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A SOLUTION TO THE YAMAGUCHI PARADOX

A. Le Yaouanc, L. Oliver and J.-C. Raynal

Laboratoire de Physique Théorique et Hautes Energies*
Université de Paris-Sud, bâtiment 211, 91405 Orsay, France

Abstract - From the point of view of the S-matrix, Yamaguchi has studied unstable particles by the weak interaction (like $K_S \to \pi\pi$). He reaches the unconventional conclusion that the decay amplitudes are solely given by the weak interaction, without final state strong interaction (FSI) phases. His conclusions are based on the hypothesis that a good approximation would be to consider only, among the eigenvalues of the S-matrix, a single resonant phase shift dominated by the weak interaction. We analyse his arguments and solve the paradox by studying the behaviour of eigenvalues and eigenvectors of a typical resonant S-matrix. We find that, in general, when the FSI phases are not small, one cannot isolate a single resonant eigenphase shift. All eigenphase shifts have a rapid jump at the resonance energy and can therefore affect the resonance behaviour. We find the result that the decay amplitudes must be affected, in the usual way, by the FSI strong phases. We discuss further the situation in which all the FSI are small, where Yamaguchi's hypothesis seems more plausible. There is then indeed a single resonant phase shift and the rest of the phase shifts are rapidly varying but small. We find that their effect is indeed negligible, but the effect of the FSI phases appears due to the quick rotation, near the resonant energy, of the S-matrix eigenstate. Lastly, we show that it is possible to construct an ad hoc S-matrix satisfying Yamaguchi's Ansatz, that is however rather unphysical.

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1- Introduction

Studying narrow unstable particles from the point of view of the S matrix, Y.Yamaguchi gets unconventional conclusions[1]. Notably, he finds that the effect of the strong interaction should be negligible on the decay amplitudes, compared to the effect of the weak interaction which is more effective at the resonance energy. On the example of the K_S, which can be described as a $\pi\pi$ resonance, he says that the standard matrix elements $< (\pi\pi)_{I,-} \mid H_W \mid K_S >$, containing the strong phase shifts, cannot be acceptable as $K_S \to (\pi\pi)_I$ amplitudes. Through the $l = 0$ outgoing strong eigenstate $(\pi\pi)_{I,-}$, these matrix elements contain a phase $e^{i\delta_\pi^{(I)}}$ where $\delta_\pi^{(I)}$ is the $l = 0$ $\pi\pi$ phase shift with the total isospin I (according to Watson theorem). On the contrary, he finds that the ratio of the decay amplitudes is essentially controlled by the weak hamiltonian H_W, with negligible effect of strong interaction.

Yamaguchi's argument seems straightforward. Consider the eigenvalues $e^{2i\delta_\alpha(E)}$ of the S(E) matrix (which is a N x N matrix for N channels). If there is an isolated resonance, apparently just one $e^{2i\delta_{res}(E)}$ of these eigenvalues is resonant. Neglecting the other components of the (diagonalized) S matrix, the state of the decay products will be given by the eigenvector corresponding to the eigenvalue $e^{2i\delta_{res}(E)}$:

$$\mid \psi > \; = \; c_0 \mid (\pi\pi)_{I=0} > + \, c_2 \mid (\pi\pi)_{I=2} > \qquad (1.1)$$

$$S \begin{pmatrix} c_0 \\ c_2 \end{pmatrix} = e^{2i\delta_{res}(E)} \begin{pmatrix} c_0 \\ c_2 \end{pmatrix} \qquad (1.2)$$

The ratio of the decay amplitudes $K_S \to (\pi\pi)_I$, $I = 0, 2$ will be c_0/c_2. Then, the point is that the phase of this ratio c_0/c_2 has nothing to do with the final interaction phase shifts. In fact, when one has time reversal symmetry (or CPT symmetry as well), this ratio is <u>real</u> (see the end of §2). For the validity of the argument, it is still useful to notice that, in accordance with the definition of an S matrix, the states $\mid (\pi\pi)_I >$ in (1.1) are free states with all (weak and <u>strong</u>) interactions removed, and the phases of decay amplitudes should come out from the S matrix itself.

Clearly, the form of $\delta_{res}(E)$ is completely irrelevant to the argument. We may for instance use a Breit-Wigner formula with a background phase shift δ_{bgn} :

$$e^{2i\delta_{res}(E)} = e^{2i\delta_{bgn}} \frac{E - E_0 - i\,\Gamma/2}{E - E_0 + i\,\Gamma/2}. \qquad (1.3)$$

The phase δ_{bgn} is not related to the phase of c_0/c_2, and is of no help in solving the problem.

As we will see, this argument resists closer inspection (wave packets). So let us call it a paradox, by which we mean a (almost) true but surprising statement. Let us phrase carefully what we call the Yamaguchi paradox. It is the following :

<u>If only one phase shift of the diagonalized S matrix affects the properties of the resonance, then the transition matrix elements don't have the strong interaction phases.</u>

The hypothesis that only one eigenphase shift is resonant sounds as an extremely plausible assumption. We should like to take it as a definition of a resonance. On the other hand, the conclusion of this statement completely contradicts widely spread knowledge and practice about resonances.

Our solution of the paradox will come in two stages. First we will see that usually, namely except when all strong interaction phases shifts are close to each other, the basic assumption in the Yamaguchi's argument breaks down. Surprisingly, there are several resonant eigenvalues of the S matrix.

Next, when the strong interaction phases shifts are all close to each other, one may indeed single out one resonant eigenvalue of the S matrix, the other eigenvalues being small (near 1), and Yamaguchi's argument apparently applies. Now, even in this case, <u>every</u> eigenvalue of S has a rapid jump at the resonant energy. Owing to the link between scattering time delays and energy derivatives of phase shifts, it seems that every eigencomponent of the S matrix should contribute to the resonant phenomenon, and that the paradox is solved. However, closer inspection shows that it is not so. We will see that the resonant eigencomponent alone indeed gives correctly the decay amplitudes. The strong interaction phases of the decay amplitudes are in fact introduced by the E dependence of the resonant <u>eigenvector</u> of S(E) (see §4 and 6). It is then this crucial E dependence which is neglected in the Yamaguchi's argument.

In fact, the diagonalization of the S(E) matrix does not seems very useful in the understanding of the narrow resonances phenomena. In the narrow width limit $\Gamma \to 0$, the analytic structure of S(E) as a function of E is very simple, with a pole in the lower complex E plane as the only singularity (cf. formula (2.21)). By contrast, its eigenvalues $e^{2i\delta_\alpha(E)}$ also have branching points (cf. formula (6.4)). These branch points are at a distance $O(\Gamma)$ from the resonant energy, and are not removed in the small width

approximation. When going around a branch point, an eigenvalue is changed into another. It is true that only one eigenvalue have the pole, and this suggests the idea of only one resonant eigenvalue. However, this eigenvalue is not unambiguously defined, because the branching points are not far from the pole, and changing the way of cutting the E plane can change the sheet in which the pole occurs. Finally, while the pole approximation appears completely justified as a narrow width approximation when applied to the S-matrix, it is not so when applied to the eigenvalues and eigenvectors of the S-matrix. (The above statements are made explicit in §6 in the two-channels case).

In section 2 we describe and solve a very simple model. It can be solved not only for the S matrix, but also for the full time evolution. On the other hand, the model has all the relevant features : two channels with non resonant phase shifts due to a strong interaction, and a stable state becoming a resonance due to a weak coupling to the two channels. The S matrix has an usual form[2] for a narrow resonance.

In section 3, using the formula for the time evolution of this resonance, we show directly that the decay rates are given by weak matrix elements multiplied by the usual additional phase due to the final state strong interaction (that is the result of all textbooks).

Y. Yamaguchi considers the resonances from the point of view of the S matrix. As a matter of fact, to have a grasp on the problem raised, one must be able to compute the decay rates of the resonance from the S matrix, i. e. without knowing the detailed time evolution. In Section 4, we analyse how this can be done, and we show that the paradox does occur, that is, if we neglect the "small" eigenphase shifts and the rotation of the S-matrix eigenvectors, then we find that the decay rates are given by weak matrix elements without the usual phase due to the strong interaction (this is the result of Yamaguchi). Yet, a full calculation with our S matrix gives the usual result.

In Section 5, we compute explicitely the eigenvalues and eigenvectors of the S matrix in our model (i. e. of the S matrix in the small width approximation[2]). One sees that if the strong phase shifts are not small, there is no way to distinguish a single resonant eigenphase shift. In case of small strong phase shifts, there is a resonant eigenphase shift which goes rapidly through 90°, and small eigenphase shifts, which stay of the order of the strong non resonant ones. So, at least in this case, the contradiction between the two results is still not understood. A possible clue is that the small eigenphase shifts have a small but rapid jump at the resonant energy. Because a rapid energy variation of a phase shift implies a large time delay in scattering, the small eigenphase shift should be in fact relevant to the

properties of the resonance, since its variation, despite being small, is as rapid as the one of the resonant eigenphase shift.

