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N=2 super Boussinesq hierarchy: Lax pairs and conservation laws

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Abstract

We study the integrability properties of the one-parameter family of $N = 2$ super Boussinesq equations obtained earlier by two of us (E.I. & S.K., Phys. Lett. B 291 (1992) 63) as a hamiltonian flow on the $N = 2$ super-W_3 algebra. We show that it admits nontrivial higher order conserved quantities and hence can give rise to integrable hierarchies only for three values of the involved parameter, $\alpha = -2,\ -1/2,\ 5/2$. We find that for the case $\alpha = -1/2$ there exists a Lax pair formulation in terms of local $N = 2$ pseudo-differential operators, while for $\alpha = -2$ the associated equation turns out to be bi-hamiltonian.

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1 Introduction

During the last few years there has been a considerable interest in the integrable evolution equations associated with the W-type algebras and superalgebras (see, e.g., [1-8]). Since the discovery [9, 10] that the classical Virasoro algebra provides the second hamiltonian structure for the KdV hierarchy, there appeared a lot of papers where various supersymmetric and W extensions of this algebra were treated along a similar line and the relevant hierarchies of the evolution equations were deduced and analyzed. In particular, integrable $N = 1$ [11], $N = 2$ [12], $N = 3$ [13] and $N = 4$ [14] supersymmetric KdV equations have been constructed, with the $N = 1, 2, 3, 4$ super Virasoro algebras as the second hamiltonian structures. In [1] it has been shown that the classical W_3 algebra (with a non-zero central charge) defines a second hamiltonian structure for the Boussinesq hierarchy. The Lax pair formulation of the latter in terms of the Gel'fand-Dikii pseudo-differential operators closely related to the hamiltonian formulation also has been given (see, e.g., [2]).

Obviously, supersymmetric extensions of the Boussinesq equation should be associated, in the above sense, with super-W_3 algebras. In ref. [8] two of us (E.I. & S.K.) have constructed, in a manifestly supersymmetric $N = 2$ superfield form, the most general $N = 2$ super Boussinesq equation for which the second hamiltonian structure is given by the classical $N = 2$ super-W_3 algebra [15]. This equation turned out to contain an arbitrary real parameter α, much like the $N = 2$ super KdV equation [12].

In this letter we address the question of existence of the whole $N = 2$ Boussinesq hierarchy, i.e. we examine whether the equation constructed in [8] admits an infinite sequence of conserved quantities in involution and a Lax pair formulation. We find that the nontrivial higher order conserved quantities exist only for *three* values of α, namely for $\alpha = -2, -1/2, 5/2$. This again highly resembles the case of $N = 2$ super KdV equation which is known to give rise to the integrable hierarchies only for three special values of the involved parameter [12]. We prove the integrability of the option $\alpha = -1/2$ by finding the Lax pair for it (in terms of the $N = 2$ pseudo-differential operators). We also show that the equation corresponding to the choice $\alpha = -2$ possesses the first hamiltonian structure. This property, together with the existence of higher-order conservation laws, suggest that the $\alpha = -2$ equation is integrable as well.

2 N=2 super Boussinesq equation

Let us first briefly recall the basic points of ref. [8] in what concerns the $N = 2$ super Boussinesq equation and its relation to the $N = 2$ super-W_3 algebra.

All the basic currents of $N = 2$ super-W_3 algebra [15, 16] are accomodated by the spin 1 supercurrent $J(Z)$ and the spin 2 supercurrent $T(Z)$, where $Z \equiv (x, \theta, \bar{\theta})$ are the coordinates of the $N = 2$, $1D$ superspace. The supercurrent $J(Z)$ generates the $N = 2$ super Virasoro algebra, while $T(Z)$ can be chosen to be primary with respect to the latter. The closed set of SOPE's for these supercurrents, such that it defines the classical $N = 2$ super-W_3 algebra, has been written down in [8]. Here we prefer an equivalent notation via the super Poisson brackets

$$\{V_A(Z_1), V_B(Z_2)\}_{(2)} = \mathcal{D}_{AB}(Z_2)\Delta(Z_{12}) \,, \tag{2.1}$$

where $V_{A=1,2} \equiv (J, T)$, and $\Delta(Z_{12})$ denotes the $N = 2$ super delta-function

$$\Delta(Z_{12}) = \theta_{12}\bar{\theta}_{12}\delta(z_1 - z_2) \,. \tag{2.2}$$

The subscript "(2)" of the super Poisson brackets indicates that they provide the second hamiltonian structure for the $N=2$ super Boussinesq equation to be defined below.

