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The Geometrical Principles of String Theory Constructions

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Abstract

The principles of string theory constructions are introduced starting at an elementary level. Through a detailed analysis of the quantization of bosonic and spinning strings, we discuss how the requirements of quantum invariance under local and global world-sheet symmetries lead to general constraints in the construction of string theories.

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THE GEOMETRICAL PRINCIPLES OF STRING THEORY CONSTRUCTIONS

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1 INTRODUCTION

In their quest for a quantum consistent unified theory of all fundamental interactions among elementary particles, theoretical particle physicists in recent years have been entertaining the idea[1] that such a theory may be constructed within the framework of superstring theories[1]. As opposed to the previous attempts in the past fifteen years, all formulated within the framework of ordinary quantum field theory, superstring theories seem to possess the very same desirable properties that stimulated previous attempts as well as the resolution of problems which led to the dismissal of these attempts[2,3,4].

One of the most appealing features, not only from a particle physicist's point of view, is that string theories provide a unique unification between gravity and all other interactions. Indeed, *all* fundamental matter particles and gauge quanta mediating *all* interactions, including the graviton, are nothing but the same fundamental object, a relativistic string in different states of excitation. This idea resolves the centuries old dream of ultimate unification: the Universe and all that it contains being made of only one fundamental object, with dynamics so rich that it leads to this infinitely large variety of physical phenomena that we observe at all energy scales in our Universe.

Obviously, our present understanding of these theories is still far too poor to be able to make any definite conclusion concerning such fascinating speculations. About four years ago[5], when string theory became the prime candidate for fundamental unification, uniqueness was regarded as a very compelling argument. Indeed, only a handful of different theories were known at the time[1]. Since then however, countless new string theories have been constructed in space-time dimensions ranging from ten to four, with a few of the four dimensional theories leading to a massless particle content resembling that of the quarks and leptons in the Standard Model.

Furthermore, some four years ago, optimism was high that realistic particle phenomenology would easily be achieved withing string theory. This optimism was based on the study of the corresponding low-energy effective field theories[2,3,4]. Recent investigations however, at the level of string theory itself, show that the problems are much more subtle, with entirely new mechanisms coming into play[4,6].

The prevailing attitude nowadays[7], is to consider that all these theories are different vacua of a few underlying fundamental theories still to be formulated - the hope being

that non-perturbative phenomena would determine which among all possible string constructions are actually dynamically generated, together with the ensuing particle content and phenomenology.

Clearly, such a state of affairs makes it even more difficult to make any experimental contact with string theories. On the other hand , this is not a good enough reason to dismiss this approach to unification altogether. After all, string theories are the best and only (so far) consistent formulation for such a quantum unification, including gravity. Moreover, we should not forget that it took many years before the basic ideas of the Standard Model were widely accepted by the scientific community, and that the detailed experimental tests of the Standard Model are only just beginning at the new big accelerators[8].

From a different point of view, the study of string theories is also responsible for a strong revival of interest in two-dimensional conformal field theories[9] and related mathematical structures.In recent years, this interest has led to fundamental developments[9] in the field of two-dimensional statistical systems at and off criticality. New results point to deep relationships between so far unrelated fields in pure mathematics[9], with completely new insights into different problems[10]. These are truely fascinating subjects in their own right.

All these reasons make the effort involved in getting acquainted with string theory worthwhile. One could even go as far as to consider that some familiarity with the field should now be part of common knowledge for any (theoretical) particle physicist, in the same manner that such familiarity is now assumed for grand unified theories, supersymmetry, and so on. Herein lies the main motivation for these lectures.

It is of course not possible to discuss in such lectures all the aspects of the subject which have emerged over the last years. What will not be discussed for example, are the following points:

- the calculation of string scattering amplitudes, at any order of perturbation theory, either through path-integral methods[11], or through operatorial methods such as the infinite Grassmannian approach[12],
- string field theory[13], i.e. the second quantization of string theory,
- particle phenomenology from string theory[4].