In Section 6, we look more closely at the effect of the small eigenphase shifts, with the unexpected result that this effect is in fact negligible. It turns out that the strong interaction phases of the weak decay amplitudes are an effect of the rapid rotation in E of the eigenvector of S(E) (in case of small strong phase shifts).

So, the conclusion of this thesis-antithesis exercise is that the usual final state interaction effect on weak transition matrix elements does occur. We have worked in a simple, physically unrealistic, model because one has explicit formulae and one just has to look at them. However the generality of the conclusion is clear, since it is based on an expression of the S matrix which, near a narrow resonance, is valid in a great variety of resonances models.

Still, as we shall discuss in Section 7, one may construct a S matrix describing a resonance for which the Y. Yamaguchi way of calculating decay rates is the right way. The point is that some very special unusual mechanism must be at work on this resonance. In fact, the apparent discrepancy with Watson theorem comes from neglecting part of the final state interaction. However, this shows that one cannot always apply blindly the strong phase shift rule. In fact, the $K_L \to \pi\pi$ decay amplitudes give an example where the final state weak interaction (through the K_S) is in competition with the final state strong interaction.

2 - A completely soluble model

The model includes a particle with two internal states, propagating in one spatial dimension with a constant velocity v independent of the energy, and a stationary state localized at a point x_0, which becomes unstable throught a weak transition to the particle. The particle is in a strong potential V_i when in state $i = 1, 2$. The hamiltonian is

$$H\Psi = H\begin{pmatrix}\psi_1\\ \psi_2\\ f\end{pmatrix} = \begin{pmatrix} -i\,v\,\dfrac{d\psi_1(x)}{dx} + V_1(x)\psi_1(x) + \gamma_1 f\,\delta(x-x_0) \\ -i\,v\,\dfrac{d\psi_2(x)}{dx} + V_2(x)\psi_2(x) + \gamma_2 f\,\delta(x-x_0) \\ \gamma_1^*\,\psi_1(x_0) + \gamma_2^*\,\psi_2(x_0) + E_0 f \end{pmatrix} \qquad (2.1)$$

The wave function Ψ has three components: $\psi_1(x)$ and $\psi_2(x)$ are the wave functions of the particle in states 1 and 2 (the analogues of $I = 0, 2$ for $\pi\pi$), and f is the amplitude of the resonant state (the analogue of K_s). The bare energy of the resonance is E_0, and its weak coupling constants to the channels 1 and 2 are γ_1 and γ_2.

This soluble model can be generalized to any number of channels with different velocities, to any number of stationary states, to different points of interaction between the various channels and stationary states, and one can also include local interactions between channels. These models may seem rather unrealistics, however they exhibit all the detailed narrow resonances physics (except threshold effects): non resonant phase shifts, interferences between close resonances. We consider here the simplest version useful for our purpose.

To compute without ambiguity the time evolution, we must be more explicit in the definition of the hamiltonian. In order for $H\Psi$ to make sense, the "function" $-iv\psi_i' + V_i\psi_i + \gamma_i\delta(x-x_0)$ must be in the Hilbert space, and at least should not have a Dirac δ-function component. This requires that ψ_i has a first order discontinuity at point x_0, such that $\gamma_i\delta(x-x_0)$ can be cancelled by the Dirac δ-function component appearing in ψ_i'. Then, the functions ψ_i may be smooth, except for a jump at x_0, related to the resonance amplitude f by :

$$\psi_i(x_0+0) - \psi_i(x_0-0) = -i\frac{\gamma_i}{v} f \qquad (2.2)$$

This jump condition will be automatically satisfied by the solution of the time evolution equation, but it shows that the quantity $\psi_i(x_0)$ entering in the formula for H is in fact not defined. We have to define it as a combination of $\psi_i(x_0+0)$ and $\psi_i(x_0-0)$. Now, when the expression $\psi_i(x_0)$ appears, it will be always a shorthand notation for

$$\psi_i(x_0) = \frac{\psi_i(x_0+0) + \psi_i(x_0-0)}{2} \qquad (2.3)$$

This choice is dictated by the requirement that H must be hermitian, that is, the diagonal matrix elements $\langle \Psi | H \Psi \rangle$ must be real. From the hermiticity of H, it follows that the time evolution operator and the S matrix are unitary.

Let us now compute the time evolution operator e^{-iHt}. Writing Ψ^0 for the wave function at time $t = 0$, the notations will be :

$$\Psi(t) = e^{-iHt}\Psi^0, \qquad \Psi^0 = \begin{pmatrix}\psi_1^0(x)\\ \psi_2^0(x)\\ f^0\end{pmatrix}, \qquad \Psi(t) = \begin{pmatrix}\psi_1(x,t)\\ \psi_2(x,t)\\ f(t)\end{pmatrix} \qquad (2.4)$$

The differential equations are :

$$i\frac{d\psi_i(x,t)}{dt} = -i\,v\frac{d\psi_i(x,t)}{dx} + V_i(x)\,\psi_i(x,t) + \gamma_i\,f(t)\,\delta(x-x_0) \qquad (2.5)$$

$$i\frac{df(t)}{dt} = E_0\,f(t) + \sum_i \gamma_i^*\,\psi_i(x_0,t) \qquad (2.6)$$

and we have the initial conditions

$$\psi_i(x,0) = \psi_i^0(x), \qquad f(0) = f^0. \qquad (2.7)$$

In fact, we shall often not specify the number of internal states, the calculations being valid for any number. As an orientation, let us consider the <u>free</u> evolution (with $\gamma_i = 0$ and $V_i(x) = 0$). It is so simple that we can write it at once :

$$\psi_i(x,t) = \psi_i^0(x-vt), \quad f(t) = f^0\,e^{-iE_0t}. \qquad (2.8)$$

A wave packet just travels at constant velocity v without any deformation.

Let us turn to the full equations. We can solve the equation (2.5) for ψ_i with a given arbitrary function $f(t)$. Let us introduce a notation :

$$\Phi_i(x',x) = \frac{1}{v}\int_x^{x'} d\xi\, V_i(\xi). \qquad (2.9)$$

The solution, including the initial condition for ψ_i, writes :

$$\psi_i(x,t) = e^{i\Phi_i(x-vt,x)}\,\psi_i^0(x-vt) - i\frac{\gamma_i}{v}\,e^{i\Phi_i(x_0,x)}\,\chi(x, x-vt; x_0)\,f(t+(x_0-x)/v) \qquad (2.10)$$

We have also introduced the notation

$$\chi(a_2,a_1;a) = \int_{a_1-a}^{a_2-a} d\xi\,\delta(\xi) = \begin{cases} +1 & \text{if } a_1 < a < a_2 \\ -1 & \text{if } a_2 < a < a_1 \\ 0 & \text{if } a \text{ is not between } a_1 \text{ and } a_2 \end{cases} \qquad (2.11)$$

This χ function produces the expected jump at $x = x_0$ (and washes out the old jump now at $x = x_0+vt$). The limit values $\psi_i(x_0+0,t)$ and $\psi_i(x_0-0,t)$ are easily obtained, the jump condition at x_0 is verified at all t, and the value at $x = x_0$ is :

$$\psi_i(x_0,t) = e^{i\Phi_i(x_0-vt,x_0)}\,\psi_i^0(x_0-vt) - i\frac{\gamma_i}{2v}\,\mathrm{sgn}(t)\,f(t) \qquad (2.12)$$

We substitute this expression of $\psi_i(x_0,t)$ into the equation (2.6) for $f(t)$. The result is the following closed equation for $f(t)$:

$$i\frac{df(t)}{dt} = [E_0 - i\frac{\Gamma}{2}\,\mathrm{sgn}(t)]\,f(t) + \sum_i \gamma_i^*\, e^{i\Phi_i(x_0-vt,x_0)}\,\psi_i^0(x_0-vt), \quad (2.13)$$

where appears the resonance width Γ given by

$$\Gamma = \sum_i \frac{|\gamma_i|^2}{v} \quad (2.14)$$

and the resonance energy E_0 is equal to the bare energy. Using $\frac{d|t|}{dt} = \mathrm{sgn}(t)$, this equation is easily solved. Taking into account the initial condition for $f(t)$, one gets :

$$f(t) = \exp\{-iE_0t - \frac{\Gamma}{2}|t|\}\,f^0 \quad (2.15)$$

$$- i\sum_i \gamma_i^* \int_0^t d\tau\, \exp\{-i\,(E_0 - i\frac{\Gamma}{2}\mathrm{sgn}(t))\,(t-\tau)\}\, e^{-i\Phi_i(x_0,x_0-v\tau)}\,\psi_i^0(x_0-v\tau)$$

We have already the expression of $\psi_i(x,t)$ for an arbitrary function $f(t)$. We just have to substitute this result for $f(t)$ in it. One gets :

$$\psi_i(x,t) = -\,i\frac{\gamma_i}{v}\, e^{i\Phi_i(x_0,x)}\,\chi(x, x-vt; x_0)\, \exp\{-i\,[E_0 - i\frac{\Gamma}{2}\mathrm{sgn}(t)]\,(t+\frac{x_0-x}{v})\}\, f^0$$

$$+ e^{i\Phi_i(x-vt,x)}\,\psi_i^0(x-vt) - \frac{\gamma_i}{v}\, e^{i\Phi_i(x_0,x)}\,\chi(x, x-vt; x_0)\sum_j \gamma_j^* \int_{(x-x_0)/v}^{t} d\tau \;\times$$

$$\times\, \exp\{-i\,[E_0 - i\frac{\Gamma}{2}\mathrm{sgn}(t)]\,(t-\tau)\}\, e^{-i\Phi_j(x_0,x-v\tau)}\,\psi_j^0(x-v\tau) \quad (2.16)$$

These formulas (2.15) and (2.16) are the solution of the equations (2.5) - (2.7) and give the full evolution.