The 2x2 super-differential operator $\mathcal{D}_{AB}$ in (2.1) encodes the full information about the structure of the classical $N=2$ super-W_3 algebra. The explicit form of its entries is as follows:

$$
\begin{aligned}
\mathcal{D}_{11} &= -\frac{c}{8}\left[\mathcal{D},\overline{\mathcal{D}}\right]\partial + \overline{\mathcal{D}}J\mathcal{D} + \mathcal{D}J\overline{\mathcal{D}} + J\partial + \partial J\,, \\
\mathcal{D}_{12} &= \overline{\mathcal{D}}T\mathcal{D} + \mathcal{D}T\overline{\mathcal{D}} + 2T\partial + \partial T\,, \\
\mathcal{D}_{21} &= \overline{\mathcal{D}}T\mathcal{D} + \mathcal{D}T\overline{\mathcal{D}} + 2T\partial + 2\partial T\,, \\
\mathcal{D}_{22} &= \frac{c}{8}\left[\mathcal{D},\overline{\mathcal{D}}\right]\partial^3 - 2J\partial^3 - 6\mathcal{D}J\overline{\mathcal{D}}\partial^2 - 6\overline{\mathcal{D}}J\mathcal{D}\partial^2 - 6\partial J\partial^2 - \overline{\mathcal{D}}(8\partial J - 5T - B^{(2)})\mathcal{D}\partial \\
&\quad -(5T - 2\left[\overline{\mathcal{D}},\mathcal{D}\right]J + B^{(2)})\left[\mathcal{D},\overline{\mathcal{D}}\right]\partial - \mathcal{D}(8\partial J + 5T + B^{(2)})\overline{\mathcal{D}}\partial \\
&\quad + \left(\frac{3}{2}\left[\overline{\mathcal{D}},\mathcal{D}\right]T - 6\partial^2 J + U^{(3)}\right)\partial - \frac{1}{2}\partial\left(5T - 2\left[\overline{\mathcal{D}},\mathcal{D}\right]J + B^{(2)}\right)\left[\mathcal{D},\overline{\mathcal{D}}\right] \\
&\quad - \left(3\partial\mathcal{D}T + 3\partial^2\mathcal{D}J + \overline{\Psi}^{(7/2)}\right)\overline{\mathcal{D}} + \left(3\partial\overline{\mathcal{D}}T - 3\partial^2\overline{\mathcal{D}}J - \Psi^{(7/2)}\right)\mathcal{D} \\
&\quad + \left(-2\partial^3 J + \partial\left[\overline{\mathcal{D}},\mathcal{D}\right]T + \frac{1}{2}\partial U^{(3)} + \frac{1}{2}\overline{\mathcal{D}}\overline{\Psi}^{(7/2)} + \frac{1}{2}\mathcal{D}\Psi^{(7/2)} - \frac{1}{4}\partial\left[\overline{\mathcal{D}},\mathcal{D}\right]B^{(2)}\right) \qquad (2.3)
\end{aligned}
$$

Here c is the central charge taking an arbitrary value at the classical level, and

$$B^{(2)}(Z),\ \Psi^{(7/2)}(Z),\ \overline{\Psi}^{(7/2)}(Z),\ U^{(3)}(Z)$$

are the composite supercurrents with spin 2, 7/2, 7/2, 3, respectively,

$$
\begin{aligned}
B^{(2)} &= \frac{8}{c}J^2\,, \\
\overline{\Psi}^{(7/2)} &= \frac{8}{c}\partial(J\mathcal{D}J) - \frac{72}{c}T\mathcal{D}J + \frac{36}{c}\left[\overline{\mathcal{D}},\mathcal{D}\right]J\mathcal{D}J + \frac{8}{c}J\mathcal{D}T - \frac{128}{c^2}J^2\mathcal{D}J + \frac{4}{c}\partial J\mathcal{D}J\,, \\
\Psi^{(7/2)} &= -\frac{8}{c}\partial\left(J\overline{\mathcal{D}}J\right) - \frac{72}{c}T\overline{\mathcal{D}}J + \frac{36}{c}\left[\overline{\mathcal{D}},\mathcal{D}\right]J\overline{\mathcal{D}}J + \frac{8}{c}J\overline{\mathcal{D}}T - \frac{128}{c^2}J^2\overline{\mathcal{D}}J - \frac{4}{c}\partial J\overline{\mathcal{D}}J, \\
U^{(3)} &= \frac{56}{c}JT - \frac{32}{c}J\left[\overline{\mathcal{D}},\mathcal{D}\right]J + \frac{128}{c^2}J^3 + \frac{120}{c}\overline{\mathcal{D}}J\mathcal{D}J\quad, \qquad (2.4)
\end{aligned}
$$