What will be discussed are the "principles of string theory construction". By first studying in detail the case of the bosonic string, we shall understand the origin and the meaning of the structure and of the consistency constraints which appear when quantizing the system. Having identified the relevant structures, we shall then show how they easily generalize to the case of spinning strings, and how these general constraints lead to the formulation of "principles for string theory construction".

Hopefully , by starting at an introductory level, newcomers to the subject will not be put off by the jargon which will be introduced in a pedestrian manner as we go along. Interested readers should find these notes a useful starting point for their own research.

Basic references may be found in Refs. 1,14,15,16, where some historical comments concerning the development of the subject from a theory for strong interactions to a theory of fundamental unification are also given. Here, we shall follow more closely Refs. 17,18,19. We shall also use units in which $\hbar = 1 = c$, and we shall take space-time to be D-dimensional, with the Minkowski metric $\eta^{\mu\nu}(\mu,\nu = 0,1,...D-1))$ of signature mostly "+" signs.

2 RELATIVISTIC BOSONIC STRINGS

2.1 Classical Lagrangian Description

2.1.1 The action principle

Although we shall discuss right away the relativistic bosonic string, the interested reader may find it useful to first consider the case of the relativistic scalar particle, through an analysis similar to the one followed in this chapter. For more details, see Refs. 17-20.

A bosonic string, being a 1-dimensional object, sweeps out a 2-dimensional surface, its world-sheet, as it propagates freely through space-time. To represent this space-time trajectory, one introduces coordinates $x^\mu(\sigma,\tau)$ giving the space-time position of the string. These quantities are D functions of two (dimensionless) world-sheet coordinates σ and τ, parametrizing the two-dimensional surface. Both for open and for closed strings, we shall take $0 \leq \sigma \leq \pi$; τ is considered as the time-evolution parameter of the system. Clearly, having both the observer's time coordinate x^o and the world-sheet parameter τ present in this formulation is a redundant feature. This is the price to pay for having an explicit space-time covariant description.

In order that the physical properties of the system do not depend neither on the observer's reference frame in space-time nor on the world-sheet parametrization (σ,τ), it is necessary that the action describing the dynamics of the bosonic string be a space-time scalar and a world-sheet scalar. The total world-sheet area between some initial and final string configurations $x_i^\mu(\sigma) = x^\mu(\sigma,\tau=\tau_i)(i=1,2)$, properly rescaled by a dimensionful quantity, is the obvious candidate for such an action.

There are two ways of expressing such an action. The first, by measuring the area using the metric induced on the world-sheet by the space-time Minkowski metric. This corresponds to the Nambu-Goto action[21], leading to non-linear equations of motion. The second way to express the action is by coupling the 2-dimensional fields $x^\mu(\sigma,\tau)$, which transform as space-time vectors and world-sheet scalars, to an intrinsic world-sheet metric. This corresponds to the so-called Polyakov action[22], originally discussed in Ref. 23, which leads to linear equations of motion.

In the following, we shall use the notation $\xi^\alpha(\alpha=0,1)$ for world-sheet coordinates, with $\xi^0=\tau$ and $\xi^1=\sigma$. As usual, partial derivatives with respect to τ (resp.σ) will be denoted by a dot (resp. a prime) above quantities.

With these notations, from the invariant space-time line element

$$ds^2 = \eta^{\mu\nu}dx_\mu dx_\nu = \eta^{\mu\nu}\partial_\alpha x_\mu \partial_\beta x_\nu d\xi^\alpha d\xi^\beta, \tag{2.1}$$

it is clear that the induced world-sheet metric $\gamma_{\alpha\beta}$ is given by the components

$$\gamma_{\alpha\beta} = \eta^{\mu\nu}\partial_\alpha x_\mu \partial_\beta x_\nu. \tag{2.2}$$