A comment is in order here. Taking $\psi_i^0 = 0$ (i. e., we start simply with the resonant state) in the formula for $f(t)$, one sees that we have a <u>pure</u> exponential decay law. On the other hand, there is a theorem saying that, at large times, the decay law becomes power like. The point is that the theorem requires that the energy spectrum is bounded below, while in the model the energy spectrum covers the whole real line. In other words, the large time power like behaviour is a threshold effect, but we have no threshold, and this is why we may have a pure exponential decay law.

Having explicit formulae for the time evolution, we know in principle everything about our system. Any question can be answered. One may, for instance, compute the wave operators $\Omega_\pm$ by a simple limiting process and, applying one of them to a normalized system of eigenfunction of the free hamiltonian, get a normalized system of eigenfunction of the full hamiltonian (we shall not do it).

To get a insight into the model, and for use in the next sections, let us now compute the S matrix. We can directly use the time dependent definition :

$$S = \lim_{t\to+\infty,\, t'\to-\infty} e^{iH_0t}\, e^{-iH(t-t')}\, e^{-iH_0t'}. \quad (2.17)$$

The free evolution is so simple that the calculation of the S matrix is reduced to an algorithm with the full evolution formulae as an input. In these formulae, replace t by $t-t'$, then replace $\psi_i^0(x)$ by $\psi_i^0(x-vt')$ and f^0 by $e^{-iE_0t'}f^0$, then replace x by $x+vt$, then take the limits $t\to+\infty$, $t'\to-\infty$.

On finds that $f\to 0$, and that the term containing f^0 in ψ_i vanishes in the limit. This means that the resonance disappears asymptotically (when $t\to+\infty$, $t'\to-\infty$), as expected, and that the S operator acts only on the particle states ψ. One obtains :

$$(S\psi)_i(x) = e^{i\Phi_i(-\infty,+\infty)}\,\psi_i(x) \quad (2.18)$$

$$-\frac{\gamma_i}{v}\, e^{i\Phi_i(x_0,+\infty)} \sum_j \gamma_j^*\, e^{-i\Phi_j(x_0,-\infty)} \int_0^{+\infty} d\tau\, \exp\{-i\,(E_0 - i\frac{\Gamma}{2})\,\tau\}\,\psi_j(x+v\tau)$$

This S operator is of course diagonal in the energy representation, as follows from general theory. Let us introduce a complete system of eigenfunctions of the free hamiltonian :

$$(\varphi_{j,E})_i(x) = \delta_{j,i}\frac{1}{\sqrt{v}}\, e^{iEx/v} \quad (2.19)$$

The wave function $\varphi_{j,E}$ describe the free particle with energy E in internal state j, with the normalization $<\varphi_{j,E}\,|\,\varphi_{j',E'}> = \delta_{j,j'}\,2\pi\,\delta(E-E')$. The $v^{-1/2}$ coefficient would be important in case of different velocities in different channels, in order to have a properly unitary matrix $S(E)$ (that is unitary for the usual scalar product in the finite dimensional Hilbert space of internal states). This $S(E)$ matrix is defined by

$$S\varphi_{j,E} = \sum_i S_{i,j}(E)\,\varphi_{i,E}. \quad (2.20)$$

Replacing $\{\psi_i\}$ by $\{(\varphi_{j,E})_i\}$ in the above expression for the S operator, we find easily the expression of the S matrix :

$$S_{i,j}(E) = e^{2i\delta_i^-}\,\{\delta_{i,j} - \frac{i}{v}\frac{\gamma_i\,\gamma_j^*}{E - E_0 + i\,\Gamma/2}\}\, e^{2i\delta_j^+} \quad (2.21)$$

$$\Gamma = \sum_i \frac{|\gamma_i|^2}{v}$$

We have introduced notations for the final state interaction phase shifts δ_i^- and the initial state interaction phase shifts δ_i^+. They are given by

$$\delta_i^- = -\frac{1}{2}\Phi_i(+\infty,x_0) = -\frac{1}{2v}\int_{x_0}^{+\infty} d\xi\, V_i(\xi), \quad (2.22)$$

$$\delta_i^+ = -\frac{1}{2}\Phi_i(x_0,-\infty) = -\frac{1}{2v}\int_{-\infty}^{x_0} d\xi\, V_i(\xi).$$

In the resonance decoupling limit, and for $E \neq E_0$, the limit $S^0_{i,j}(E) = \lim_{\gamma_i \to 0} S_{i,j}(E)$ is the diagonal strong interaction S matrix, given by

$$S^0_{i,j}(E) = e^{2i\delta_i^0}\,\delta_{i,j} \tag{2.23}$$

The "strong interaction" phase shifts $\delta_i^0 = \delta_i^- + \delta_i^+$ are given by

$$\delta_i^0 = -\frac{1}{2}\Phi_i(+\infty,-\infty) = -\frac{1}{2v}\int_{-\infty}^{+\infty} d\xi\, V_i(\xi). \tag{2.24}$$

We can see now that this model describes something like a perfect narrow resonance. The background phase shifts (not the eigenphase shifts) and the width are not only slowly varying for E near E_0, they are just <u>constants</u>. In fact, the expression (2.21) is the general form of a resonant S-matrix in the narrow witdh approximation[2]. This form is dictated by the following requirements

- S(E) is a meromorphic function of E in the whole complex plane which has only one pole (located in the lower half-plane), this pole being simple, and S(E) goes to a constant when $E \to \infty$,
- the residue of the pole is a matrix of rank one,
- and of course, S(E) is unitary when E is real.

The virtue of our model is that this ideal approximate resonant S-matrix is the <u>exact</u> S matrix of the given hamiltonian, and moreover the time evolution of the system is known (see eq. (2.15) and (2.16)).

Concerning the problem of the decay amplitudes phases, the Yamaguchi paradox is particularly striking when one has time reversal symetry. In this case, the S matrix satisfies $TST = S^\dagger$, which means that it is <u>symmetrical</u> in a real basis (a basis for which $T\Psi = \Psi$). Such a symmetrical unitary matrix can be diagonalized by a <u>real</u> unitary matrix and, according to Yamaguchi, it follows that the decay amplitudes are real (relative to each over).

We cannot implement T symmetry in our model, except by introducing other channels with opposite velocity. However, it is easy to obtain a symmetrical S matrix. We have just to fulfill the following conditions :

$$\gamma_i \text{ real}, \qquad \delta_i^- = \delta_i^+ \; (= \tfrac{1}{2}\delta_i^0) \tag{2.25}$$

It is readily found that this is due to a PT symetry, P being the parity reflexion with respect to the point x_0. Indeed, the equality of the initial and final strong phase shifts is obtained if the potentials satisfy

$$V_i(2x_0-x) = V_i(x) \tag{2.26}$$

and then, if the weak coupling constants are real, our hamiltonian satisfies (PT)H(PT) = H, T being just the complex conjugation of the wave functions and resonance amplitude. Moreover our basis is real (up to a common phase which does not affect the matrix elements) :

$$PT\, e^{-iEx_0/v}\,\varphi_{j,E} = e^{-iEx_0/v}\,\varphi_{j,E} \tag{2.27}$$

since P exchanges $x_0 \to -x_0$ and T amounts to take the complex conjugate.

3 - Decay rates and decay amplitudes from time evolution

To discuss with certainty the phases of the decay amplitudes, we must first of all relate them to quantities that can be directly measured. Then these quantities can be computed in our explicit model, and the phases can be determined (or at least the relative phases). These quantities are of course the decay rates. However, the decay rates to the internal states $i = 1, 2$ are not enought. To be specific, let us consider the $K_S \to \pi\pi$ problem. The two internal states of the particle in our model corresponds to the strong interaction diagonal isospin states $I = 0, 2$ of the $\pi\pi$ system. However, due to electromagnetism, the measured $\pi\pi$ states are in fact $\pi^0\pi^0$ and $\pi^+\pi^-$. Then, the decay rates $K_S \to \pi^0\pi^0$ and $K_S \to \pi^+\pi^-$ are given by

$$R_{00} = \frac{1}{3}|A_0 - \sqrt{2}\,A_2|^2, \qquad R_{+-} = \frac{1}{3}|\sqrt{2}\,A_0 + A_2|^2, \tag{3.1}$$

and they clearly depend on the relative phase of the $I = 0$ and 2 decay amplitudes A_0 and A_2.