where

$$\theta_{12} = \theta_1 - \theta_2 \quad,\quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2 \quad,\quad Z_{12} = z_1 - z_2 + \frac{1}{2}\left(\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1\right) \qquad (2.5)$$

and the covariant spinor derivatives are defined by

$$
\begin{aligned}
\mathcal{D}_\theta = \frac{\partial}{\partial\theta} - \frac{1}{2}\bar{\theta}\partial_x\,, \qquad \bar{\mathcal{D}}_\theta = \frac{\partial}{\partial\bar{\theta}} - \frac{1}{2}\theta\partial_x\,, \\
\{\mathcal{D},\bar{\mathcal{D}}\} = -\partial_x\,, \qquad \mathcal{D}^2 = \bar{\mathcal{D}}^2 = 0\,. \qquad (2.6)
\end{aligned}
$$

The $N=2$ super Boussinesq equation can be defined as the system of two $N=2$ superfield equations for the supercurrents T, J with the $N=2$ super-W_3 algebra (2.1), (2.3), (2.4) as the second hamiltonian structure. In other words, it amounts to the following set of evolution equations:

$$\dot{T} = \{T,H\}_{(2)} \quad,\quad \dot{J} = \{J,H\}_{(2)} \qquad (2.7)$$

or, in the condensed notation,

$$\dot{V}_A = \mathcal{D}_{AB} \frac{\delta H}{\delta V_B} \ . \tag{2.8}$$

The hamiltonian H in (2.7) and (2.8) is given by

$$H = \int dZ \left(T + \alpha J^2 \right) \ . \tag{2.9}$$

We emphasize that (2.9) is the most general hamiltonian which can be constructed out of J and T under the natural assumptions that it respects $N = 2$ supersymmetry and has the same dimension 2 as the hamiltonian of the ordinary bosonic Boussinesq equation. Note the presence of the free parameter α in (2.9).

Now, using the Poisson brackets (2.1) and the definitions (2.3), (2.4), it is straightforward to find the explicit form of the $N = 2$ super Boussinesq system:

$$\begin{aligned}
\dot{T} &= -2J''' + \left[\overline{\mathcal{D}}, \mathcal{D}\right] T' + \frac{80}{c} \partial \left(\overline{\mathcal{D}} J \mathcal{D} J \right) - \frac{32}{c} J' \left[\overline{\mathcal{D}}, \mathcal{D}\right] J - \frac{16}{c} J \left[\overline{\mathcal{D}}, \mathcal{D}\right] J' + \frac{256}{c^2} J^2 J' \\
&+ \left(\frac{40}{c} - 2\alpha \right) \overline{\mathcal{D}} J \mathcal{D} T + \left(\frac{40}{c} - 2\alpha \right) \mathcal{D} J \overline{\mathcal{D}} T + \left(\frac{64}{c} + 4\alpha \right) J' T + \left(\frac{24}{c} + 2\alpha \right) J T' \ , \\
\dot{J} &= 2T' + \alpha \left(\frac{c}{4} \left[\overline{\mathcal{D}}, \mathcal{D}\right] J' + 4 J J' \right) \ .
\end{aligned} \tag{2.10}$$

The bosonic sector of eqs. (2.10) is a coupled system of equations for two spin 2 currents and the spin 1 $U(1)$ Kac-Moody current. As was shown in ref. [8], the standard Boussinesq equation decouples from this system only for $\alpha = -4/c$.

Note that the dependence on c in (2.10) is unessential and it can be removed by rescaling the superfields T and J. For definiteness, in what follows we will put $c = 8$. On the contrary, the dependence on α is crucial for achieving integrability: in the next section we will see that only for three special values of this parameter the above system results in integrable hierarchies.

