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# Fourier transforms on groups and quantum relativistic kinematics. I. Global particles

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**Abstract**

Although quantum relativistic physics gives a central role to invariance by the Poincaré group $\mathbf{P}$, it still seldom uses the Fourier transformation on this group. It is shown here that this transformation is a natural tool of quantum relativistic kinematics. Some basic notions are recalled : the Fourier transform of a function defined on a group $G$ ; an operator-valued measure on the dual $\widehat{G}$ ; the inverse Fourier-Stieltjes transform of such a measure. The state of a system of $n$ free particles is represented by a statistical operator $W$, which defines an operator-valued measure on $\widehat{\mathbf{P}}^n$ ($\widehat{\mathbf{P}}$ is the dual of the Poincaré group). The inverse Fourier-Stieltjes transform of that measure is called the *characteristic function* of the system ; it is a function on $\mathbf{P}^n$. The main notion is that of *global characteristic function* : it is the restriction of the characteristic function to the diagonal subgroup of $\mathbf{P}^n$ ; it represents the state of the system, considered as a single particle. The main properties of characteristic functions, and particularly of global characteristic functions, are studied.

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## I. Introduction

By the usual Fourier transformation, a function is represented as a combination of exponentials. Why exponentials, and not another family of functions? Because the function $e^{i\omega t}$ defines a unitary representation of the translation group. In the frequency analysis of a signal, there is indeed an implicit reference to the signal passing through a linear filter, invariant by the translation group [1].

In a physical theory invariant by a group $G$ larger than the translation group, one should therefore expect to meet the Fourier transformation on $G$. The Fourier transformation on a group has been long known to mathematicians [2, 3]. Up to now, however, physicists hardly used the resources of this mathematical theory. It will be shown here that it is a natural and necessary tool of quantum relativistic kinematics. The relevant group $G$ is the universal covering group $\mathbf{P}$ of the Poincaré group (or inhomogeneous $SL(2, \mathbf{C})$).

First some basic notions on Fourier transformation will be recalled. Then the *characteristic function* [4, 5] of a one-particle state will be defined : it is a function on $\mathbf{P}$, which turns out to be the inverse Fourier-Stieltjes transform of the operator-valued measure on the dual $\widehat{\mathbf{P}}$, defined by the statistical operator (usually called "density matrix") of the state.

The characteristic function of an $n$ particle state will be defined in an analogous manner : it is a function on $\mathbf{P}^n$. The restriction of this function to the diagonal subgroup of $\mathbf{P}^n$ is the *global characteristic function* [5] of the state. Its physical meaning is simple and important : it characterizes the *global state* of the system, i.e. the state of the $n$ particles, considered as a single particle. This notion is an elementary quantum mechanical one ; particle physics uses it implicitly, especially in the study of resonances in production reactions, but it does not seem to have ever been explicitly defined.

An appendix shows that the two types of characteristic functions which have been introduced belong respectively to two mathematical objects - the Fourier algebra and the Fourier-Stieltjes algebra - which play an important role in harmonic analysis.

## II. The Fourier transform

### A. The Fourier transform of a function

The mathematical properties stated in this article are meant to be applied to the case where the group $G$ is $\mathbf{P}$ = inhomogeneous $SL(2,\mathbf{C})$ (the universal covering group of the Poincaré group), or else a direct power $\mathbf{P}^n$ of this group. (One might as well consider the case where $G$ also contains discrete symmetries like $P$, $C$, $T$, and an internal symmetry group. About $T$, see paper II). They will be stated mostly under a rather general form ; the groups $\mathbf{P}$ and $\mathbf{P}^n$, as well as their extensions to discrete and internal symmetries, belong to the class of groups which will be considered. (For brevity's sake, $\mathbf{P}$ will sometimes be called simply the "Poincaré group").

Let $G$ be a separable unimodular type I Lie group [6]. The invariant measure on $G$ (Haar measure) is defined up to a factor ; this factor will be chosen once for all, and the so fixed Haar measure will be denoted by $dg$.

The set of the equivalence classes of irreducible continuous unitary representations of $G$ is called the *dual* of $G$, denoted by $\widehat{G}$. (In the following, we shall say "representation" instead of "unitary continuous representation"). In each equivalence class we shall choose a certain representation, hence $\widehat{G}$ will be considered as a set of representations. In the representation $\chi \in \widehat{G}$, the element $g \in G$ is represented by the unitary operator $U_{\chi,g}$ ; the space of this representation is denoted by $\mathcal{H}_\chi$.

The Plancherel measure on $\widehat{G}$ will be denoted by $d\chi$ ; its normalization is fixed as soon as the normalization of the Haar measure is chosen.

The definition of an *operator field* on $\widehat{G}$ amounts to giving, for almost all $\chi \in \widehat{G}$, an operator $A_\chi$ on the space $\mathcal{H}_\chi$. Let $\phi$ be a complex valued function on $G$. If $\phi$ is integrable, the *Fourier transform* of $\phi$ is the operator field on $G$ defined by

$$\mathcal{T}(\phi)_\chi = \int_G \phi_g \, U_{\chi,g} \, dg \quad . \tag{1}$$

(For reasons to appear later, the names of variables will be written as indices).