To determine, without any theoretical prejudice, the relative phase between A_0 and A_2, would require doing interferences between $\pi^0\pi^0$ and $\pi^+\pi^-$, an almost impossible task. However, working in a model, we are in a more fortunate position. We may assume that the decay rate to any superposition $\sum_i a_i \,|\, i>$ of internal states can be measured (the a_i being arbitrary). If we find that these decay rates $R(\{a_i\})$ (which generalize R_{00} and R_{+-}) are given, for any a_i, by a formula

$$R(\{a_i\}) = |\textstyle\sum_i a_i A_i|^2 \tag{3.2}$$

with some complex numbers A_i, then these numbers are defined to be the decay amplitudes, and are completely determined by the rates (up to a common phase). Indeed, one can measure the rate repeatedly, varying a_1 and a_2, until $R(a_1,a_2) = 0$.

Since the model is completely solved, we can answer any question. The aim of this rather academic discussion was to ascertain the question. We have to compute the decay rate to an arbitrary superposition $\sum_i a_i \,|\, i>$ of internal states, then extract the amplitudes, then look at their phases.

The time evolution of the resonance produced at $t = 0$ is given by the formulae (2.15) and (2.16) of the previous Section by taking $\psi_i^0 = 0$, $f^0 = 1$, and $t \geq 0$:

$$f(t) = \exp\{-i\,(E_0 - i\tfrac{\Gamma}{2})\,t\} \tag{3.3}$$

$$\psi_i(x,t) = -\,i\,\frac{\gamma_i}{v}\, e^{i\Phi_i(x_0,x)}\,\chi(x_0 < x < x_0+vt)\,\exp\{-\,i\,(E_0 - i\tfrac{\Gamma}{2})\,(t+\tfrac{x_0-x}{v})\}$$

The physical quantity which is directly given by the Postulates of Quantum Mechanics is the probability $P(\{a_i\},t)$ for the presence of the particle in internal state $\sum_i a_i \,|\, i>$ at time t:

$$P(\{a_i\},t) = \int_{-\infty}^{+\infty} dx\, |\textstyle\sum_i a_i\, \psi_i(x,t)|^2 \tag{3.4}$$

The transition rate is just given by the time derivative of $F(\{a_i\},t)$. Clearly, we only have to compute the overlaps $<\psi_j(t)\,|\,\psi_i(t)>$. One has

$$<\psi_j(t)\,|\,\psi_i(t)> = \frac{\gamma_j^*\gamma_i}{v^2}\int_{x_0}^{x_0+vt} dx\; e^{i[\Phi_j(x,x_0)-\Phi_i(x,x_0)]}\,\exp\{-\frac{\Gamma}{v}(x_0-x+vt)\} \tag{3.5}$$

$$= \frac{\gamma_j^*\gamma_i}{v}\int_0^t d\tau\; e^{i[\Phi_j(x_0+v(t-\tau),x_0)\,-\,\Phi_i(x_0+v(t-\tau),x_0)]}\; e^{-\Gamma\tau}$$

To proceed further, we have to make the specific assumption of short ranged strong interaction potentials $V_i(x)$, and to consider times t such that vt is much greater than this range r_{strg}. Then $\Phi_i(x_0+v(t-\tau),x_0)$ is equal to the constant $\Phi_i(+\infty,x_0) = -\,2\delta_i^-$ when τ is in an interval $[0, t-\frac{r_{strg}}{v}]$ which is almost all the integration interval $[0, t]$. These times can still be small relative to the lifetime, provided the following condition is fullfilled :

$$r_{strg} \ll \frac{v}{\Gamma} \tag{3.6}$$

This condition means that the decay products go away the strong interaction in a time short compared to the lifetime. It is of course fulfilled in the case of narrow resonance that we are considering.

One obtains (assuming $t \gg \frac{r_{strg}}{v}$) :

$$<\psi_j(t)\,|\,\psi_i(t)> = \frac{\gamma_j^*\gamma_i}{v\Gamma}\,\{e^{-2i\delta_j^-}\,e^{2i\delta_i^-}\,(1-e^{-\Gamma t}) + O(\frac{\Gamma r_{strg}}{v})\,e^{-\Gamma t}\} \tag{3.7}$$

and, for the derivative :

$$\frac{d}{dt}<\psi_j(t)\,|\,\psi_i(t)> = \frac{\gamma_j^*\gamma_i}{v}\,\{e^{-2i\delta_j^-}\,e^{2i\delta_i^-} + O(\frac{\Gamma r_{strg}}{v})\}\,e^{-\Gamma t}. \tag{3.8}$$

Thus, up to corrections completely negligible in the limit of narrow resonance, we have the following result for the decay rates :

$$R(\{a_i\},t) = |\textstyle\sum_i a_i \frac{\gamma_i}{\sqrt{v}}\, e^{2i\delta_i^-}|^2\; e^{-\Gamma t}, \tag{3.9}$$

and, the a_i being arbitrary, the decay amplitudes are unambiguously recognized as given by

$$A_i = \frac{\gamma_i}{\sqrt{v}}\, e^{2i\delta_i^-}\, e^{-\Gamma t/2} \tag{3.10}$$

So, using the formula for the time evolution of the resonance, we find that the decay rates are given by weak matrix elements multiplied by the usual additional phase due to the final state strong interaction (that is the result of all textbooks).

Notably, if the S matrix is symmetrical (see the discussion at the end of Section 2), γ_i is real and one has $\delta_i^- = \frac{1}{2}\delta_i^0$,

$$A_i = \frac{\gamma_i}{\sqrt{v}}\, e^{i\delta_0^-}\, e^{-\Gamma t/2} \qquad (3.11)$$

but this does not imply at all that the decay amplitudes are real.

In §4 we will relax the condition $\psi_i^0 = 0$, $f^0 = 1$, and see that the same decay amplitudes are obtained if we allow some $\psi_i^0 \neq 0$.

4 - Decay rates and decay amplitudes from S matrix

Perhaps the reader is completely convinced by the very standard argument of the previous Section, but not the writer. For instance, we assumed that the initial state was $\psi_i^0 = 0$ and $f^0 = 1$. Perhaps, a small component $\psi_i^0 \neq 0$ can radically change the result ?

This flaw will be cured if we can deduce the decay rates from the S matrix, without using the detailed time evolution as we did in §3. By this, the decay rates will be proved to be independent of any assumption concerning the initial state of the resonance. And, up to a common phase, the decay amplitudes are deduced from the decay rates (see Section 3).

Now, the S matrix gives us the state of the system after complete decay of the resonance. Apparently, it is too late to measure the decay rates. However, we can do archaeology. Suppose that, at a very large time $T >> 0$, we measure the position of the particle, and we find it at a point $x+vT$. Then we know that it was at x_0 at the time $t = \frac{x_0-x}{v}$. If we know further that the particle was incident at x_0 around time 0, and if the time t is positive and unexpectedly large, then, we can infer that the detected particle is a decay product of a resonance, and that the decay occurred around time t (if t is large < 0, something went wrong).

Let us translate this in more usual language. We need an ingoing wave packet $\psi(x)$, of spreading $\Delta x << v/\Gamma$ and peaked around $x = x_0$, in order to know that the detected particle is the product of a decay at some time t. The outgoing wave packet $S\psi$ gives the probability density $|(S\psi)(x)|^2$ for the particle to be at the point $x+vT$ at a large positive time T. Due to the resonance, this outgoing wave packet has a widely spread component, of spreading v/Γ. This component is weak, but spatially well separated from the main component which stays around $x = x_0$ with the spreading Δx of the ingoing wave packet.

The number of outgoing particles found in the space interval between $x+vT$ and $x+vT+dx$, given by $|(S\psi)(x)|^2dx$, is also the number of decays between times $\frac{x_0-x}{v}$ and $\frac{x_0-x-dx}{v}$. Finally, given the S operator, the decay rate to an arbitrary state $\sum_i a_i\, |\, i >$ is given by

$$R(\{a_i\},t) = v\, |\textstyle\sum_i a_i\, (S\psi)_i(x_0-vt)|^2 \qquad (4.1)$$

where $\{\psi_i(x)\}$ is any normalized wave packet, peaked around x_0, and of spreading $\Delta x << v/\Gamma$. The formula is valid for times $t >> \Delta x/v$. In fact, we have the decay rates up to a common factor, which is the probability of having produced the resonance. The corresponding decay amplitudes, also up a common factor, are given by

$$A_i(t) = (S\psi)_i(x_0-vt). \qquad (4.2)$$

for $t >> \Delta x/v$. The arbitrariness of the a_i has allowed to obtain all the relative phases between the decay amplitudes A_i.

We are now in position to explain more fully the problem raised by Y. Yamaguchi. Let us assume for clarity that the S matrix is symmetrical. Then it can be diagonalized by a real transformation :

$$S(E) = K(E)\,(e^{2i\delta_1(E)}, e^{2i\delta_2(E)}, \ldots)\,K(E)^{\dagger} \qquad (4.3)$$

Here, K(E) is a real unitary matrix, with the eigenvectors as columns. Next, it is extremely natural to assume that only one eigenphase shift is resonant (we are considering an isolated resonance), say $\delta_1(E)$. Then, the other diagonal components are slowly varying functions of E, and they will not contribute to the widely spread component of the outgoing wave function. So, it is completely justified to use the following approximation :

$$S_{i,j}(E) \approx e^{2i\delta_1(E)}\,K_{i,1}(E)\,K_{j,1}(E), \qquad (4.4)$$

where $K_{i,1}(E)\,K_{j,1}(E)$ is real. But, in this expression, the final state strong interaction phases $\delta_i = \frac{1}{2}\delta_i^0$ have completely disappeared. In fact, only one phase δ_1 appears, common to all channels. Thus, we reach the conclusion of Y. Yamaguchi, that all the decay amplitudes must be real with respect to each other. This result is in complete contradiction with the result of the previous section. We will see which is the flaw in the next sections.