The Nambu-Goto action[21] then reads

$$S[x^\mu] = \frac{-1}{2\pi\alpha'}\int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma(-det\gamma_{\alpha\beta})^{1/2} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \mathcal{L}(\dot{x},x'), \tag{2.3}$$

with

$$\mathcal{L}(\dot{x},x') = \frac{-1}{2\pi\alpha'}\sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}. \tag{2.4}$$

Here, α' is a dimensionful quantity, of dimension a lenght square, which thus determines the physical scales in the system, such as the scale of the mass spectrum of the bosonic

string. The choice of sign in the factor $(-det\gamma_{\alpha\beta})^{1/2}$ is due to the fact that the induced metric $\gamma_{\alpha\beta}$ has a world-sheet signature $(-+)$. This also guarantees that no point of the string moves faster than the speed of light.[14,17]

Introducing an intrinsic world-sheet metric $g_{\alpha\beta}$, of signature $(-+)$, the Polyakov action[22,23] is given as:

$$S[x^\mu, g_{\alpha\beta}] = \frac{-1}{4\pi\alpha'}\int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma\sqrt{-detg_{\alpha\beta}}\; g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}. \tag{2.5}$$

A priori, one could also add to it the usual Einstein-Hilbert action term

$$\frac{1}{4\pi\kappa}\int d^2\xi\sqrt{-g}R^{(2)}, \tag{2.6}$$

or a world-sheet cosmological term

$$\frac{-1}{4\pi\alpha'}\mu^2\int d^2\xi\sqrt{-g}. \tag{2.7}$$

However, the former term is a topological invariant in two dimensions (up to surface terms), which thus does not contribute to the equations of motion (in 2 dimensions, the Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R^{(2)}$ vanishes identically). The latter term is consistent with the classical equations of motion only if $\mu^2 = 0$.

At the quantum level, using a path-integral formulation[11], it may be seen that the Einstein-Hilbert action sets the strength of string interactions, whereas the cosmological term is induced by 1-loop short-distances singularities on the world-sheet[24]. It then has to be properly renormalized[24] so as to maintain the local Weyl symmetry of the classical Polyakov action (see below).

Let us remark here that the action (2.5) may also be used to represent the propagation of the bosonic string in a curved space-time background of metric $G_{\mu\nu}(x)$. This is easily done by replacing $\eta_{\mu\nu}$ in (2.5) by $G_{\mu\nu}(x)$, leading to the usual action of a non-linear sigma model in two dimensions. Such a formulation is the starting point of a much studied approach to the problem of space-time compactification in string theory[25].

By construction, both the Nambu-Goto and Polyakov actions are invariant under space-time Poincaré transformations and world-sheet reparametrizations. From the 2-dimensional field theoretical point of view, the former symmetry is a global internal symmetry, whereas the latter is a world-sheet gauge symmetry.

By Noether's theorem, associated to space-time translations and generalized rotations (i.e. rotations in space and Lorentz boosts), there exist locally conserved energy- and angular-momentum world-sheet currents, whose conserved charges, giving the total energy- and angular-momentum of the bosonic string, generate Poincaré transformations of the system. Clearly, the existence of such quantities follows from the fact that the bosonic string is propagating in a space-time with isometry precisely the Poincaré group. For a generic curved space-time, with arbitrary metric $G_{\mu\nu}$, such quantities would not exist, unless the space-time metric has some special symmetries.

In contradistinction, the world-sheet reparametrization gauge symmetry is always present for any string theory. Actually, it is precisely this gauge invariance, when imposed at the quantum level, which is responsible for all the profound and beautiful properties of string theory (this should become clear as we proceed with the discussion). Also, as we shall see in chapter 4, the formulation of "principles for string theory

construction" immediately follows from the requirement of this gauge invariance in the quantized theory.

