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Abstract

Inequalities for Gram determinants are used to obtain inequalities of the r.m.s. emittances in the case of coupled 2,3 dimensional motions of particle beams. It is also generalized to the case of motion with a larger number of degrees of freedom. As a consequence, due to coupling of the degrees of freedom, the increase of the product of projected emittances is a general phenomenon.

1. Introduction

In a one-dimensional motion of a particle beam the r.m.s. emittance ε_1 is defined as the square root of the determinant of the σ matrix. The latter is formed by the second-order statistical centered moments of the particle population in the two-dimensional phase space. The r.m.s. emittance ε_1 is given by[1]:

$$\varepsilon_1 = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle x\ x'\rangle^2} \tag{1}$$

where x,x' are the two conjugate coordinates in phase space with respect to the barycenter of the population and <...> designates the statistical average of the enclosed expression over the particle population of the beam.

In the case of coupling between the two directions of betatron motion and eventually between betatron and synchrotron motions, one has to consider a 2 or 3 dimensional motion. If one would have to take account of a coupling with spin motion, the motion would become 4-dimensional.

The generalization of the r.m.s. emittance to a p-dimensional motion is straightforward. The σ matrix is a p×p matrix formed by the second-order statistical centered moments: $\langle x_\alpha x_\beta\rangle$, $\langle x_\alpha x'_\beta\rangle$ and $\langle x'_\alpha x'_\beta\rangle$ (α, β = 1,...,p). The r.m.s. emittance $\varepsilon_{1,\ldots,p}$ is the square root of the determinant of the σ matrix. Its main feature is its invariance in a linear motion.

For each degree of freedom α one can still define a r.m.s. emittance ε_α as:

$$\varepsilon_\alpha = \sqrt{\langle x_\alpha^2\rangle\langle x'^2_\alpha\rangle - \langle x_\alpha\ x_\alpha'\rangle^2} \tag{2}$$

that is the projected emittance in the x_α, x'_α subspace . If the p degrees of freedom would be independent, the p-dimensional emittance $\varepsilon_{1\ldots p}$ is just the product of the p emittances ε_α, α = 1,...,p. Here it is shown that the p-dimensional emittance $\varepsilon_{1\ldots p}$ is in fact lower than this product when the motion is coupled and then satisfies the inequality:

$$\varepsilon_{1,\ldots,p} \leq \varepsilon_1 \varepsilon_2 \ldots \varepsilon_p \tag{3}$$

becoming an equality only if the motion is uncoupled. It is the generalization of a result obtained by L. Rivkin[2] in the case of a 2-dimensional motion:

$$\varepsilon_{x,y} \leq \varepsilon_x \varepsilon_y \tag{4}$$

where x and y designate the two coordinates of a betatron motion as usual.

For a motion with more than 2 degrees of freedom, the inequality (3) is in fact the two ends of a series of inequalities. These are similar to the Faguet's inequalities[3] that generalize the Hadamard inequality for Gram determinants. One will mainly consider the 2-dimensional and 3-dimensional motions that are the most usual cases in particle beam dynamics. One leaves the general proof to the Appendix. In Section 2. one again derives the inequality (4) for a 2-dimensional motion using one Faguet's inequality. For 3-dimensional motion the series of inequalities reduce to a double inequality that is derived in Section 3.

2. Emittance inequality for a 2-dimensional motion

The proof is obtained using a first inequality of Gram determinants derived by M. Faguet[3] as a Lemm.

The phase space coordinates of a particle are x, x', y, y'. The number of particles in the beam is n and their coordinates are numbered by the index i running from 1 to n.

In R_n (the real vector space of dimension n), let one introduce 4 vectors **U**, **U'**, **V**, **V'**. The coordinates of **U** are: $u_1, u_2, \ldots, u_n$, where $u_i = x_i/\sqrt{n}$ is proportional to the coordinate x_i of the i-th particle (i = 1,...,n). Similarly, the coordinates of **U'**, **V** and **V'** are respectively: $u'_i = x'_i/\sqrt{n}$, $v_i = y_i/\sqrt{n}$ and $v'_i = y'_i/\sqrt{n}$. With that notation the 10 second-order moments $\langle x^2\rangle$, $\langle xx'\rangle$, $\langle xy\rangle$, $\langle xy'\rangle$,... are the 10 scalar products of the vectors **U**, **U'**, **V**, **V'**. For instance:

$$\langle xy\rangle = \frac{1}{n}\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} u_i . v_i = \mathbf{U}.\mathbf{V} \tag{5}$$

and the determinant of the σ matrix is just the Gram determinant of the 4 vectors **U**, **U'**, **V**, **V'** :

$$\det \sigma = \begin{vmatrix} \mathbf{U}.\mathbf{U} & \mathbf{U}.\mathbf{U'} & \mathbf{U}.\mathbf{V} & \mathbf{U}.\mathbf{V'} \\ \mathbf{U'}.\mathbf{U} & \mathbf{U'}.\mathbf{U'} & \mathbf{U'}.\mathbf{V} & \mathbf{U'}.\mathbf{V'} \\ \mathbf{V}.\mathbf{U} & \mathbf{V}.\mathbf{U'} & \mathbf{V}.\mathbf{V} & \mathbf{V}.\mathbf{V'} \\ \mathbf{V'}.\mathbf{U} & \mathbf{V'}.\mathbf{U'} & \mathbf{V'}.\mathbf{V} & \mathbf{V'}.\mathbf{V'} \end{vmatrix} = G(\mathbf{U},\mathbf{U'},\mathbf{V},\mathbf{V'}) \tag{6}$$

The Faguet's inequality, obtained as a Lemm, is:

$$G(\mathbf{X},\mathbf{Y},\mathbf{Z})G(\mathbf{X}) \leq G(\mathbf{X},\mathbf{Z})G(\mathbf{X},\mathbf{Y}) \tag{7}$$

where **X**, **Y**, **Z** are any three sets of vectors in R_n. Let here apply that inequality for a void set **X** = **Ø**, with G(**X**) = 1, and **Y** = (**U**,**U'**), **Z** = (**V**,**V'**):

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$$G(\mathbf{Y},\mathbf{Z}) \leq G(\mathbf{Y})G(\mathbf{Z}) \tag{8}$$

i.e.:

$$G(\mathbf{U},\mathbf{U}',\mathbf{V},\mathbf{V}') \leq G(\mathbf{U},\mathbf{U}')G(\mathbf{V},\mathbf{V}') \tag{9}$$

Taking the square root of the latter (Gram determinants are never negative), one obtains the inequality (4), previously derived by L. Rivkin[2].

The inequalities (4) and (9) become equalities if the vectors **V**, **V'** are both orthogonal to the vectors **U**, **U'**, i.e. the four coupling moments $\langle xy \rangle$, $\langle xy' \rangle$, $\langle x'y \rangle$, $\langle x'y' \rangle$ vanish. Inversely, it can easily be shown that the equalities hold only if either the vectors **U**, **U'** or **V**, **V'** are linearly dependent (i.e. the emittance ε_x or ε_y vanishes), or **U**, **U'** are both orthogonal to **V**, **V'** (i.e. the four coupling moments vanish).

The inequality (4) means that, in a linear motion for which the emittance $\varepsilon_{x,y}$ is a constant, the coupling leads to an increase of the product of the two projected emittances ε_x and ε_y of a beam initially uncoupled. The coupling is usually detrimental to the beam qualities.

3. Emittance double inequality for 3-dimensional motion

The proof is obtained in the same way using the Faguet's inequality (7). Now, the phase space coordinates of a particle are x, x', y, y', z, z'. In R_n let one introduce 6 vectors **U**, **U'**, **V**, **V'**, **W**, **W'** with the coordinates: $u_i = x_i/\sqrt{n}$, $u'_i = x'_i/\sqrt{n}$, $v_i = y_i/\sqrt{n}$, $v'_i = y'_i/\sqrt{n}$, $w_i = z_i/\sqrt{n}$ and $w'_i = z'_i/\sqrt{n}$. Here again, the 21 second-order moments are the 21 scalar products of these 6 vectors and the determinant of the σ matrix is their Gram determinant. One first applies the Faguet' inequality (7) with $\mathbf{X} = \emptyset$, $\mathbf{Y} = (\mathbf{U}, \mathbf{U}')$ and $\mathbf{Z} = (\mathbf{V},\mathbf{V}', \mathbf{W}, \mathbf{W}')$:

$$G(\mathbf{U}, \mathbf{U}',\mathbf{V},\mathbf{V}',\mathbf{W}, \mathbf{W}') \leq G(\mathbf{V},\mathbf{V}',\mathbf{W}, \mathbf{W}')G(\mathbf{U}, \mathbf{U}') \tag{10}$$

and a second time with $\mathbf{X} = (\mathbf{U}, \mathbf{U}')$, $\mathbf{Y} = (\mathbf{V},\mathbf{V}')$ and $\mathbf{Z} = (\mathbf{W}, \mathbf{W}')$:

$$G(\mathbf{U}, \mathbf{U}',\mathbf{V},\mathbf{V}',\mathbf{W}, \mathbf{W}')\ G(\mathbf{U}, \mathbf{U}') \leq G(\mathbf{U}, \mathbf{U}',\mathbf{W}, \mathbf{W}')\ G(\mathbf{U}, \mathbf{U}',\mathbf{V},\mathbf{V}') \tag{11}$$

Multiplying the two preceding inequalities (10) and (11) and taking the square root, one obtains:

$$G(\mathbf{U}, \mathbf{U}',\mathbf{V},\mathbf{V}',\mathbf{W}, \mathbf{W}') \leq \sqrt{G(\mathbf{U}, \mathbf{U}',\mathbf{V},\mathbf{V}')\ G(\mathbf{V},\mathbf{V}',\mathbf{W}, \mathbf{W}')\ G(\mathbf{U}, \mathbf{U}',\mathbf{W}, \mathbf{W}')} \tag{12}$$

i.e. in terms of r.m.s. emittances:

$$\varepsilon_{x,y,z} \leq \sqrt{\varepsilon_{x,y}\ \varepsilon_{y,z}\ \varepsilon_{z,x}} \tag{13}$$

where $\varepsilon_{x,y,z}$ is the emittance of the 3-dimensional motion and $\varepsilon_{x,y}$, $\varepsilon_{y,z}$, $\varepsilon_{z,x}$ are the projected emittances of the 2-dimensional motions x-y, y-z, z-x respectively.

Finally, using the inequality (4) for the latter emittances, one obtains the double inequality:

$$\varepsilon_{x,y,z} \leq \sqrt{\varepsilon_{x,y}\,\varepsilon_{y,z}\,\varepsilon_{z,x}} \leq \varepsilon_x\,\varepsilon_y\,\varepsilon_z \tag{14}$$

Hereagain, the equality holds if, and only if, either one projected emittance ε_x, ε_y or ε_z vanishes or all the 12 coupling moments vanish.
In the 6-dimensional phase space $\varepsilon_{x,y,z}$ is proportional to the volume of a hyperellipsoid surrounding the particles. The physical meaning of this double inequality is that the square of that volume is smaller than the product of the volumes of the projections on the 4-dimensional subspaces (x, x', y, y'), (y, y', z, z'), (z, z', x, x') and of the volumes of the projections on the 2-dimensional subspaces (x, x'), (y, y'), (z, z'). Therefore, in a linear motion the coupling can only increase these products of projected emittances. It generalizes that property well-known for a coupled 2-dimensional motion.

APPENDIX

Hereafter, one doesn't distinguish between a coordinate x and its conjugate momentum x'. The phase space coordinates of a particle are x_α where the index α runs from 1 to 2p (p is the number of degrees of freedom of the motion). One will order the coordinates such that x_{2k} is the conjugate momentum of x_{2k-1}. To distinguish the coordinates of different particles, $x_{\alpha i}$ will designate the coordinates of the i-th particle. The number of particles is n and the index i runs from 1 to n.

In R_n let one introduce 2p vectors $\mathbf{U}_\alpha$ with the coordinates: $u_i = x_{\alpha i}/\sqrt{n}$, proportional to the coordinate $x_{\alpha i}$ of the i-th particle. With that notation the second-order moment $\langle x_\alpha x_\beta \rangle$ is the scalar product $\mathbf{U}_\alpha.\mathbf{U}_\beta$. The determinant of the σ matrix is the Gram determinant of the 2p vectors $\mathbf{U}_\alpha$: $G(\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{2p})$.