To proceed further, we need a simple formula relating the decay amplitudes to the S matrix. We now turn to this problem. The starting point is formula (4.2). We have to identify the widely spread part of the outgoing wave function $(S\psi)(x)$. This can be done with precision only by considering a limiting process in which $\Gamma \to 0$. So, we must consider a family of models, depending on some parameters, and see what happens when $\Gamma \to 0$ and E_0 stays constant. We expect that, provided we take Γ as an energy scale, the widely spread part of $(S\psi)(x)$ has a definite limit, while its main part shrinks to 0. We may consider the function $g_\Gamma(\xi) = (S\psi)(\xi/\Gamma)$ and look for a limit. However, due to the real part of the resonance energy, $g_\Gamma(\xi)$ has a phase $e^{iE_0 v^{-1}\xi/\Gamma}$ which prevents a limit. We should also divide by Γ, since the probability of production and the decay rates are proportional to the width. So, we redefine our reduced wave function $g_\Gamma(\xi)$ as follows :

$$g_\Gamma(\xi) = \frac{1}{\Gamma} e^{-iE_0 v^{-1}\xi/\Gamma}\,(S\psi)(\xi/\Gamma) \qquad (4.5)$$

The easiest way to look at a limit of $g_\Gamma(\xi)$ is to use a little distribution theory, that is, we integrate with an arbitrary smooth function $h(\xi)$, and consider the limit of the integral. Let us introduce the wave functions $\tilde{\psi}_i(E)$ in the energy representation :

$$\psi_i(x) = \int \frac{dE}{2\pi}\,\tilde{\psi}_i(E)\,\frac{1}{\sqrt{v}}\,e^{iEx/v}. \qquad (4.6)$$

One has

$$(S\psi)_i(x) = \sum_j \int \frac{dE}{2\pi}\,S_{i,j}(E)\,\tilde{\psi}_j(E)\,\frac{1}{\sqrt{v}}\,e^{iEx/v}. \qquad (4.7)$$

Then, we may compute our integral :

$$\int d\xi\, h(\xi)\, g_\Gamma(\xi) = \frac{1}{\Gamma}\int d\xi\, h(\xi) \sum_j \int \frac{dE}{2\pi}\,S_{i,j}(E)\,\tilde{\psi}_j(E)\,\frac{1}{\sqrt{v}}\,e^{i(E-E_0)v^{-1}\xi/\Gamma} \qquad (4.8)$$

$$= \sum_j \frac{1}{\Gamma}\frac{1}{\sqrt{v}}\int \frac{dE}{2\pi}\,\tilde{h}((E-E_0)v^{-1}/\Gamma)\,S_{i,j}(E)\,\tilde{\psi}_j(E)$$

$$= \sum_j \frac{1}{\sqrt{v}}\int \frac{de}{2\pi}\,\tilde{h}(ev^{-1})\,S_{i,j}(E_0+e\Gamma)\,\tilde{\psi}_j(E_0+e\Gamma)$$

We have introduced the Fourier transform $\tilde{h}(\eta) = \int d\xi\, h(\xi)\, e^{i\eta\xi}$, and replaced the energy integration variable E by a reduced energy $e = (E-E_0)/\Gamma$.

Clearly, we need a limit $\hat{S}(e) = \lim_{\Gamma\to 0} S(E_0+e\Gamma)$ (remember that the function S(E) already depends on Γ). For the S matrix of our model, given in Section 2, $S(E_0+e\Gamma)$ is in fact independent of Γ, so that this limit exists trivially. Then we have

$$\lim_{\Gamma\to 0} \int d\xi\, h(\xi)\, g_\Gamma(\xi) = \sum_j \frac{1}{\sqrt{v}}\int \frac{de}{2\pi}\,\tilde{h}(ev^{-1})\,\hat{S}_{i,j}(e)\,\tilde{\psi}_j(E_0). \qquad (4.9)$$

This proves that the reduced outgoing wave function $g_\Gamma(\xi)$ has a limit as a distribution. One sees also that this limit does not depend on the shape of the ingoing wave function ψ. It depends on ψ only throught the finite set of coefficients $\tilde{\psi}_j(E_0)$. We can write the limit as follows :

$$g(\xi) = \lim_{\Gamma\to 0} g_\Gamma(\xi) = \sum_j \frac{1}{\sqrt{v}}\int \frac{de}{2\pi}\,e^{iev^{-1}\xi}\,\hat{S}_{i,j}(e)\,\tilde{\psi}_j(E_0) \qquad (4.10)$$

where the integral must be understood as a Fourier transform in the sense of distributions. Let us apply this result to a simple one-channel Breit-Wigner resonance. One has

$$\hat{S}(e) = \frac{e - i/2}{e + i/2} = 1 - \frac{i}{e + i/2} \qquad (4.11)$$

$$g(\xi) = \{\sqrt{v}\,\delta(\xi) - \frac{1}{\sqrt{v}}\,\theta(-\xi)\,e^{\xi/2v}\}\,\tilde{\psi}(E_0) \qquad (4.12)$$

and, for $(S\psi)(x)$, we obtain the following approximation :

$$(S\psi)(x) \approx \Gamma\, e^{iE_0 v^{-1}x}\, g(\Gamma x)$$

$$= \{\sqrt{v}\,\delta(x) - \frac{\Gamma}{\sqrt{v}}\,\theta(-x)\exp[i\,(\frac{E_0}{v} - i\,\frac{\Gamma}{2v})\,x]\}\,\tilde{\psi}(E_0) \qquad (4.13)$$

One sees that the non resonating part of the outgoing wave function has shrinked to a δ function, and any information about it is lost, but the resonating part is correctly obtained.

Returning to physics, we can write :

$$(S\psi)_i(x) = \Gamma\, e^{iE_0 v^{-1}x}\, g_\Gamma(\Gamma x) \approx \Gamma\, e^{iE_0 v^{-1}x}\, g(\Gamma x) \qquad (4.14)$$

$$\approx \Gamma\, e^{iE_0 v^{-1}x} \sum_j \frac{1}{\sqrt{v}} \int \frac{de}{2\pi}\, e^{icv^{-1}\Gamma x}\, S_{i,j}(E_0+e\Gamma)\, \tilde{\psi}_j(E_0) = \sum_j \frac{1}{\sqrt{v}} \int \frac{dE}{2\pi}\, e^{iEv^{-1}x}\, S_{i,j}(E)\, \tilde{\psi}_j(E_0)$$

Moreover, in $(S\psi)_i(x_0\text{-}vt)$, we may neglect x_0 since we are limited to times t such that Γt remains finite in the limit $\Gamma \to 0$.

The result now is quite simple. We find that, up to a common factor, <u>the decay amplitudes are just given, as a function of time, by the Fourier transform of the S matrix with respect to the energy</u> :

$$A_i(t) = \sum_j \int \frac{dE}{2\pi}\, e^{-iEt}\, S_{i,j}(E)\, \tilde{\psi}_j(E_0). \qquad (4.15)$$

Let us recapitulate some minimal conditions that we encounter, in order that a S matrix describes a narrow resonance, with decay amplitudes given by its Fourier transform. First, the S matrix should depend on parameters, let one of them be Γ, and let us write $S(\Gamma; E)$ to remember this dependency. Next, there must be a limit $\hat{S}(e) = \lim_{\Gamma\to 0} S(\Gamma, E_0+e\Gamma)$, where E_0 can depend on the parameters, but must also have a limit. Next, the Fourier transform $A(\tau) = \int \frac{de}{2\pi}\, e^{-ie\tau}\, \hat{S}(e)$ must be a function outside $\tau = 0$, and it must vanish for $\tau < 0$ (the resonance cannot decay before it has been produced). Finally, for each $\tau > 0$, the matrix $A(\tau)$ must be of rank one (for an isolated resonance, in order that the decay rates be independent of the production process). Then, the decay amplitudes (up to a common factor) are given by

$$A_i(t) = \sum_j A_{i,j}(\Gamma t)\, \tilde{\psi}_j(E_0). \qquad (4.16)$$

This formula is valid for $t > 0$, the singular part at $t = 0$ being a residuum of the non resonant part of the outgoing wave function. Compared to the previous formula, the S matrix is replaced by its resonating part $S^r(\Gamma; E)$ defined by

$$S^r(\Gamma; E) = \hat{S}((E-E_0)/\Gamma). \qquad (4.17)$$

So, we have also a way to wash out all the slowly varying, non resonant parts, of the S matrix. We just have to compute the limit $\hat{S}(e)$, and we are done with the small Γ approximations. Then we have to compute the Fourier transform of this fixed function $\hat{S}(e)$.