Among world-sheet gauge transformations, we have local world-sheet reparametrizations in the same connected component of the diffeomorphism group as the identity transformation. These reparametrizations are generated by repeated applications of infinitesimal reparametrizations, thus preserving the orientation of the world-sheet. Correspondingly, there should exist two quantities, associated to the two arbitrary functions $\xi^\alpha(\xi)$ specifying such transformations, which generate infinitesimal reparametrizations and which vanish at all times for physical solutions. In other words, associated to world-sheet reparametrization invariance, we have two constraints generating the local gauge symmetry of the system. As we shall see, imposing this local gauge invariance at the quantum level leads to a constraint on the number of world-sheet degrees of freedom used in the construction of string theories.

There also exist world-sheet gauge transformations which are not connected to the identity transformation, so called disconnected or global gauge transformations. Depending on the world-sheet topology, there may exist orientation preserving global reparametrizations. These symmetries lead to a new type of constraint at the quantum level, not existing in ordinary field theory, which goes by the name of modular invariance. One may also impose invariance under orientation reversing global reparametrizations, thus describing unoriented string theories (indeed, both the linear and non-linear actions above are invariant under such transformations). At the quantum level, this leads to further constraints on the physical spectrum of the corresponding theories. Both these constraints and modular invariance will be discussed in chapter 4.

When compared to the Nambu-Goto action, the Polyakov action (2.5) actually possesses one more local symmetry (when $\mu^2 = 0$) under local Weyl rescalings of the intrinsic metric $g_{\alpha\beta}$:

$$g_{\alpha\beta}(\xi) \to e^{\phi(\xi)} g_{\alpha\beta}(\xi). \tag{2.8}$$

These transformations lead to local shifts in the conformal mode of the metric $g_{\alpha\beta}$, by the arbitrary world-sheet function $\phi(\xi)$. Hence, one expects one more constraint, involving the generator of local Weyl rescalings. Actually, this constraint combines with the previous two generators of local reparametrizations into the equation of motion for the metric $g_{\alpha\beta}$ which is an auxiliary field:

$$T_{\alpha\beta} = 0. \tag{2.9}$$

Here, $T_{\alpha\beta}$ is the world-sheet energy-momentum tensor of the scalar fields x^μ, with:

$$T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x_\mu - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x_\mu. \tag{2.10}$$

The constraint $T_{\alpha\beta} = 0$ is thus a consequence of local reparametrization and Weyl invariance. Its solution expresses the intrinsic metric $g_{\alpha\beta}$ as a Weyl rescaling of the induced metric $\gamma_{\alpha\beta}$. When substituted in the Polyakov action, one then recovers the Nambu-Goto action, thus showing that at the classical level, these two formulations of the relativistic bosonic string are indeed entirely equivalent.

This need not be true at the quantum level however, due to the conformal anomaly in two dimensions. In a quantization approach which explicitly preserves reparametrization gauge invariance but not Weyl invariance, such as Polyakov's path-integral approach[22], one finds that the conformal mode of the metric $g_{\alpha\beta}$ couples dynamically unless a specific condition on the number of degrees of freedom is met. In the case of

the bosonic string discussed here, this condition fixes the number of scalar fields x^μ (or the dimension of space-time) as $D = 26$.

In a quantization approach which explicitly preserves Weyl invariance but not reparameterization invariance, such as the canonical approach which we shall use in these lectures[20], this latter symmetry is not realized at the quantum level unless the same condition ensuring decoupling of the conformal mode in Polyakov's path-integral is met[26]. Hence, when this condition is satisfied, both the linear and non-linear approaches lead to equivalent descriptions of the system, also at the quantum level. In these lectures, we shall only consider the construction of string theories in which the conformal mode decouples, i.e. reparametrization *and* Weyl invariant string theories. It is also possible to construct reparametrization but not Weyl invariant string theories, in which the conformal mode is dynamical[27]. Recently, more progress has been made in that direction[28].

For these reasons, we shall now concentrate on the non-linear Nambu-Goto action, and use canonical Hamiltonian methods to quantize the system. The same results actually also follow for the linear action when applying the same methods[20].