3 Conservation laws

Now we turn to the basic theme of the present paper, the analysis of integrability of the set (2.10). The standard signal of integrability is the presence of an infinite sequence of mutually commuting nontrivial conserved quantities. In this section we report on the results of our study of the issue of existence of the higher order conserved quantities for (2.10) with $c = 8$.

In searching for such objects we made use of the standard method of undetermined coefficients. One considers an integral of degree n constructed from all the possible independent densities of degree n, each multiplied by an undetermined coefficient. (Two densities are dependent if their difference is a total (super)derivative). The coefficients are then fixed by requiring the integral to be a conservation law, that is time-independent.

In this way, with the heavy use of the symbolic manipulation program Mathematica [17], we have found the following first six conserved quantities:

$$\begin{aligned}
H_1 &= \int dZ \ J \ , \\
H_2 &= \int dZ \ \left(T + \alpha J^2 \right) \ ,
\end{aligned}$$

$$
\begin{aligned}
H_3 &= \delta \int dZ \left(JT + a_1 J^3 + a_2 \mathcal{D}J\overline{\mathcal{D}}J \right) , \\
H_4 &= \int dZ \left(T^2 + b_1 TJ^2 + b_2 TJ' + b_3 T\mathcal{D}\overline{\mathcal{D}}J + b_4 J^4 + b_5 J^2 \mathcal{D}\overline{\mathcal{D}}J + b_6 JJ'' \right) , \\
H_5 &= \int dZ \left(T\mathcal{D}\overline{\mathcal{D}}T + c_1 T^2 J + c_2 TJ^3 + c_3 TJJ' + c_4 TJ\mathcal{D}\overline{\mathcal{D}}J + c_5 T\mathcal{D}J\overline{\mathcal{D}}J + c_6 TJ'' \right. \\
&\quad \left. + c_7 J^5 + c_8 J^3 \mathcal{D}\overline{\mathcal{D}}J + c_9 J^2 J'' + c_{10} J^2 \mathcal{D}\overline{\mathcal{D}}J' + c_{11} J\mathcal{D}\overline{\mathcal{D}}J\mathcal{D}\overline{\mathcal{D}}J + c_{12} J\mathcal{D}\overline{\mathcal{D}}J'' \right) , \\
H_6 &= \delta \int dZ \left(d_1 T^3 + d_2 TT'' + d_3 T^2 J^2 + d_4 T^2 J' + d_5 \mathcal{D}T\overline{\mathcal{D}}TJ + d_6 T^2 \mathcal{D}\overline{\mathcal{D}}J \right. \\
&\quad + d_7 TJ^4 + d_8 TJ^2 J' + d_9 TJ^2 \mathcal{D}\overline{\mathcal{D}}J + d_{10} TJ\mathcal{D}J\overline{\mathcal{D}}J + d_{11} TJJ'' + d_{12} TJ'J' \\
&\quad + d_{13} TJ'\mathcal{D}\overline{\mathcal{D}}J + d_{14} T\mathcal{D}J'\overline{\mathcal{D}}J + d_{15} T\mathcal{D}J\overline{\mathcal{D}}J' + d_{16} T\mathcal{D}\overline{\mathcal{D}}J\mathcal{D}\overline{\mathcal{D}}J + d_{17} TJ''' \\
&\quad + d_{18} T\mathcal{D}\overline{\mathcal{D}}J'' + d_{19} J^6 + d_{20} J^4 \mathcal{D}\overline{\mathcal{D}}J + d_{21} J^3 J'' + d_{22} J^3 \mathcal{D}\overline{\mathcal{D}}J' + d_{23} J^2 \mathcal{D}\overline{\mathcal{D}}J\mathcal{D}\overline{\mathcal{D}}J \\
&\quad \left. + d_{24} J^2 J''' + d_{25} J^2 \mathcal{D}\overline{\mathcal{D}}J'' + d_{26} J\mathcal{D}\overline{\mathcal{D}}J\mathcal{D}\overline{\mathcal{D}}J' + d_{27} JJ^{IV} \right) ,
\end{aligned}
\tag{3.1}
$$

Here

$$
\delta \equiv \delta_{\alpha,-2} + \delta_{\alpha,5/2} \, . \tag{3.2}
$$

The most striking result of this exercise is that the nontrivial higher-order H_n ($n \geq 3$) exist if and only if the parameter α takes one of the following three values:

$$
\alpha = -2, \; -1/2, \; 5/2 \, . \tag{3.3}
$$

We have then verified that H_3 and H_6 exist only for the two values of $\alpha : \alpha = -2, \; 5/2$. Notice that the special value of α at which the Boussinesq equation in the bosonic sector decouples from two other equations is present among those in (3.3): for the choice $c = 8$ it is just $\alpha = -1/2$.