For a set of m vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ in R_n, M. Faguet[3] has derived the following series of inequalities of their Gram determinants:

$$\begin{aligned}
G(\mathbf{x}_1,\cdots,\mathbf{x}_m) &\leq \left\{\prod_{k=1}^{m} G(\mathbf{x}_1,\cdots,\mathbf{x}_{k-1},\mathbf{x}_{k+1},\cdots,\mathbf{x}_m)\right\}^{\frac{1}{m-1}} \\
&\leq \left\{\prod_{1\leq k<l\leq m} G(\mathbf{x}_1,\cdots,\mathbf{x}_{k-1},\mathbf{x}_{k+1},\cdots,\mathbf{x}_{l-1},\mathbf{x}_{l+1},\cdots,\mathbf{x}_m)\right\}^{\frac{1.2}{(m-1)(m-2)}} \leq \cdots \\
&\cdots \leq \left\{\prod_{1\leq s_1<\cdots<s_r\leq m} G(\mathbf{x}_{s_1},\mathbf{x}_{s_2},\cdots,\mathbf{x}_{s_r})\right\}^{\frac{1.2\cdots(m-r)}{(m-1)(m-2)\cdots r}} \leq \cdots \\
&\cdots \leq \left\{\prod_{1\leq s_1<s_2\leq m} G(\mathbf{x}_{s_1},\mathbf{x}_{s_2})\right\}^{\frac{1}{m-1}} \leq \prod_{s=1}^{m} G(\mathbf{x}_s)
\end{aligned} \tag{A1}$$

The two end terms of the series are just the well-known Hadamard inequality:

$$G(\mathbf{x}_1,\cdots,\mathbf{x}_m) \leq \prod_{s=1}^{m} G(\mathbf{x}_s) \tag{A2}$$

These inequalities are not convenient for the 2p vectors $\mathbf{U}_\alpha$ as they don't take into account that each doublet $\mathbf{U}_{2k-1}$, $\mathbf{U}_{2k}$ correspond to a pair of conjugate coordinates. In fact, in the derivation of the series of inequalities (A1) by M. Faguet nothing would be changed if one formally replaces each vector **x** by a doublet of vectors. Therefore, one can readily write a similar and convenient series of inequalities for the 2p vectors $\mathbf{U}_\alpha$:

$$
\begin{aligned}
G(\mathbf{U}_1,\cdots,\mathbf{U}_{2p}) &\le \left\{\prod_{k=1}^{p} G(\mathbf{U}_1,\cdots,\mathbf{U}_{2k-1},\mathbf{U}_{2k+1},\cdots,\mathbf{U}_{2p})\right\}^{\frac{1}{p-1}} \\
&\le \left\{\prod_{1\le k<l\le p} G(\mathbf{U}_1,\cdots,\mathbf{U}_{2k-1},\mathbf{U}_{2k+1},\cdots,\mathbf{U}_{2l-1},\mathbf{U}_{2l+1},\cdots,\mathbf{U}_{2p})\right\}^{\frac{1.2}{(p-1)(p-2)}} \le \cdots \\
\cdots &\le \left\{\prod_{1\le s_1<\cdots<s_r\le p} G(\mathbf{U}_{2s_1-1},\mathbf{U}_{2s_1},\mathbf{U}_{2s_2-1},\mathbf{U}_{2s_2},\cdots,\mathbf{U}_{2s_r-1},\mathbf{U}_{2s_r})\right\}^{\frac{1.2\cdots(p-r)}{(p-1)(p-2)\cdots r}} \le \cdots \\
\cdots &\le \left\{\prod_{1\le s_1<s_2\le m} G(\mathbf{U}_{2s_1-1},\mathbf{U}_{2s_1},\mathbf{U}_{2s_2-1},\mathbf{U}_{2s_2})\right\}^{\frac{1}{p-1}} \le \prod_{s=1}^{p} G(\mathbf{U}_{2s-1},\mathbf{U}_{2s})
\end{aligned}
\tag{A3}
$$

In the particular cases p = 2 and 3 one recovers the inequalities (9) and (12). These inequalities generalize at any degree of freedom the increase of the projected-emittance products as a result of the coupling on a beam initially uncoupled.

References

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