If the function $\chi \to \mathrm{Tr}[\mathcal{T}(\phi)_\chi U_{\chi,g}]$ is integrable on $\widehat{G}$ for any $g \in G$, equation (1) can be inverted :

$$\phi_g = \int_{\widehat{G}} \mathrm{Tr}\,[\mathcal{T}(\phi)_\chi \; U_{\chi,g^{-1}}]\, d\chi \quad . \tag{2}$$

**B. The inverse Fourier-Stieltjes transform of an operator-valued measure**

Mathematicians usually start with a function on a group and define its Fourier transform, which is an operator field on the dual ; the inverse Fourier transformation allows one to come back to the function. But for the problems of interest here the starting point is an object on the dual. The statistical operator which describes the state of a particle is indeed related to an irreducible representation of the invariance group, and therefore to a point of the dual. We shall also introduce states which have a mass spectrum (the global states) ; here we shall have an operator field on the dual. The notion which includes these two different situations is an operator-valued measure on the dual, which has an inverse Fourier-Stieltjes transform [5, 7]. Let us now proceed to define these notions.

Let $\mu$ be a positive scalar measure on $\widehat{G}$, and let $F$ be an operator field defined on the support of $\mu$ : to any $\chi \in \mathrm{supp}(\mu)$ there corresponds an operator $F_\chi$ on $\mathcal{H}_\chi$. The couple $\mathcal{M} = (\mu, F)$ defines an operator-valued measure on $\widehat{G}$. If the operator $F_\chi$ is positive and trace class $\mu$-almost everywhere, and if the function $\chi \to \mathrm{Tr}(F_\chi)$ is $\mu$-integrable, one says that the operator-valued measure $\mathcal{M}$ is *trace class.*

If $\mathcal{M} = (\mu, F)$ is a trace class operator-valued measure on $G$, its inverse Fourier-Stieltjes transform $\phi$ is defined by a Stieltjes integral :

$$\phi_g = \int_{\widehat{G}} \mathrm{Tr}\,[F_\chi \; U_{\chi,g^{-1}}]\, d\mu(\chi) \quad . \tag{3}$$

If the measure $\mu$ is absolutely continuous with respect to the Plancherel measure, one can assume without restriction that it is equal to the Plancherel measure ; the operator field $F$ then appears as the *spectral density* of the operator-valued measure $\mathcal{M}$ with respect to the Plancherel measure (otherwise stated, as its Radon-Nikodym derivative). If $\mu$ is equal to the Plancherel measure, definition (3) becomes

$$\phi_g = \int_{\widehat{G}} \mathrm{Tr}\,[F_\chi \; U_{\chi,g^{-1}}]\, d\chi \quad , \tag{4}$$

and we get back formula (2).

We shall need also another particular case : the case when $\mu$ is a Dirac measure at the point $\chi \in \widehat{G}$, whereas the field $F$ reduces to a single positive trace class operator $W$ on $\mathcal{H}_\chi$. The operator-valued measure thus defined will be called the Dirac measure $W$ at the point $\chi$. Its inverse Fourier-Stieltjes transform is given by

$$\phi_g = \mathrm{Tr}\,[W U_{\chi,g^{-1}}] \quad . \tag{5}$$

## C. Examples

### SU(2)

For a compact group such as $SU(2)$, the dual is discrete ; in equation (2), the integration reduces to a summation. The natural normalisation of $dg$ is defined by putting the total measure of the group equal to 1 ; as a result, the Plancherel measure of a representation is equal to its dimension.

For $SU(2)$, the Haar measure is defined by

$$\int_G f_g \; dg = (8\pi^2)^{-1} \int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta \; d\beta \int_0^{2\pi} d\gamma \; f(\alpha,\beta,\gamma)$$

where $\alpha$, $\beta$, $\gamma$ are the Euler angles.

The dual is the set of the possible values of the angular momentum :

$$\widehat{G} = \{0, 1/2, 1, 3/2, \cdots\} \quad .$$

The Plancherel measure consists in masses $2j+1$ at the points $j$ :

$$p_j = 2j+1 \quad . \tag{6}$$

For any $j \in \widehat{G}$, the space $\mathcal{H}_j$ has dimension $2j+1$ and the operator $U_{j,g}$ is defined by the matrix

$$< m|U_{j,g}|m' > = D^j_{mm'}(g)$$

where the $D^j_{mm'}$ are the well known matrix elements of the $j$ representation of $SU(2)$. The Fourier transformation sends the function $f$ on $G$ onto the operator field defined thus : to any $j$ corresponds the operator

$$\mathcal{T}(f)_j = \int_G f_g \; U_{j,g} \; dg. \tag{7}$$

The Fourier inversion formula expresses the function $f$ in terms of the operator field $\mathcal{T}(f)$ :

$$f_g = \sum_j \mathrm{Tr}\,[\mathcal{T}(f)_j \; U_{j,g^{-1}}]\, p_j \quad . \tag{8}$$

It can be obtained directly by using the orthogonality relations of the matrix elements of the representations of $SU(2)$. The form of equation (8) makes obvious the fact that it is a particular case of equation (2).

