamma \ : \ \phi \rightarrow \phi' = (\gamma)\phi(\gamma) \quad . \tag{32}$$

The transformation law of a characteristic distribution is the same :

$$\gamma \ : \ T \rightarrow T' = (\gamma)T(\gamma) \quad . \tag{33}$$

Finally, let us note that as the characteristic distributions are the inverse Fourier transforms of positive operator fields, they are positive-definite distributions [17].

B. Characterization of homogeneous systems and kenemes

1) An n-particle state which is invariant by all the transformations of the invariance group will be called a *homogeneous state.* Let T be the characteristic distribution of a homogeneous state, it has the following property :

$$\text{for any } \gamma \in G_d \quad , \qquad (\gamma)T(\gamma) = T \quad . \tag{34}$$

An n-particle system which can annihilate has been called a *keneme* ; the state of a keneme will be called a *closed state.* (The term refers to the completion property, see farther).

If $n = 1$, one has a trivial keneme : a "particle" whose states transform by the trivial representation, denoted by ω, of the invariance group G. This representation is defined by

$$\text{for any } g \in G \quad , \qquad U_{\omega,g} = I \quad . \tag{35}$$

Hence the characteristic function of the trivial keneme is constant ; one may say also that it is invariant by left and right translations :

$$\text{for any } \gamma \in G \quad , \qquad (\gamma)\phi = \phi(\gamma) = \phi \quad . \tag{36}$$

Let us now consider a non-trivial keneme ($n > 1$). As it can annihilate, its global particle is a trivial keneme. If its state could be described by a characteristic function ϕ, as would be the case if the group G were compact, the restriction of this function to the diagonal subgroup G_d would be a constant. Now we have a theorem (reference [18], corollary 32.6, p. 257) which says (with our present notations) : let ϕ be a positive-definite function on G^n, and let G_0 be the set of the elements y of G^n such that $\phi(y) = \phi(e)$. Then G_0 is a subgroup of G^n, and the function ϕ is invariant by right and left translations by the elements of G_0. From this theorem it follows that if ϕ would be constant on the diagonal subgroup G_d, the latter would be included in G_0 ; hence the characteristic function ϕ would be invariant by the right and left translations by the elements of the diagonal subgroup :

$$\text{for any } \gamma \in G_d \quad , \qquad (\gamma)\phi = \phi(\gamma) = \phi \quad . \tag{37}$$

The converse is obvious : if ϕ has the property (37), it is constant on the diagonal subgroup.

We know, however, that in fact the keneme does not have a characteristic function. (Because its global particle is a trivial keneme, its global state is described by a density of statistical operator which is a multiple of the identity operator, and which therefore is not trace class, see Section III). Not knowing *a priori* whether the restriction to the diagonal subgroup of the characteristic distribution of the keneme can be defined, we cannot be sure that it makes sense to say that this restriction is a constant. But property (37), which for a function is equivalent to being a constant over the diagonal subgroup, still makes sense for a distribution. Hence we shall adopt this property as a characterization of kenemes.

Summarizing : kenemes are characterized by their characteristic distributions being invariant by left and right translations by the elements of the diagonal subgroup :

$$\text{for any } \gamma \in G_d \quad , \qquad (\gamma)T = T(\gamma) = T \quad . \tag{38}$$

As to homogeneous systems, they are characterized by their characteristic distributions being invariant by internal automorphisms of the group G^n, induced by the elements of the diagonal subgroup :

$$\text{for any } \gamma \in G_d \quad , \qquad (\gamma)T(\gamma) = T \quad . \tag{39}$$

C. Relations between kenemes and homogeneous systems

1) The characteristic properties (38) and (39) immediately show that *any keneme is a homogeneous system* ; the converse is false.

2) Let us show that *any subsystem of a homogeneous system is a homogeneous system.* That would be trivially true if the characteristic distributions were functions. Recall indeed (see (I), equation (19)) that if we have a system of n particles and if we neglect (n-p) particles among the n, one gets the characteristic function of the second system from that of the first one by putting equal to e (the neutral element of the group) the variables associated with the neglected particles. Now it is easily seen that if property (39) holds true for a function, it still holds true for the restriction of this function obtained by putting equal to e some of the variables.

It will remain for us to discuss the existence of the restriction of a characteristic distribution to a subgroup defined by putting equal to e some of the variables. We shall say that the distribution T is *localizable* with respect to the variables $g_{p+1}, \cdots, g_n$ if the restriction defined by putting these variables equal to e exists. If such is the case we shall say that $g_{p+1}, \cdots, g_n$ are *function variables*, whereas $g_1, \cdots, g_p$ are *distribution variables.* Any function variable can be considered as a distribution variable ; the converse is false.

To express conveniently this type of properties, we shall use one of the following two notations.

Notation a. One writes only the function variables. The distribution T, localizable with respect to the variables $g_{p+1}, \cdots, g_n$, will be denoted by $T_{g_{p+1} \cdots g_n}$. It can be smeared over a test function which depends on the distribution variables $g_1, \cdots, g_p$; the result is a function whose value reads $< T_{g_{p+1} \cdots g_n}, \phi >$.