With this new tool at hand, let us return to the Y. Yamaguchi approximation to the S matrix, namely, only one resonant eigenvalue and a slowly varying eigenvector with energy. Let us assume for simplicity that the resonating eigenphase shift δ_1 is given by a Breit-Wigner formula. For $\hat{S}(e)$, one gets :

$$\hat{S}_{i,j}(e) = \frac{e - i/2}{e + i/2}\, K_{i,1}(E_0)\, K_{j,1}(E_0) \qquad (4.18)$$

For the Fourier transform, one gets, in the hypothesis of Yamaguchi :

$$A_{i,j}(\tau) = \{\delta(\tau) - \theta(\tau)\, e^{-\tau/2}\}\, K_{i,1}(E_0)\, K_{j,1}(E_0), \qquad (4.19)$$

and for the decay amplitudes, one gets

$$A_i(t) = \theta(t)\, K_{i,1}(E_0)\, e^{-\Gamma t/2}\, \{\textstyle\sum_j K_{j,1}(E_0)\, \tilde{\psi}_j(E_0)\}, \qquad (4.20)$$

where we have assumed $K_{i,1}(E)\, K_{j,1}(E) \approx K_{i,1}(E_0)\, K_{j,1}(E_0)$ (slowly varying eigenvector with energy). So, <u>we confirm that they are real with respect to each other</u> (assuming that the S matrix is symmetrical, for instance by T invariance).

Let us apply this treatment to the S matrix of our model given by (2.21). For $\hat{S}(e)$, one gets

$$\hat{S}_{i,j}(e) = e^{2i\delta_i^-} \left\{ \delta_{i,j} - \frac{\gamma_i \gamma_j^*}{v\Gamma}\, \frac{i}{e + i/2} \right\} e^{2i\delta_j^+} \qquad (4.21)$$

where one may consider $\frac{\gamma_i \gamma_j^*}{v\Gamma}$ as constant when $\Gamma \to 0$. For the Fourier transform, one gets

$$A_{i,j}(\tau) = e^{2i\delta_i^-} \left\{ \delta_{i,j}\, \delta(\tau) - \frac{\gamma_i \gamma_j^*}{v\Gamma}\, \theta(\tau)\, e^{-\tau/2} \right\} e^{2i\delta_j^+}, \qquad (4.22)$$

and for the decay amplitudes, one gets

$$A_i(t) = \theta(t)\, \gamma_i\, e^{2i\delta_i^-}\, e^{-\Gamma t/2}\, \{\textstyle\sum_j \gamma_j^*\, e^{2i\delta_j^+}\, \tilde{\psi}_j(E_0)\}. \qquad (4.23)$$

So, <u>we confirm that they have the final state strong interaction phase shifts</u>. In fact, we reobtain the result of Section 3, but now without any hypothesis about the initial state of the resonance, i. e. without specifying what we understand for the resonant state, invoking only scattering theory that deals with asymptotic states.

The solution of this mystery will be given in the next two sections.

ase of small strong phase shifts, effect of rotation of eigenvectors

vill see now that in the case of small strong phase shifts, Yamaguchi's conclusions idated by the rotation of the S-matrix eigenvectors.

us restrict to the case of two channels. By introducing the one-dimensional al projectors $P_\pm(E)$ on the eigenvectors corresponding to the eigenvalues $e^{2i\delta_\pm(E)}$ omposition of the S(E) matrix writes

$$S(E) = e^{2i\delta_+(E)} P_+(E) + e^{2i\delta_-(E)} P_-(E). \qquad (6.1)$$

t to consider the contribution of each term to the decay amplitude. The matrices are related to the matrix $K(E)_{i,\alpha}$ in formula (5.10) (here $\alpha = \pm$) by

$$P_\pm(E)_{i,j} = K(E)_{i,\pm} K(E)^*_{j,\pm}, \qquad (6.2)$$

e eigenvectors $u(E)_{i,\alpha}$ in formula (5.7) by

$$P_\pm(E)_{i,j} = u(E)_{i,\pm} u(E)^*_{j,\pm}, \qquad (6.3)$$

eigenvalue equation $\det(S_{ij} - e^{2i\delta}\,\delta_{ij}) = 0$ has been reduced to eq. (5.6) in the section and is of second degree in $tg(\delta)$. We can write the solutions :

$$= -\frac{\frac{\Gamma_1+\Gamma_2}{2}(1+tg^2\delta_0) \pm \sqrt{[2(E-E_0)tg\delta_0 - \frac{\Gamma_1-\Gamma_2}{2}(1-tg^2\delta_0)]^2 + \Gamma_1\Gamma_2(1+tg^2\delta_0)^2}}{2(E - E_0 + \frac{\Gamma_1-\Gamma_2}{2} tg\delta_0)} \qquad (6.4)$$

mula assumes $\delta_1^0 = -\delta_2^0 = \delta^0$. From eq. (5.6) one sees that the general case is d by adding a same constant phase to δ_1^0, δ_2^0 and $\delta_\pm(E)$. Let us recall the notations ve two constants strong phase shifts δ_1^0, δ_2^0, and two eigenphase shifts $\delta_\pm(E)$ onding to the two eigenvalues $e^{2i\delta_\pm(E)}$ of the S-matrix. Then this result can be ed into the expression (5.7), (5.8) for the eigenvectors, and the projectors can be ed. In order to write the result in manageable form, we need to introduce some s :

$$\beta_i = \frac{\gamma_i}{\sqrt{v}}\frac{1}{\sqrt{\Gamma}}, \qquad B_i = |\beta_i|^2, \quad B = B_1 - B_2, \quad e = \frac{E - E_0}{\Gamma} \qquad (6.5)$$

B_2 are the branching ratios. The S matrix writes as follows :

$$= \begin{pmatrix} e^{2i\delta_1^-}\frac{2e - iB}{2e + i}e^{2i\delta_1^+}, & -2i\, e^{2i\delta_1^-}\frac{\beta_1\beta_2^*}{2e + i}e^{2i\delta_2^+} \\ -2i\, e^{2i\delta_2^-}\frac{\beta_2\beta_1^*}{2e + i}e^{2i\delta_1^+}, & e^{2i\delta_2^-}\frac{2e + iB}{2e + i}e^{2i\delta_2^+} \end{pmatrix} \qquad (6.6)$$

urther introduce :

$$\delta = \delta_1^0 - \delta_2^0, \quad s = \sin(\delta), \quad c = \cos(\delta) \qquad (6.7)$$

$$e' = es - \frac{Bc}{2}, \qquad e'' = ec + \frac{Bs}{2}, \qquad R = \sqrt{e'^2 + B_1B_2}$$

$$\delta' = \delta - 2\delta_1^+ + 2\delta_2^+ = (\delta_1^- - \delta_1^+) - (\delta_2^- - \delta_2^+)$$

Then the eigenvalues write :

$$e^{2i\delta_\sigma(E)} = e^{i(2\delta_1^0-\delta)}\frac{2e'' - 2i\sigma R}{2e + i} \qquad (\sigma = \pm 1) \qquad (6.8)$$

and the projectors write :

$$P_\sigma(E) = \begin{pmatrix} \frac{R - \sigma e'}{2R}, & \sigma\frac{e^{i\delta'}}{2R}\beta_1\beta_2^* \\ \sigma\frac{e^{-i\delta'}}{2R}\beta_2\beta_1^*, & \frac{R + \sigma e'}{2R} \end{pmatrix} (\sigma = \pm 1) \qquad (6.9)$$

Having obtained these expressions, it is easy to verify that they satisfy the required conditions $P_+ + P_- = 1$ and $SP_\sigma = e^{2i\delta_\sigma}P_\sigma$.

There is an ambiguity of π in the definition of δ. Adding π to δ exchanges the $\pm$ terms. When $|\delta_i^0| << 1$, taking the small value for δ (and the positive square root for R), the large eigenphase shift is δ_+ and the small one is δ_-.

At this point, we may reconsider the argument of Y.Yamaguchi. Let us assume that a good approximation to the S matrix is given by just one diagonal component :

$$S_{i,j}(E) \approx e^{2i\delta_+(E)} K(E)_{i,+} K(E)^*_{j,+} = e^{2i\delta_+(E)} P_+(E)_{i,j}. \qquad (6.10)$$

The decay amplitudes are given by the Fourier transform of $S_{i,j}(E)$. To isolate the rapidly varying part of $S_{i,j}(E)$, the safest way is to consider the limit $\hat{S}_{i,j}(e) = \lim_{\Gamma\to 0} S_{i,j}(E_0+e\Gamma)$. By this method, at the end of Section 4, we obtained real decay amplitudes. However, it was assumed that $\lim_{\Gamma\to 0} P_+(E_0+e\Gamma) = P_+(E_0)$, which amounts assuming that $P_+(E)$ is a slowly varying function. We can see now that this was not correct, since $P_\pm(E_0+e\Gamma)$ is a function of e, in fact <u>independent</u> of Γ as we see from (6.7) - (6.9). This means that the eigenvectors are rapidly rotating for E near E_0, and this must be taken into account in order to obtain properly the properties of the resonance. The neglect of this effect is the flaw in Yamaguchi's Ansatz in the case of small strong phase shifts.