But we shall need, rather than the inverse Fourier transform (8), the inverse Fourier-Stieltjes transform of an operator-valued measure, especially that of the Dirac measure at the point $j$. This transform is a particular case of equation (5) :

$$f_g = \mathrm{Tr}\,[W U_{j,g^{-1}}] \quad . \tag{9}$$

**Poincaré group P**

The above defined group **P** is locally compact, but not compact : its total Haar measure is infinite. Hence the dual is not discrete. In physics one uses most often the part of the dual which corresponds to the irreducible representations with positive squared mass and energy. Let us remark at once that the representations which differ from the ones just mentioned only by having a negative energy are as simple, they can be obtained for instance by taking the complex conjugate of the operator for positive energy. These two classes of representations are relevant for the study of vacuum (article II) ; hence their products (which contain some representations with $m^2 < 0$) are relevant also. The following considerations hold for all the mentioned classes of representations. However, we shall always take as examples the well known representations with $m^2 > 0$ and positive energy.

In order to define the Haar measure and to fix its normalization, let us write with Nghiem [8, 9] an element of $SL(2, \mathbf{C})$ under the canonical form

$$A = U H_k \quad .$$

In this equation $U$ denotes an element of $SU(2)$ ; $H_k$ is the unique hermitian positive matrix which sends the vector $k$ onto the vector (1,0,0,0) ; finally, $k$ is the image of the vector (1,0,0,0) by $A^{-1}$. We can now define the Haar measure on $SL(2,\mathbf{C})$ : it is given by

$$d^6A = d^3U \; d^3\mathbf{k}/k_0 \quad ,$$

where $d^3U$ is the Haar measure on $SU(2)$. Hence the Haar measure on $\mathbf{P}$ is given by

$$dg = d^4a \; d^6A \quad . \tag{10}$$

This normalization of the Haar measure fixes that of the Plancherel measure. In particular, the latter reads on the representations with positive $m^2$ thus ($s$ denotes the spin value) :

$$d\chi = \left[2(2\pi)^4\right]^{-1} (2s+1)m^2 dm^2 \quad . \tag{11}$$

The Fourier transform of a function on $\mathbf{P}$ is the operator field on $\widehat{\mathbf{P}}$ defined by equation (1). The Fourier inversion formula is given by equation (2). The extension to the case $G = \mathbf{P}^n$ is straightforward.

Practically, to compute the operator $\mathcal{T}(\phi)_\chi$ and to evaluate the trace in equation (2) one uses an improper basis in the space $\mathcal{H}_\chi$. See reference [5], and for more details Nghiem's papers [8, 9].

Finally, the inverse Fourier-Stieltjes transform of the operator-valued Dirac measure $W$ at the point $\chi$ is given by equation (5).

## III. Characteristic functions

### A. One particle

Let us first consider a spin $j$, i.e. a particle of spin $j$, of which we consider only the spin degrees of freedom. Its state is defined by a statistical operator $W$ on the space $\mathcal{H}_j$ (whose dimension is $2j+1$) of the representation $j$ of $SU(2)$. To this state one may also associate the Dirac operator-valued measure $W$ at the point $j$.

We define the *characteristic function* of the state as the inverse Fourier-Stieltjes transform of the latter operator-valued measure. This transform is the function $\phi$ given by equation (9). As $W$ is trace class, the function is defined everywhere on the group ; the trace of $W$ is equal to the value of the characteristic function at the neutral element of the group :

$$\mathrm{Tr}\ W = \phi_e \quad . \tag{12}$$

The positivity of the operator $W$ is equivalent to the fact that $\phi$ is a positive-definite function [6, 10].

It should be noticed that the characteristic function is not the inverse Fourier transform of the operator field defined by putting $W$ at the point $j$, zero elsewhere ; the latter is given by equation (8), not equation (9) ; it is the same function, but multiplied by $(2j+1)$. The choice made will be justified later ; let us already remark that it alone can be extended to a non-compact group.

Let us now consider a particle transforming according to the representation $\chi$ of the group $G = \mathbf{P}$. Its state is represented by a statistical operator $W$ on the space $\mathcal{H}_\chi$. The associated operator-valued measure is the Dirac measure $W$ at the point $\chi$. Again, we define the characteristic function of the state as the inverse Fourier-Stieltjes transform of the latter operator-valued measure. It is given by equation (5). As $W$ is trace class, this function is defined everywhere on $G$.

As for a spin, the positivity of $W$ is equivalent to the fact that $\phi$ is a positive-definite function. The trace of $W$ is again given by equation (12), where $e$ denotes the neutral element of $G$.