The corresponding values of the coefficients in H_3 – H_6 are given in Tables I – IV.

TABLE I. Coefficients of H_3.

α	a_1	a_2
-2	-5/4	-5/2
5/2	1	2

TABLE II. Coefficients of H_4.

α	b_1	b_2	b_3	b_4	b_5	b_6
-2	-4	4	8	4	-16	4
-1/2	2	4	8	-1/2	-1	1
5/2	14	-8	-16	17/2	-31	7

TABLE III. Coefficients of H_5.

α	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}
-2	-3/2	15/4	-10	-20	-25/2	5/2	-483/160	497/24	-77/16	21/4	-21/2	21/4
-1/2	1	0	0	0	-5	0	1/5	4/3	-2	-7/2	7	-1
5/2	-3/2	-15/2	25/2	25	10	-5/2	-33/10	113/6	-13/4	21/4	-21/2	4

TABLE IV. Coefficients of H_6.

α	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
-2	-1/9	1/6	2/3	-2/3	1	-4/3	-4/3	20/3	40/3
5/2	1/117	1/26	23/78	-7/39	5/13	-14/39	31/39	-101/39	-202/39
d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}
16	-10/3	-8/3	-16/3	-2	2	-16/3	2/3	4/3	8/9
-55/13	1	28/39	61/39	5/13	-5/13	61/39	-2/13	-4/13	5/18
d_{20}	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}	d_{27}		
-28/3	8/3	-32/9	32/3	-16/9	-14/3	20/3	2/3		
-89/39	115/234	-94/117	94/39	-73/234	-11/13	4/3	3/26		

The existence of these first higher order nontrivial conservation laws is a very strong indication of the complete integrability of the $N = 2$ super Boussinesq equation for the three values of α indicated in eq. (3.3) and, hence, the existence of $N = 2$ super Boussinesq hierarchies in these cases. In the next sections we will present the Lax pair for the $\alpha = -1/2$ case and the first hamiltonian structure for the $\alpha = -2$ case.

4 Lax pairs

In this Section we construct a Lax pair for the $N = 2$ super Boussinesq equation (2.10) (with $c = 8$).

We start from the general multi-parameter form of the third order Lax operator

$$L = \partial^3 + A_1 \cdot \partial^2 + A_2 \cdot [\mathcal{D}, \overline{\mathcal{D}}]\partial + A_3 \cdot \mathcal{D}\partial + A_4 \cdot \overline{\mathcal{D}}\partial + A_5 \cdot \partial + A_6 \cdot [\mathcal{D}, \overline{\mathcal{D}}] + A_7 \cdot \mathcal{D} + A_8 \cdot \overline{\mathcal{D}} + A_9 \ , \quad (4.1)$$

where $A_1, \ldots, A_9$ are arbitrary polynomials in J, T and their (super)derivatives with suitable dimensions and $U(1)$ properties. This means, in particular, that L must be a $U(1)$ singlet. For example, the first several coefficients in L can be parametrized as follows:

$$\begin{aligned} A_1 &= k^{(1)} J \ , \\ A_2 &= k^{(2)} J \ , \\ A_3 &= k^{(3)} \overline{\mathcal{D}} J \ , \\ A_4 &= k^{(4)} \mathcal{D} J \ , \\ A_5 &= k_1^{(5)} J^2 + k_2^{(5)} T + k_3^{(5)} \partial J + k_4^{(5)} [\mathcal{D}, \overline{\mathcal{D}}] J \quad \text{etc.} \ , \end{aligned} \quad (4.2)$$

where k are numerical coefficients. We search for those values of these parameters for which the Lax equation

$$L_t = \beta \left[L^{\frac{2}{3}}_{\geq 1} \ , L \right] \quad (4.3)$$

reproduces the $N = 2$ super Boussinesq set (2.10). Here, the subscript " ≥ 1" denotes a strictly differential part of the operator[1]. For $\beta = -3$ we have found the following two solutions, both