Now, the above expressions for the components of S(E) are writen as functions of e $= (E-E_0)/\Gamma$, and they are independent of Γ. Thus, they cannot be simplified in the limit of narrow width, and, apparently, we have to take the Fourier transform of these complicated expressions.

However, in the case $s << 1$ of small strong interaction phase shifts, one may notice that the square root R (defined in (6.7)) varies slowly by comparison to the resonant

denominator. In fact, the branch points $e = e_{b,\pm} = \frac{1}{s}(\frac{Bc}{2} \pm i\sqrt{B_1B_2})$ of R are far from the real axis, compared to the pôle $e = e_p = -i/2$. This means that we can expand R (and 1/R) in powers of s, but we will now see that this fails to give any effect of the small diagonal component. When doing this, we find that the contribution of the small diagonal component is unexpectedly small. The coefficients in the expansion are polynomials P(e) and one has Fourier transforms like

$$\int_{-\infty}^{+\infty}\frac{de}{2\pi}\,\frac{P(e)}{e+i/2}\,e^{-ie\tau} = Q(i\frac{d}{d\tau})\,\delta(\tau) + \int_{-\infty}^{+\infty}\frac{de}{2\pi}\,\frac{P(-i/2)}{e+i/2}\,e^{-ie\tau} \qquad (6.11)$$

where $Q(e) = \frac{P(e)-P(-i/2)}{e+i/2}$ is a polynomial. The term $Q(i\frac{d}{d\tau})\,\delta(\tau)$ do not contribute to the decay amplitudes, and one sees that one may replace e by -i/2 everywhere in the expansion. This means that we can obtain the whole expansion by just keeping the pôle at $e_p = -i/2$. Then, the values at $e = -i/2$ are easily computed :

$$2R|_{e=-i/2} = c + iBs, \qquad (6.12)$$

$$(2e+i)e^{2i\delta_\sigma}|_{e=-i/2} = e^{i(2\delta_1^0-\delta)}\,(2e'' - 2i\sigma R)|_{e=-i/2}$$
$$= -i\,e^{i(2\delta_1^0-\delta)}\,(c + iBs)(1+\sigma) \qquad (6.13)$$

For $\sigma = -1$, the result is that the pole is not present in $e^{2i\delta_-}$, nor in $e^{2i\delta_-(E)}\,P_-(E)$. This could be foreseen, since the residue of the S(E) matrix has only one non-vanishing eigenvalue. But this implies that the small diagonal component gives a <u>zero contribution to all orders</u> in $s = \sin(\delta_1^0 - \delta_2^0)$.

We see now that, to all orders in $s = \sin(\delta_1^0 - \delta_2^0)$, the full decay amplitudes are given by the large diagonal component of S(E) alone, since the contribution of $e^{2i\delta_-(E)}\,P_-(E)$ has just been shown to vanish. So, the possibility that the effect of the strong final-state-interaction phases is restored by the small diagonal components of the S-matrix is ruled out because this effect is O(1) in $s = \sin(\delta_1^0 - \delta_2^0)$.

Therefore all the effect should come from the large component alone, and there must be a flaw in the reasoning of the apparently straigtforward argument of the introduction. We will see now that the effect will come from the <u>rapid variation with the energy</u> of the main <u>eigenvector.</u>

At first sight, it is not quite clear why the variation of a real quantity should produce complex phases (see the argument of the introduction). To obtain this effect, one must use the precise link between decay amplitudes and Fourier transform of the S-matrix, exposed in detail in §4. Let us see by a simple calculation how this occurs. Consider some S matrix elements of the form

$$M_i(E) = (1 + c_i\frac{E-E_0}{\Gamma})\,e^{2i\delta(E)}, \qquad e^{2i\delta(E)} = 1 - \frac{i\Gamma}{E-E_0+i\Gamma/2} \qquad (6.14)$$

where the coefficients c_i are <u>real</u> constants. At first sight, there is only one phase $\delta(E)$ in this problem, and one may expect that the corresponding decay amplitudes will be real with respect to each other. However, let us compute the Fourier transforms. One has

$$M_i(E) = 1 - ic_i + c_i\frac{E-E_0}{\Gamma} - (1 - i\frac{c_i}{2})\frac{i\Gamma}{E-E_0+i\Gamma/2}, \qquad (6.15)$$

$$\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\,M_i(E)\,e^{-iEt} = (1 - ic_i - c_i\frac{E_0-i\frac{d}{dt}}{\Gamma})\,\delta(t) - \theta(t)\,(1 - i\frac{c_i}{2})\,\Gamma\,e^{-i(E_0-i\Gamma/2)t} \qquad (6.16)$$

So, the decay amplitudes are proportional to $1 - i\frac{c_i}{2}$, and the real coefficients c_i produce phases.

As exposed above, the expansion in s fails to give an estimation of the contribution of the small diagonal component. However, we can obtain for it the following expression when $\Gamma\frac{|\beta_1\beta_2|}{|s|}|t| >> 1$.

$$\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\,e^{2i\delta_-(E)}\,P_-(E)\,e^{-iEt} \approx \frac{1}{\sqrt{2\pi}}\frac{\sqrt{|\beta_1\beta_2|}}{\sqrt{|s|}}\,\frac{i\eta|\beta_1|e^{i\delta_2^0} - |\beta_2|e^{i\delta_1^0}}{|\beta_1|e^{-i\delta_2^0} + i\eta|\beta_2|e^{-i\delta_1^0}}\ \times$$

(6.17)

$$\times\ \frac{1}{2}\begin{pmatrix} e^{i\delta'}\beta_1/|\beta_1| \\ i\eta\beta_2/|\beta_2| \end{pmatrix}\left(e^{-i\delta'}\frac{\beta_1^*}{|\beta_1|},\ i\eta\frac{\beta_2^*}{|\beta_2|}\right)\sqrt{\frac{\Gamma}{|t|}}\,e^{-i(E_0+\frac{Bc}{2s}\Gamma)t}\,e^{-\Gamma\frac{|\beta_1\beta_2|}{|s|}|t|}$$

(η = sign(t/s)). This result is not valid near the values $|s| = 1$, $|\beta_1| = |\beta_2|$, where the pôle coincides with one of the two branch points. The $\delta_\pm$ solutions are distinguished by chosing $\delta = \delta_1^0 - \delta_2^0$ such that $|\delta| < \frac{\pi}{2}$.

One sees on this expression that the small component vanishes more quickly than any power of s, but contributes at times of order $s\Gamma^{-1}$. This shows that the small component contributes to the decay amplitudes at short times, and cannot explain their strong interaction phases. At a fixed time, it vanishes exponentially in the limit $\delta_1^0 - \delta_2^0 \to 0$.

For the large diagonal component, at $t \neq 0$, we have :

$$\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\,e^{2i\delta_+(E)}\,P_+(E)\,e^{-iEt} = -\,\Gamma\begin{pmatrix} e^{2i\delta_1^-}\beta_1 \\ e^{2i\delta_2^-}\beta_2 \end{pmatrix}\left(e^{2i\delta_1^+}\beta_1^*,\ e^{2i\delta_2^+}\beta_2^*\right)\,\theta(t)\,e^{-i(E_0-i\frac{1}{2}\Gamma)t}$$
$$-\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\,e^{2i\delta_-(E)}\,P_-(E)\,e^{-iEt} \qquad (6.18)$$

where the second term given by (6.17) is non-analytic in s at $s = 0$. Here one may still notice a causality drawback of the approximation $S(E) \approx e^{2i\delta_+(E)} P_+(E)$, since the corresponding decay amplitudes do not vanish at $t < 0$. This is due to the lack of analyticity of the separate diagonal components in the upper half complex E plane, where we have a branch point of the square root R.

The conclusion of this section is that, even in the case of small strong phase shifts, where we can single out a resonant eigenvalue of the S-matrix, one obtains the usual formulas for the decay amplitudes including the strong phase shifts, as a result of the effect of the rotation of the S-matrix eigenstate.

7 - Concluding remarks

We have seen that, using a usual form for the $S(E)$ matrix near a narrow resonance, the Yamaguchi paradox does not arises. There are two reasons for that. First, all the phase shifts of the diagonalized $S(E)$ matrix have a rapid variation near the resonant energy. Second, the eigenvectors also have a rapid variation.

Still, there is an intriguing possibility to construct a S matrix describing a resonance for which the decay amplitudes are lacking the final state strong interaction phase shifts. From our conclusions above, we only have to fullfill two conditions : only one resonating eigenphase shift, and slowly varying eigenvectors. We can do that by the following construction. First define $S'(E)$ from the usual expression, but replacing the very small width Γ by say $100\,\Gamma$:

$$S'_{i,j}(E) = e^{2i\delta_i^-} \left\{ \delta_{i,j} - \frac{i}{v} \frac{100\,\gamma_i\,\gamma_j^*}{E - E_0 + 100\,i\,\Gamma/2} \right\} e^{2i\delta_j^+} \qquad (7.1)$$

Since $100\,\Gamma$ and $100\,\gamma_i\,\gamma_j^*$ are still very small, strong interaction experiments will miss the structure near E_0, and give $S^0_{i,j}(E) = e^{2i\delta_i^0}\,\delta_{i,j}$, $\delta_i^0 = \delta_i^- + \delta_i^+$, for the S matrix.