### B. Several particles

Let us now consider the case of $n$ spins $j_1, \cdots, j_n$. The state of the system is represented by an operator $W$ on the space

$$\mathcal{H}_{j_1 \cdots j_n} = \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_n} \quad .$$

This case is analogous to that of a single spin : one should only replace the group $SU(2)$ by $SU(2)^n$, a group whose elements read $(g_1 \cdots g_n)$. The elements of the dual

of $SU(2)^n$ read $(j_1, \cdots, j_n)$ ; the operator $U_{j_1 \cdots j_n, g_1 \cdots g_n}$ is the tensor product

$$U_{j_1 \cdots j_n, g_1 \cdots g_n} = U_{j_1, g_1} \otimes \cdots \otimes U_{j_n, g_n} \quad .$$

The characteristic function of the state $W$ reads

$$\phi_{g_1 \cdots g_n} = \mathrm{Tr}\left\{W\left[U_{j_1 \cdots j_n, g_1^{-1} \cdots g_n^{-1}}\right]\right\} \quad . \tag{13}$$

As in the case $n = 1$, this equation defines the inverse Fourier-Stieltjes transform of an operator-valued measure, namely the Dirac measure $W$ at the point $(j_1, \cdots, j_n)$ of the dual.

Let us consider now $n$ particles, corresponding to the irreducible representations $\chi_1, \cdots, \chi_n$ of the group $\mathbf{P}$. The state of the system is represented by an operator $W$ on the space

$$\mathcal{H}_{\chi_1 \cdots \chi_n} = \mathcal{H}_{\chi_1} \otimes \cdots \otimes \mathcal{H}_{\chi_n} \quad .$$

This case is analogous to that of a single particle : one should only replace the group $\mathbf{P}$ by $\mathbf{P}^n$, a group whose elements write $(g_1 \cdots g_n)$. The elements of the dual of $\mathbf{P}^n$ read $(\chi_1, \cdots, \chi_n)$ ; the operator $U_{\chi_1 \cdots \chi_n, g_1 \cdots g_n}$ is the tensor product

$$U_{\chi_1 \cdots \chi_n, g_1 \cdots g_n} = U_{\chi_1, g_1} \otimes \cdots \otimes U_{\chi_n, g_n} \quad . \tag{14}$$

The characteristic function of the state $W$ reads

$$\phi_{g_1 \cdots g_n} = \mathrm{Tr}\left\{W\left[U_{\chi_1 \cdots \chi_n, g_1^{-1} \cdots g_n^{-1}}\right]\right\} \quad . \tag{15}$$

As in the case $n = 1$, this equation defines the inverse Fourier-Stieltjes transform of an operator-valued measure, namely the Dirac measure $W$ at the point $(\chi_1, \cdots, \chi_n)$ of the dual.

## IV. Properties of the characteristic functions

The properties and the physical meaning of characteristic functions have been studied in detail in reference [5]. Let us recall here some essential results.

## A. Meaning of the characteristic function

As shown by equation (5), the value of the characteristic function $\phi$ of a state at the element $g$ of the group is the expectation value in this state of the operator $U_{X,g^{-1}}$. This definition is analogous to that of the characteristic function of a random variable $X$ : its value at $t$ is the expectation value of $e^{itX}$. Furthermore in the case $G = \mathbf{P}$, the restriction of $\phi$ to the translation subgroup, $a \rightarrow \phi(a, e)$, turns out to be the characteristic function (in the usual sense of probability theory) of the energy-momentum vector. If indeed we write an element of $\mathbf{P}$ under the form $(a, A)$ with $a \in \mathbf{T}$ (the translation subgroup) and $A \in SL(2, \mathbf{C})$, the operator $U(a, e)$ can be written $\exp(ia_\mu P^\mu)$. (The neutral elements of $\mathbf{P}$ and of $SL(2, \mathbf{C})$ will be both denoted by $e$). Hence the expectation values of the components of the energy-momentum vector are given by

$$< P^\mu >= [i/\phi_e]\, \partial\phi_{a,e}/\partial a_\mu|_{a=0} \quad . \tag{16}$$

(We did not assume that the statistical operator is normalized ; we have used equation (12)).

More generally, the expectation values of the dynamical variables of the system can be obtained by applying to the characteristic function suitable differentiable operators, which correspond to elements of the universal enveloping algebra of the Lie algebra of the Poincaré group.

Let us define the partial Fourier transform [9] (on the translation subgroup) of the characteristic function $\phi$ :

$$\bar{\phi}_{k,A} = \int e^{ika}\ \phi_{a,A}\ d^4a \quad .$$

As $\phi(a, e)$ is the characteristic function (in the probabilistic sense) of the energy-momentum vector, the restriction of $\bar{\phi}$ defined by putting $A = e$ is the (unnormalized) probability density of the energy-momentum vector.

Let us now consider the restriction of $\bar{\phi}$ defined by giving to $k$ a fixed value and restraining $A$ to the little group of $k$, i.e. to the subgroup of the elements of $SL(2, \mathbf{C})$ which leave $k$ fixed : $Ak = k$. It can be shown [5], thanks to Nghiem's results [9],

that this restriction gives the spin characteristic function (cf. Section III.A) of the particle. Explicitly, the spin characteristic function is given by

$$\omega_{k,g} = \bar{\phi}_{k,H_k^{-1}gH_k} \tag{17}$$

where $H_k$ has been defined in Section II.C, and $g$ is an element of $SU(2)$.