[1]We have checked that the Lax formulation of (2.10) in a more customary form $L_t = \beta \left[L^{\frac{2}{3}}_{+} \ , L \right]$, where " $+$" means restriction to the positive and zero parts of the $N = 2$ super pseudo-differential operator $L^{\frac{2}{3}}$, does not exist at all. The Lax representation of the type (4.3) was proposed earlier, e.g., for the $N = 2$ super KdV equation in ref. [18].

corresponding to the same value of $\alpha = -1/2$:

$$
\begin{aligned}
L^{(1)} &= \partial^3 + 3J\partial^2 - 3\overline{\mathcal{D}}J\mathcal{D}\partial + \left(2J^2 - T + \frac{3}{2}\partial J - \frac{1}{2}[\mathcal{D},\overline{\mathcal{D}}]J\right)\partial \\
&\quad + \left(\overline{\mathcal{D}}T - 4J\overline{\mathcal{D}}J - 2\partial\overline{\mathcal{D}}J\right)\mathcal{D}\,, \qquad (4.4)\\
L^{(2)} &= \partial^3 + \frac{3}{2}J\partial^2 + \frac{3}{2}J[\mathcal{D},\overline{\mathcal{D}}]\partial - 3\overline{\mathcal{D}}J\mathcal{D}\partial + \left(J^2 - \frac{1}{2}T + \frac{3}{4}\partial J - \frac{1}{4}[\mathcal{D},\overline{\mathcal{D}}]J\right)\partial \\
&\quad + \left(J^2 - \frac{1}{2}T + \frac{3}{4}\partial J - \frac{1}{4}[\mathcal{D},\overline{\mathcal{D}}]J\right)[\mathcal{D},\overline{\mathcal{D}}] + \left(\overline{\mathcal{D}}T - 4J\overline{\mathcal{D}}J - 2\partial\overline{\mathcal{D}}J\right)\mathcal{D}\,. \qquad (4.5)
\end{aligned}
$$

The solutions for the choice $\beta = 3$ in (4.3) can be obtained from (4.4)-(4.5) through the substitutions

$$J \to -J\,,\ T \to T\,,\ \mathcal{D} \to \overline{\mathcal{D}}\,,\ \overline{\mathcal{D}} \to \mathcal{D}\,.$$

They yield a Lax pair which is conjugate of (4.3).

Let us define the $N = 2$ super residue of a generic $N = 2$ super pseudo-differential operator

$$\mathcal{A} = \sum_{i=-\infty}^{M} (\beta_i + \gamma_i\mathcal{D} + \bar{\gamma}_i\overline{\mathcal{D}} + \rho_i[\mathcal{D},\overline{\mathcal{D}}])\partial^i \qquad (4.6)$$

as the coefficient of $[\mathcal{D},\overline{\mathcal{D}}]\partial^{-1}$:

$$\mathrm{Res}\ \mathcal{A} = \rho_{-1}\,. \qquad (4.7)$$

Then, following the reasoning of [12], we can show that (4.3) implies the equation

$$\frac{d}{dt}\int \mathrm{Res}\ L^{k/3}dZ = 0\,. \qquad (4.8)$$

This gives an infinite number of conservation laws. We have checked, that for $L^{(1)}$ all residues of the operators Res $L^{k/3}$ are equal to zero, so $L^{(1)}$ is a degenerated Lax operator, while for $L^{(2)}$ the expressions Res $L^{k/3}$ reproduce the conserved quantities for $\alpha = -1/2$ independently found in the previous section. Note that, despite the non-self-conjugacy of the operators L and $L^{k/3}$, the integrals in eq. (4.8) are real: the imaginary parts of the integrands in all cases prove to be full derivatives.

Thus we have proved the integrability of the $N = 2$ super Boussinesq equation for $\alpha = -1/2$. Its correct Lax form is given by eq. (4.3) with $\beta = -3$ and the Lax operator $L^{(2)}$ (4.5) (or its conjugate, with $\beta = 3$ in (4.3)). The Lax operators for the other cases listed in eq. (3.3) (if existing) cannot be represented by local super-differential operators (nonlocal Lax formulations for super KdV equations were considered, e.g., in [19]).