Next, consider the expansion of $S'(E)$ in diagonals components :

$$S'_{i,j}(E) = \sum_k K_{i,k}(E)\, e^{2i\delta_k(E)}\, K^{-1}_{k,j}(E) = \sum_k P^k_{i,j}(E)\, e^{2i\delta_k(E)} \qquad (7.2)$$

where $K(E)$ is a diagonalizing unitary matrix, $P^k(E)$ is the projector $P^k_{i,j}(E) = K_{i,k}(E)\, K^{-1}_{k,j}(E)$ on the eigenvector corresponding to the eigenvalue $e^{2i\delta_k(E)}$. Assuming that the strong phase shifts δ_i^0 are small, only one eigenphase shift, say $\delta_1(E)$ is large. However, all eigenphase shifts and all eigenvectors have slow variation on the scale Γ.

Finally, replace the eigenphase shift $\delta_1(E)$ by a resonating phase shift $\delta_{res}(E)$, given by

$$e^{2i\delta_{res}(E)} = \frac{E - E_0 - i\,\Gamma/2}{E - E_0 + i\,\Gamma/2}, \qquad (7.3)$$

in an interval of width about $10\,\Gamma$ around E_0, and joining smoothly to $\delta_1(E)$ for $|E-E_0| > 100\,\Gamma$ (see fig. 2). Our S matrix will be :

$$S(E) = S'(E) + P^1(E)\,(e^{2i\delta_{res}(E)} - e^{2i\delta_1(E)}) \qquad (7.4)$$

and its diagonal expansion writes

$$S(E) = P^1(E)\, e^{2i\delta_{res}(E)} + \sum_{k>1} P^k(E)\, e^{2i\delta_k(E)}. \qquad (7.5)$$

This S matrix is unitary, since we have kept eigenvalues of modulus 1, and on a strong interaction scale, that is for $|E-E_0| \gg 100\,\Gamma$, it is equal to $S^0(E)$. Now, the only rapidly varying part is $e^{2i\delta_{res}(E)}$ and, except at short times of order $\frac{1}{100}\Gamma^{-1}$, we have

$$\int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, S(E)\, e^{-iEt} \approx -\Gamma P^1(E_0)\, \theta(t)\, e^{-i(E_0 - i\frac{1}{2}\Gamma)t} \qquad (7.6)$$

The corresponding decay amplitudes, up to a common factor are given by

$$A_i = K_{i,1}(E_0) \qquad (7.7)$$

Assuming for clarity that the S matrix is symmetrical by some antilinear symmetry, the diagonalizing K(E) matrix is real, and the decay amplitudes are real with respect to each over. So, we have constructed an S matrix describing a resonance for wich the usual expectation, that the decay amplitudes have the strong interaction phases $e^{2i\delta_i^-} = e^{i\delta_i^0}$, is in default. As we see in Fig. 2, the two small eigenphase shifts have an energy variation much slower compared to the large resonant phase shift $\delta_1(E)$, than in Fig. 1.

Now, it is of course of interest to identify the physical mechanism at work behind our example. We would like to know when the phases are present and when they are not. Actually, the mechanism seems rather involved. First, we must have a "strong" weak interaction responsible for a large mixing of the strong channels near energy E_0. Next, the weak interaction resonance must be coupled to one and only one of these mixed channels, which is a very special circumstance. One may also notice that the "strong" weak interaction cannot be local since (see Section 6) the Fourier transform of S(E) extends to times $t < 0$ of order $\frac{1}{100}\Gamma^{-1}$.

All this appears quite unphysical. But it reminds us that the final state interaction includes not only the strong interaction, but also the part of the weak one not involved in the decay process. We have an example of this with the $K^0\bar{K}^0$ system. Consider the amplitudes A^L_{+-} and A^L_{00} for $K_L \to \pi^+\pi^-$ and $K_L \to \pi^0\pi^0$. Introduce the amplitudes $A^L_I e^{i\delta_I} = \langle (\pi\pi)_I \mid H_W \mid K_L \rangle$, where $(\pi\pi)_I$ is the outgoing eigenstate of the strong interaction hamiltonian for isospin I, and δ_I is the strong interaction $\pi\pi$ phase shift. Apparently, it is just a matter of Clebsch-Gordan coefficients to write :

$$A^L_{+-} = \frac{\sqrt{2}\, A^L_0\, e^{i\delta_0} + A^L_2\, e^{i\delta_2}}{\sqrt{3}}, \qquad A^L_{00} = \frac{A^L_0\, e^{i\delta_0} - \sqrt{2}\, A^L_2\, e^{i\delta_2}}{\sqrt{3}}. \qquad (7.8)$$

However, these formulae are not in agreement with the Lee-Oehme-Wu-Yang[3] generalization of the Weisskopf-Wigner theory. In fact, we have disregarded the $\pi\pi$ final state interaction through the K_S resonance :

$$K_L \to \pi\pi \xrightarrow{(K_S)} \pi\pi$$

This interaction is weak but effective in mixing the isospin channels and changing the phases. Its effect is of course taken into account in the Lee-Oehme-Wu-Yang theory.

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Figure captions

Fig. 1

The heavy lines are the phases of the eigenvalues of a standard S(E) matrix accross a narrow resonance, assuming three channels. When the weak interaction is neglected, the phase shifts δ_i^0 (i = 1, 2, 3 for three channels) of the already diagonal $S^0(E)$ matrix are constants shown by the horizontal dotted lines. The B_i's are the assumed branching ratios.

Fig. 2

The heavy lines are the phases of the eigenvalues of an S(E) matrix, artificially constructed in such a way that the three decay amplitudes are real with respect to each other.

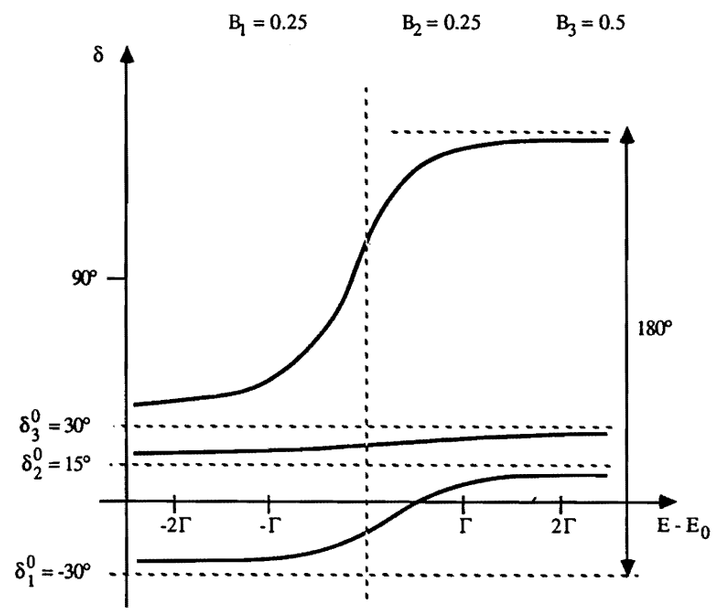
$B_1 = 0.25$
$B_2 = 0.25$
$B_3 = 0.5$
δ
90°
180°
$\delta_3^0 = 30°$
$\delta_2^0 = 15°$
$\delta_1^0 = -30°$
-2Γ
-Γ
Γ
2Γ
E - E0

FIG. 1

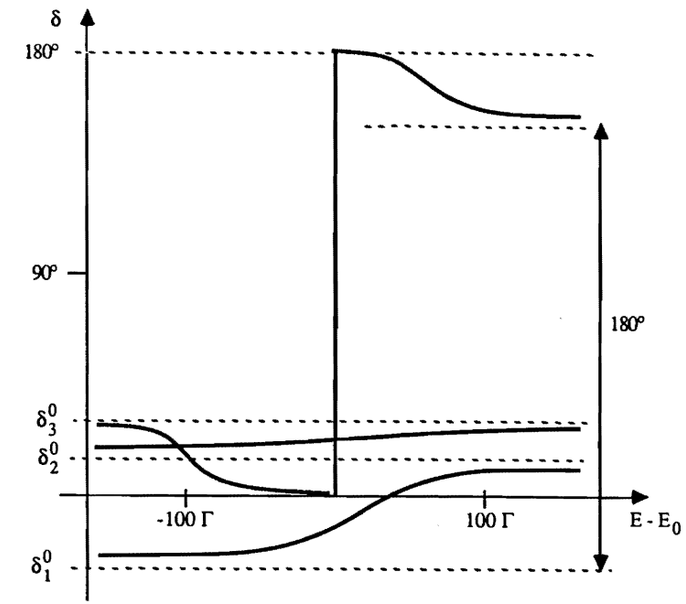
δ
180°
90°
180°
δ_3^0
δ_2^0
δ_1^0
-100 Γ
100 Γ
E - E0

FIG. 2

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