### B. Transformation law

Let us consider a one-particle state defined by its statistical operator $W$ and by its characteristic function $\phi$ ; let $\chi$ be the irreducible representation associated to the particle. The element $\gamma$ of the invariance group transforms this state into another one, whose statistical operator is

$$W' = U_{\chi,\gamma} W U_{\chi,\gamma^{-1}} \quad .$$

Equation (5) then gives the new characteristic function $\phi'$ :

$$\phi'_g = \mathrm{Tr}\,[U_{\chi,\gamma} W U_{\chi,\gamma^{-1}}\ U_{\chi,g^{-1}}] = \mathrm{Tr}\,[W U_{\chi,\gamma^{-1}}\ U_{\chi,g^{-1}}\ U_{\chi,\gamma}] \quad ,$$

and finally

$$\phi'_g = \phi_{\gamma^{-1}g\gamma} \quad .$$

This transformation can be generalized to the case of an $n$ particle state. If there is no interaction between the particles, we might consider $\mathbf{P}^n$ as the invariance group $G$, because in this case the dynamical variables relative to each single particle are separately conserved. However, the interesting cases are those where there is an interaction somewhere ; for instance, the state might be an asymptotic ingoing or outgoing state. The invariance group is then the diagonal subgroup $G_d$ of $G$, defined by

$$g_1 = \cdots = g_n \quad .$$

The transformation law reads

$$\phi'_{g_1\cdots g_n} = \phi_{\gamma^{-1}g_1\gamma\cdots\gamma^{-1}g_n\gamma} \quad . \tag{18}$$

### C. Inclusive characteristic functions

Let us note finally that the characteristic function of a system of particles also allows one to express the inclusive characteristic functions. An inclusive state of a system of $n$ particles is obtained by neglecting some of them, for instance those numbered by $p+1, \cdots, n$. One has to take the partial trace of the $n$ particle statistical operator with respect to the degrees of freedom corresponding to the $(n-p)$ neglected particles. This operation becomes very simple with characteristic functions :

$$\phi^{(p)}_{g_1 \cdots g_p} = \phi^{(n)}_{g_1 \cdots g_p e \cdots e} \quad . \tag{19}$$

One puts equal to $e$ (the neutral element of the group) the variables corresponding to neglected particles.

## V. Global characteristic functions

### A. The state of a global particle

Let us consider a system of $n$ particles described by irreducible representations of the invariance group. What are the properties of this system, if we consider it as a single particle ? (In the following, this particle will be called the *global particle*). This problem is often approached, although not quite explicitly. For instance, one may study the motion of the center of mass. One may also consider the total angular momentum of the system, or else its effective mass (i.e. the mass of the global system) ; the latter case occurs in the experimental study of resonances in production processes.

To get the state of the global particle, one has to take the partial trace [11] of the statistical operator of the $n$ particle system over all degrees of freedom, except those which correspond to the global motion. This operation, however, is not always easy to carry out. Let us show that it is very simple if one uses characteristic functions.

Let $\phi$ be the characteristic function of the system : it is a function on $\mathbf{P}^n$, defined by

$$\phi_{g_1 \cdots g_n} = \mathrm{Tr}\left\{W\left[U_{\chi_1, g_1^{-1}} \otimes \cdots \otimes U_{\chi_n, g_n^{-1}}\right]\right\} \quad . \tag{20}$$

Let us now consider the restriction $\phi_{glob}$ of $\phi$ to the diagonal subgroup. Clearly, this is the characteristic function of the global particle (or, as we shall call it, the *global characteristic function*). Indeed, the restriction to the diagonal subgroup means that one considers only the combined motions of the system. Besides, definition (20) implies that the expectation values of the dynamical variables $P_\mu$, $M_{\mu\nu}$, which correspond to elements of the Lie algebra of the group, are obtained by adding the expectation values of the $n$ single particle variables. (See equation (16) for the variable $P_\mu$).

To study the global characteristic function in more detail, let us consider the operator field which is its Fourier transform :

$$\rho_\chi = \int_G \phi_{glob\ g}\ U_{\chi,g}\ dg \quad . \tag{21}$$

This field is the spectral density of an operator-valued measure. Let $K$ be a Borel part of $G$ : the operator-valued measure assigns to it the operator

$$W(K) = \int_K^{\oplus} \rho_\chi d\chi \tag{22}$$

on the space

$$\mathcal{H}(K) = \int_K^{\oplus} \mathcal{H}_\chi d\chi \quad . \tag{23}$$

The operator $W(K)$ will be called a *conditional statistical operator* (it is conditioned by $K$).