5 First hamiltonian structure

In the previous section we have found a Lax pair for the $N = 2$ super Boussinesq equation with $\alpha = -1/2$. Here we study for which values of α the set (2.10) can be given a first hamiltonian structure.

The first hamiltonian structure for the ordinary Boussinesq equation can be obtained from the second one by shifting the stress tensor by a constant. Here we will use the same idea.

In the case at hand there is a substantial freedom compared to the bosonic case since one can shift the supercurrents J and T by independent constants. However, a close inspection of the second Hamiltonian structure (2.1), (2.3) shows that only shifting the supercurrent T yields a self-consistent structure:

$$\{V_A(Z_1), V_B(Z_2)\}_{(1)} = \tilde{\mathcal{D}}_{AB}(Z_2)\Delta(Z_{12}) \,, \tag{5.1}$$

where now

$$\begin{aligned}
\tilde{\mathcal{D}}_{11} &= 0 \,, \\
\tilde{\mathcal{D}}_{12} &= 2\partial \,, \\
\tilde{\mathcal{D}}_{21} &= 2\partial \,, \\
\tilde{\mathcal{D}}_{22} &= -5\left[\mathcal{D}, \overline{\mathcal{D}}\right]\partial + 7J\partial + 9\mathcal{D}J\overline{\mathcal{D}} + 9\overline{\mathcal{D}}J\mathcal{D} + 8\partial J \,.
\end{aligned} \tag{5.2}$$

It is easy to check that this super Poisson structure together with the proper degree hamiltonian H_4 from the set (3.1) reproduce the $N = 2$ super Boussinesq equation only for $\alpha = -2$

$$\widetilde{H} = -\frac{1}{2}(H_4)_{(\alpha=-2)} = -\frac{1}{2}\int dZ \left(T^2 - 4TJ^2 + 4TJ' + 8T\mathcal{D}\overline{\mathcal{D}}J + 4J^4 - 16J^2\mathcal{D}\overline{\mathcal{D}}J + 4JJ''\right). \tag{5.3}$$

In this case eq. (2.10) can be represented in the form which follows from (2.8) via the substitutions $\mathcal{D}_{AB} \to \tilde{\mathcal{D}}_{AB}$, $H \to \widetilde{H}$. This proves that our $N = 2$ super Boussinesq equation is bi-Hamiltonian for $\alpha = -2$. In a number of cases the existence of two hamiltonian structures for the same equation already implies an infinite tower of higher order conservation laws (see, e.g., [9]). It would be of interest to see is it true in the present case, i.e. whether the existence of higher order conserved quantities for the $\alpha = -2$ $N = 2$ super Boussinesq equation can be traced to its bi-hamiltonian nature.

The proof of integrability of the $N = 2$ super Boussinesq equation for the remaining value of $\alpha = 5/2$ is an open problem. For this case we were not able to find neither a Lax pair nor a first hamiltonian structure.

6 Conclusion

In this paper we have presented the results of our study of the integrability properties of the $N = 2$ supersymmetric Boussinesq equation (2.10) with the $N = 2$ super-W_3 algebra as the underlying second hamiltonian structure. We have found that the integrable $N = 2$ super Boussinesq hierarchies can exist only for the three special values (3.3) of the free parameter α. The integrability in the case $\alpha = -1/2$ stems from the existence of the Lax pair, while in the case $\alpha = -2$ it could be a consequence of the presence of two hamiltonian structures. It is as yet unknown how to account for the integrability of the case with $\alpha = 5/2$. Perhaps, one should consider non-local Lax operators along the lines of ref. [19].

Finally, we wish to point out once more the analogy with the $N = 2$ super KdV equation [12] which is integrable also only for the three values of the corresponding free parameter, $a = -2, 4, 1$. However, this analogy is not quite literal: e.g., both cases $a = -2, 4$ are known to possess Lax formulations in terms of local pseudo-differential operators. The origin of this strange resemblance is not clear to us, since the underlying second hamiltonian structures of

both systems are essentially different: it is the $N = 2$ super-W_3 algebra in the case of super Boussinesq and the $N = 2$ super Virasoro algebra in the case of super KdV.

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Note added. After this paper has been submitted for publication, we became aware of a preprint by Yung [20] where the existence of the three distinguished values of the parameter α has been also established and the relevant conserved quantities, up to H_6, have been presented.

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