Notation b. One writes all the variables : the function variables as lower indices, the distribution variables as upper indices. The same distribution as above

will be written $T^{g_1 \cdots g_p}_{g_{p+1} \cdots g_n}$. Smearing it over the same test function as above, one gets the function of $g_{p+1,\cdots,g_n}$

$$< T^{g_1 \cdots g_p}_{g_{p+1} \cdots g_n}, \phi_{g_1 \cdots g_p} >$$

(one applies the Einstein convention).

Now let $T(n)$ be the characteristic distribution of a homogeneous system of n particles. If $T(n)$ is localizable with respect to the variables $g_{p+1}, \cdots, g_n$, we shall admit that the inclusive characteristic distribution (i.e., the distribution defined by neglecting the (n-p) last particles) reads in notation a :

$$T(p) = T(n)_{g_{p+1}=\cdots=g_n=e} \quad .$$

The announced property immediately follows : if the system described by $T(n)$ is homogeneous, the system described by $T(p)$ is homogeneous.

As an important particular case, *any subsystem of a keneme is a homogeneous system.*

3) Let $H(n)$ be the characteristic distribution of a homogeneous n-particle state. Let us define the distribution $F(n+1)$, localizable with respect to the variable g_{n+1}, by

$$F(n+1)_{g_{n+1}} = (g_{n+1}) \ H(n) = H(n) \ \left(g_{n+1}^{-1}\right) \quad . \tag{40}$$

This definition implies, as is readily checked, that $F(n+1)$ is invariant by left and right diagonal translations (see equation (38)). It can also be shown that if $H(n)$ is a positive-definite distribution invariant by diagonal internal automorphisms (see equation (39)), then $F(n+1)$ is also a positive-definite distribution.

$F(n+1)$ is a (right or left) translate of $H(n)$ by an element of the diagonal subgroup. Its definition immediately implies that $H(n)$ is the inclusive characteristic distribution defined by putting equal to e the variable g_{n+1} of $F(n+1)$:

$$H(n) = F(n+1)_{g_{n+1}=e} \quad . \tag{41}$$

Furthermore, one has the following unicity property : if a right and left invariant distribution $F(n+1)$ is related to $H(n)$ by equation (41), then it is given in terms

of H(n) by the relation (40). This is an immediate consequence of the invariance properties of F.

All these properties can be given the following physical interpretation : the homogeneous system described by $H(n)$ can be completed into a keneme described by $F(n+1)$. Thus we have shown that any n-particle homogeneous distribution can be completed to a $(n+1)$-particle *keneme distribution.* The distribution $F(n+1)$ defined by equation (40) will be called the *completed distribution* of the distribution $H(n)$.

VI. The characteristic distributions of vacuum

It must first be stressed that our formalism is a purely kinematical one, which is essentially unable to distinguish between a particle in the usual sense of the word and a global particle, composed of particles of any kinds. In the following, by the word "particle" it will be understood any object which carries the dynamical variables related to the invariance group.

A. The vacuum keneme distributions

Here there is no description of vacuum *per se* nor of vacuum as a whole. Instead, to all processes involving N particles we assume that one can associate a unique N-particle keneme. (See Section II.C). The characteristic distribution of this keneme will be denoted by $F(N)$ and called the *N-particle vacuum keneme distribution.*

As we know, these distributions have the following invariance property :

$$\text{for any } g \in G \quad , \qquad (g)F(N) = F(N)(g) = F(N) \quad . \tag{42}$$

$F(N)$ is a positive-definite distribution ; hence it has a Fourier transform, which is an operator-valued measure with slow growth on $\widehat{G}^N$ (see reference [17], theorem 4.1). This measure defines a conditional N-particle statistical operator, which will be denoted by $Y(N)$ and called the *N-particle vacuum keneme statistical operator.*

The measure in question does not have a density. Indeed, as it describes a keneme it is concentrated on the part of $\widehat{G}^N$ defined by the condition :

$$\chi_1 \otimes \cdots \otimes \chi_N \quad \text{weakly contains the trivial representation } \omega \quad . \tag{43}$$

(An irreducible representation χ is said to be weakly contained in a representation R of the group G, if χ is present when one writes R as a direct integral of primary representations, i.e. of representations which are multiples of irreducible representations). Equation (43) defines a part of $\widehat{G}^N$ whose measure is zero ; if there would exist a density it would be concentrated on a part of measure zero, hence equivalent to zero.

B. The homogeneous distributions of vacuum

The subsystems of the N-particle keneme will be represented by a set of statistical operators $Z_N(n)$ $(n = 1, \cdots, (N-1))$.

On the other hand in quantum statistical mechanics [19], the state of a system of N identical particles is represented by a set of n-particle reduced statistical operators $(n = 1, \cdots, N)$, denoted by $W(n)$. The operator $W(N)$ gives a complete description of the state of the system. Each operator $W(n)$ is obtained by averaging over $(N-n)$ particles.