The operator-valued measure just defined is another representation of the state of the global particle. The mathematical object which has an immediate physical meaning is not the *density of statistical operator* $\rho_\chi$, but rather the operator $W(K)$. The latter is dimensionless, as are the probabilities computed from it, whereas $\rho_\chi$ has the dimension $M^{-4}$ because $d\chi$ has the dimension $M^4$, see equation (11). On the dual another measure than the Plancherel measure might be used, for instance $dm^2$ ; in such a case the density of statistical operator would be modified, but $W(K)$

would not be. From this point of view, the expression (22) of the statistical operator is analogous to that of the power radiated by a source in the frequency range $[\nu_1, \nu_2]$ :

$$W([\nu_1, \nu_2]) = \int_{\nu_1}^{\nu_2} (dW/d\nu) d\nu \quad .$$

This equation expresses the fact that the radiated power is an additive function of the frequency interval. If instead of the frequencies one uses the wavelengths, the spectral density of power becomes $dW/d\lambda = (\nu^2/c)\ (dW/d\nu)$, but the radiated power is of course not changed.

On the other hand, the dynamical variables of the global particle are represented by operator fields of the form

$$Q \ : \ \chi \rightarrow Q_\chi \quad .$$

The expectation value of the dynamical variable $Q$ is given in terms of the statistical operator (22) by

$$< Q >_K = \int_K \mathrm{Tr}\,[Q_\chi\ \rho_\chi]\, d\chi \quad . \tag{24}$$

If we use another measure on the dual, the operator field $Q$ will not be modified. This is due to the fact that $Q_\chi$ depends on the $\chi$ (the mass and spin, say) of the global particle, whereas $W$ has the additivity property characteristic of a measure : if $K_1$ and $K_2$ are two disjoint Borel parts of $\widehat{G}$, one has

$$W(K_1 \cup K_2) = W(K_1) \oplus W(K_2) \quad .$$

$Q_\chi$ is analogous to physical quantities like the refraction index or the absorption coefficient, which are not densities of additive functions of the frequency interval, and which are therefore independent of the measure used on the frequency axis.

The representation of a state by an operator-valued measure on the dual of the invariance group is not a familiar one ; it is, however, unavoidable for a system like the global particle, which has the same dynamical variables as an elementary sytem [12], but is described by a reducible representation of the invariance group.

Indeed, the definition of the global particle implies that its algebra of observables is isomorphic to that of an elementary system. Now the center of the universal enveloping algebra of the Lie algebra of the Poincaré group is generated by the

elements $P_\mu P^\mu$ and $W_\mu W^\mu$. For an elementary system, the representation of the group is irreducible and the operators which represent $P_\mu P^\mu$ and $W_\mu W^\mu$ are multiples of the identity operator. For a global particle the representation is no longer irreducible, it is only multiplicity free ; but the operators which represent the elements of the center of the universal enveloping algebra still commute with all the dynamical variables (in other terms, they still belong to the center of the algebra of observables). Physically, this means that $P_\mu P^\mu$ and $W_\mu W^\mu$ are superconserved : they define continuous superselection rules. Hence any state of a global particle is an incoherent superposition of states with fixed values of $\chi$. This is indeed precisely what is expressed by equation (22). Besides, if we take for $K$ the whole dual we get an unconditional statistical operator :

$$W = \int_{\widehat{G}}^{\oplus} \rho_\chi \, d\chi \quad . \tag{25}$$

In the just mentioned formalism of the algebra of observables, all observables are represented by operators on the same Hilbert space, which could be here only the unconditional space

$$\mathcal{H} = \int_{\widehat{G}}^{\oplus} \mathcal{H}_\chi \, d\chi \quad . \tag{26}$$

Such a representation is useful in certain cases, but here it would oblige us to assign to dynamical variables spectral densities, which are devoid of direct physical meaning.

Let us point out at last that the formalism just introduced fits in well with the methods of experimental analysis of global particles, which are used for instance in the study of strong processes. In such a case the measurements are conditioned by the choice of an interval of effective mass, and often also by the choice of a value for the total angular momentum of the system. This was especially conspicuous, for instance, in an experiment by Baton and Laurens [13] : they studied dipions $\pi^-\pi^0$ in the mass region of the $\rho$ meson ; their statistics was high enough to allow them conditioning their measurements by different mass intervals, whose width was most often smaller than that of the $\rho$ meson. But for any value of the mass width chosen, the measured dynamical variables are always functions of the mass of the global particle, whereas the elements of density matrix which can be measured depend

additively of the mass interval.

For those reasons the dynamical variables must be represented by operator fields, and the state must be represented by an operator-valued measure on the dual which allows conditional statistical operators such as (22) to be defined.

## B. Conservation of the global particle

The value of the characteristic function of a particle (simple particle or global particle) at the element $g^{-1}$ of the invariance group is the expectation value of the operator of the representation of the invariance group. For a simple particle this operator is $U_{\chi,g}$ ; for a global particle it is given by the tensor product of the $n$ irreducible representations, i.e. by equation (14) (where one must put every $g_i$ equal to $g$), or equivalently by an operator field $\chi \to U_{\chi,g}$ . Now the operators of the representation of the invariance group are conserved quantities, as are those which represent the elements of the Lie algebra (the former are the exponentials of the latter). Hence the expectation values of those operators are also conserved quantities. This implies that in any tranformation process, the global characteristic function of the system is conserved. We shall use in article II a particular case of this : if $n$ real particles transform into a virtual particle, the characteristic function of the virtual particle is equal to the global characteristic function of the initial $n$ particle system.

For a process described by an $S$ matrix, one can give a formal proof of the conservation of the global characteristic function. Let $g \to U_g$ be the tensor product of the $n$ irreducible representations corresponding to the $n$ initial particles. The $S$ matrix commutes with the $U_g$. The initial global characteristic function reads

$$\phi(in)_g = \mathrm{Tr}\,(W_{in} U_{g^{-1}}) \quad .$$

The final characteristic function reads

$$\begin{aligned}
\phi(out)_g &= \mathrm{Tr}\,(W_{out}\, U_{g^{-1}}) \\
&= \mathrm{Tr}\left(S W_{in}\, S^{-1}\, U_{g^{-1}}\right) \\
&= \mathrm{Tr}\left(W_{in}\, S^{-1}\, U_{g^{-1}}\, S\right) \\
&= \mathrm{Tr}\,(W_{in}\, U_{g^{-1}}) = \phi(in)_g \quad .
\end{aligned}$$

## C. Some details about the meaning of global characteristic functions

In reference [5] it has been checked by elementary calculations (with use of the explicit forms of the matrix elements of the representations of $SU(2)$ and $\mathbf{P}$, respectively) that the restrictions of the characteristic functions to the diagonal subgroups describe indeed the global state. These computations will not repeated here, but we shall now recall their main results. They show how the restriction of the characteristic function to the diagonal subgroup is reflected on the statistical operator, on the other side of the Fourier transformation.

### Example 1 : SU(2)

Let us consider first the case of two spins $j_1$ and $j_2$. The complete state of the system is described by a statistical operator $W$ which can be written either in the basis of individual states $|m_1 m_2>$, or in the basis of global states $|JM>$. To go from one basis to the other one one uses the Clebsch-Gordan coefficients $< m_1 m_2 | JM >$. In the basis $|JM>$, the operator $W$ is not diagonal in general. To pass from $W$ to $W_{glob}$, we must eliminate the elements which are non-diagonal with respect to $J$ :

$$W_{glob} = \oplus_J \, W_J \tag{27}$$

with

$$< M|W_J|M' > = < JM|W|JM' > \quad . \tag{28}$$

Otherwise stated, the operator $W_{glob}$ is obtained from $W$ by diagonal truncation with respect to $J$ : if we call $\Pi_J$ the projector onto the subspace defined by the value $J$ for the total angular momentum, one has

$$W_J = \Pi_J W \Pi_J \quad . \tag{29}$$

If one computes the characteristic function $f$ corresponding to the operator (27) :

$$f_g = \sum_J \mathrm{Tr}\left[W_J \, U_J\left(g^{-1}\right)\right], \tag{30}$$

one can see that it is equal to the diagonal restriction of the complete characteristic function.

Let us now consider the case of more than two spins. We have to do with a degenerate case : the common eigenspaces of $J^2$ and $J_z$ are now more than one-dimensional. We shall denote the vectors of the collective basis by $|JMa>$, where $a$ stands for one or several degeneracy parameters. The vectors of the individual basis will be denoted by $|m_1 \cdots m_n>$, or $|(m_i)>$ for short. The equations analogous to (27)-(28) read :

$$W_{glob} = \oplus_J \, W_J \tag{31}$$

with

$$< M|W_J|M' >= \sum_a < JMa|W|JM'a > \quad . \tag{32}$$

Here going from $W$ to $W_J$ is more complicate : equation (29) is replaced by

$$W_J = \mathrm{Tr}_a \left[\Pi_J W \Pi_J\right] . \tag{33}$$

We have taken the partial trace with respect to the degeneracy parameters $a$.

Here again, one can check explicitly that the characteristic function corresponding to the operator (31) is equal to the diagonal restriction of the complete characteristic function.

**Example 2 : Poincaré group**

The main difference with the $SU(2)$ case is due to the fact that the dual is no longer discrete. In the definition of the global statistical operator $W_{glob}$, the direct sum of equation (27) is replaced by a direct integral [4, 6] :

$$W_{glob} = \int_{\widehat{G}}^{\oplus} \rho_\chi \, d\chi \quad . \tag{34}$$

To make explicit the definition of $\rho_\chi$, we shall use continuous pseudobases in the spaces $\mathcal{H}_\chi$. We shall use the results and the notations of Joos [14]. Let us take the case $n = 2$. A one-particle state will be denoted by $|k>$, where $k$ stands for

both the energy-momentum and the spin or helicity index. Similarly, a two-particle state will be denoted by $|k_1 k_2>$. As we have almost always a degenerate case (the only exception is the case where at least one of the spins is zero), a global state will always be denoted by $|\chi k\eta>$, where $\eta$ stands for one or several degeneracy parameters. The Clebsch-Gordan coefficients (whose explicit expressions are given by Joos) will be denoted by $< k_1 k_2|\chi k\eta >$. The expression of an element of the global pseudobasis is

$$|\chi k\eta >= \sum_{sp} \int d^3k_1[2\omega_1]^{-1} d^3k_2[2\omega_2]^{-1} |k_1 k_2 >< k_1 k_2|\chi k\eta >$$

where one has put

$$\omega_i = \left(\mathbf{k}_i^2 + m_i^2\right)^{1/2} \quad ;$$

here $\mathbf{k}$ stands for the space part of $k$ ; $\sum_{sp}$ denotes summation over the spin indices contained in $|k_1 k_2 >$.