The main differences of the present formalism with statistical mechanics are the following : a) there exists no statistical operator which would give a complete description of vacuum ; b) $Z_N(n)$ is defined for any (N, n) such that $n < N$; c) $Z(n)$ is not a usual statistical operator, but a conditional statistical operator, of the type studied in Section IV.

As the operator $Z_N(n)$ describes global particles, the operator-valued measure defined by it has a density. (This means that vacuum contains no real particles : to them would correspond Dirac measures). The operator-valued measure will be supposed to be with slow growth, hence it has an inverse Fourier transform $H_N(n)$, which is the characteristic distribution of $Z_N(n)$.

The invariance of vacuum by the invariance group implies that $Z_N(n)$ and $H_N(n)$ have the invariance properties characteristic of a homogeneous system. Hence as we know,

$$\text{for any } g \in G \quad , \qquad (g)H_N(n)(g) = H_N(n) \quad . \tag{44}$$

Among the various $H_N(n)$ for a given N, the distribution $H_N(N-1)$ plays a special role : as equations (40) and (41) define a one-to-one correspondence between $F(N)$ and $H_N(N-1)$, the latter contains the same information as its completed distribution $F(N)$, which is the N-particle vacuum keneme distribution. The distribution $H_N(N-1)$ will be denoted simply by $H(N-1)$, and it will be called the $(N-1)$-*particle vacuum open distribution.* Accordingly, the statistical operator $Z_N(N-1)$ will be denoted by $Z(N-1)$ and called the $(N-1)$-*particle vacuum open statistical operator.*

C. The case n = 1.

We have seen (Section III) that as a result of the homogeneity of vacuum, the one-particle vacuum open statistical operator has the form

$$Z(1) = \int_{\widehat{G}}^{\oplus} \sigma_\chi \, I_\chi \, d\chi \quad . \tag{45}$$

Here I_χ stands for the identity operator on the space $\mathcal{H}_\chi$, and $\chi \to \sigma_\chi$ is a positive-valued function, which will be called the *spectral function.*

The one-particle vacuum open distribution is the inverse Fourier transform of the measure defined by equation (45). Formally, it reads

$$H(1)_g = \int_{\widehat{G}} \sigma_\chi \, \mathrm{Tr}\,[U_{\chi,g^{-1}}]\, d\chi \quad . \tag{46}$$

But we know that the operator $U_{\chi,g}$ is almost never trace class, because the representation χ is almost everywhere infinite dimensional. This is why $H(1)$ is not a function, but a distribution.

The distribution which gives a meaning to equation (46) is called the character of the representation χ. In the case where G is the Poincaré group $\mathbf{P}$, the characters have been computed by Joos and Schrader [20, 21] and by Fuchs and Renouard [22]. Let us call Δ_χ the character of the representation χ, defined formally by

$$\Delta_\chi^g = \mathrm{Tr}\,[U_{\chi,g}] \quad .$$

Then the spectral representation (46) of $H(1)$ reads

$$H(1) = \int_{\widehat{G}} \widehat{\Delta}_\chi \, \sigma_\chi \, d\chi \quad . \tag{47}$$

D. Simple examples

1) Assume that the vacuum particles are without interaction. Then the N-particle vacuum open statistical operator $Z(N)$ is the tensor product of N one-particle statistical operators $Z(1)$. Likewise, the N-particle vacuum open distribution $H(N)$ is the tensor product of N distributions $H(1)$:

$$H(N)^{g_1 \cdots g_N} = \prod_{i=1}^{N} H(1)^{g_i} \quad . \tag{48}$$

2) On the other hand, the simplest example of spectral density is a constant spectral density, say equal to one. The corresponding distribution $H(1)$ is a Dirac measure at the neutral element of the group :

$$H(1) = \delta_e \tag{49}$$

Indeed, by equation (30) the Fourier transform of this measure is the operator-valued measure $(\rho_\chi, d\chi)$ such that

$$< \delta, \widehat{\theta} >= \theta_e = \int_{\widehat{G}} \mathrm{Tr}\, [\mathcal{T}(\theta)_\chi \, \rho_\chi] \, d\chi \quad . \tag{50}$$

But on the other hand the Fourier inversion formula for θ reads

$$\theta_g = \int_{\widehat{G}} \mathrm{Tr}\, [\mathcal{T}(\theta)_\chi \, U_{\chi, g^{-1}}] \, d\chi$$

and we have therefore

$$\theta_e = \int_{\widehat{G}} \mathrm{Tr}\, [\mathcal{T}(\theta)_\chi] \, d\chi \quad . \tag{51}$$

By comparing equations (50) and (51) one checks that the operator-valued measure which satisfies (50) is indeed

$$\rho_\chi = \mathbf{1}_\chi \quad . \tag{52}$$

A vacuum which has such a spectral function can be called a white vacuum.

The other extreme case would be to take for the spectral function a Dirac mesure at the point ω of the dual. This would represent a desertic vacuum.

If, on the other hand, we would take a Dirac measure at the point χ of the dual, we would no longer have to do with a vacuum, but with a gas of real particles.

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