Let us call, as in equation (24), $\rho$ the operator field "density of global statistical operator". The (improper) matrix elements of $\rho_\chi$ can be expressed in terms of those of the complete statistical operator $W$ :

$$< k|\rho_\chi|k' >= \sum_\eta < \chi k\eta|W|\chi k'\eta > \quad . \tag{35}$$

As shown by equation (21), the characteristic function corresponding to the operator-valued measure whose density is $\rho_\chi$ is the inverse Fourier transform of the field $\rho$. It can be checked explicitly that the latter is equal to the diagonal restriction of the complete characteristic function.

The cases $n > 2$ differ from the case $n = 2$ only by some complications ; $\eta$ becomes a continuous parameter. The Clebsch-Gordan coefficients for any $n$ have been computed by Klink and Smith [15, 16].

Let us point out that all the results stated about the global state and the global characteristic function follow from the sole assumption of relativistic invariance. The properties of these objects are elementary kinematical properties. Their meaning did not appear clearly in references [4, 5], where they were mixed with a dynamical assumption about resonances.

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# Appendix : Functional spaces

We have introduced two types of characteristic functions. The functions which describe one or several particles are the inverse Fourier-Stieltjes transforms of Dirac measures on the dual of a group (the invariance group, or a direct power of it) ; the functions which describe global particles are the inverse Fourier transforms of operator fields on the dual of the invariance group. Let us show that these two types of functions belong respectively to two types of functional spaces. These spaces, which play an important role in harmonic analysis, have been introduced by Eymard [17] in 1964. (For a recent review, see Eymard [18]).

The *Fourier-Stieltjes algebra* $B(G)$ of a locally compact group $G$ is the vector space of the functions over $G$ defined by : $g \rightarrow < \xi | U_g | \eta >$, where $U$ is a representation (not necessarily irreducible) of the group. This space is also a Banach algebra, for a product and a norm which need not be specified here.

The *Fourier algebra* $A(G)$ of a locally compact group $G$ is the closure in $B(G)$ of the intersection of $B(G)$ with the space $L(G)$ of the continuous functions with compact support on $G$. All the functions of $A(G)$ tend to zero at infinity (reference [17], proposition (3.7)).

These definitions may look somewhat abstract ; they will become more familiar by specifiying the relations of $A(G)$ and $B(G)$ with the operator fields and the operator-valued measures on the dual.

Let $\mathcal{L}_1(\widehat{G})$ be the vector space of the trace class operator fields on $\widehat{G}$, i.e. of the operator fields $\chi \rightarrow A_\chi$ such that the function $\chi \rightarrow \mathrm{Tr}(|A_\chi|)$ exists and is integrable. The relation of almost everywhere equality (for the Plancherel measure) is an equivalence relation, and the space of classes modulo this relation is denoted by $L_1(\widehat{G})$. We shall say for simplicity's sake that $L_1(\widehat{G})$ is the space of integrable fields on $\widehat{G}$.

It follows immediately from the definition that an integrable field has an inverse Fourier transform :

$$\mathcal{T}^{-1}(A)_g = \int_{\widehat{G}} \mathrm{Tr}\,[A_\chi\, U_{\chi,g^{-1}}]\, d\chi \quad .$$

Let $\mathcal{M} = (\mu, F)$ be a trace class operator-valued measure on $\widehat{G}$ (see Section

II.B), and let $M(\widehat{G})$ be the space of all trace class operator-valued measures. We know that $\mathcal{M}$ has an inverse Fourier-Stieltjes transform.

We can now characterize $A(G)$ and $B(G)$ by using Fourier transforms. The Fourier algebra $A(G)$ is the image by the inverse Fourier transformation of the space $L_1(\widehat{G})$. The Fourier-Stieltjes algebra $B(G)$ is the image by the inverse Fourier-Stieltjes transformation of the space $M(\widehat{G})$.

It follows then from Section III that the characteristic function of an $n$ particle state ($n \geq 1$) belongs to the Fourier-Stieltjes algebra of the $n^{\text{th}}$ direct power of the invariance group.

According to equation (21) the global characteristic functions are the inverse Fourier transforms of elements of $L_1(\widehat{G})$. Hence they belong to the Fourier algebra of the invariance group.

The difference between the two cases is due to the fact that the characteristic function of an $n$ particle state corresponds to fixed values of the masses ; on the contrary, a global particle has a continuous mass spectrum.

We can see thus that the Fourier and Fourier-Stieltjes algebras might have been tailor-made for quantum mechanics.

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