Considering the Nambu-Goto action (2.3), the conserved world-sheet currents associated to space-time translations and generalized rotations are easily found to be respectively

$$P^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha x^\mu)}, \quad M^\alpha_{\mu\nu} = P^\alpha_\mu x_\nu - P^\alpha_\nu x_\mu, \tag{2.11}$$

with

$$\partial_\alpha P^\alpha_\mu = 0, \quad \partial_\alpha M^\alpha_{\mu\nu} = 0. \tag{2.12}$$

We have:

$$P^0_\mu = \frac{-1}{2\pi\alpha'} \left((\dot{x}x')^2 - \dot{x}^2 x'^2\right)^{-1/2} [(\dot{x}x')x'_\mu - x'^2 \dot{x}_\mu], \tag{2.13a}$$

$$P^1_\mu = \frac{-1}{2\pi\alpha'} \left((\dot{x}x')^2 - \dot{x}^2 x'^2\right)^{-1/2} [(\dot{x}x')\dot{x}_\mu - \dot{x}^2 x'_\mu]. \tag{2.13b}$$

The corresponding conserved charges are

$$P_\mu = \int_0^\pi d\sigma P^0_\mu, \quad M_{\mu\nu} = \int_0^\pi d\sigma M^0_{\mu\nu}, \tag{2.14}$$

giving the total energy -and angular-momentum of the string for any physical solution.

The variation of the action under infinitesimal variations $\delta x^\mu(\sigma,\tau)$ in the coordinates, leaving the initial and final configurations fixed (i.e. $\delta x^\mu(\sigma,\tau_i) = 0, i = 1,2$), leads to the equations of motion for the system and to boundary conditions in σ. The equations of motion precisely express the local conservation of the energy-momentum world-sheet current P^α_μ. The boundary conditions in σ appear since we are dealing with a two-dimensional field theory on a space with boundary.

For a simply-connected space-time, two choices of boundary conditions are possible, respectively:

$$\text{- open strings} \;:\; P^1_\mu(\sigma = 0,\pi,\tau) = 0, \tag{2.15a}$$

$$\text{- closed strings} \;:\; x^\mu(\sigma = 0,\tau) = x^\mu(\sigma = \pi,\tau). \tag{2.15b}$$

Note that the boundary conditions for open strings imply that there is no flow of energy -nor angular-momentum at the end points.

Finally, from the expressions for P^a_μ, it is straightforward to check that the following equations

$$[P^0_\mu \pm \frac{\partial_\sigma x_\mu}{2\pi\alpha'}]^2 = 0, \tag{2.16a}$$

$$[P^1_\mu \pm \frac{\partial_\tau x_\mu}{2\pi\alpha'}]^2 = 0, \tag{2.16b}$$

are identically satisfied. Actually, these are the constraints which we expect as consequences of local reparameterization invariance. Indeed, it may be seen, by performing an arbitrary infinitesimal reparametrization

$$\xi^a \to \tilde{\xi}^a = \xi^a - \eta^a(\xi), \tag{2.17}$$

that these constraints follow from reparametrization invariance of the Nambu-Goto action, and that the quantities in (2.16) are the corresponding generators.

However, from our previous discussion we would expect to have only two constraints whereas we seem to obtain four. Actually, the constraints (2.16a) and (2.16b) are not independent. They are related to one another by orientation preserving global reparameterizations exchanging the role of σ and τ. Later, we shall need to consider the constraints (2.16a) only.

Let us remark here that in the case of the open string, due to the boundary conditions (2.15a), the constraints (2.16b) reduce to

$$\dot{x}^2(\sigma = 0, \pi, \tau) = 0. \tag{2.18}$$

This implies that the end points move at the speed of light for any classical physical solution.

2.1.2 The conformal gauge

The general solution to the non-linear equations of motion of the Nambu-Goto action does not seem to be obtained easily. Moreover, it would involve two arbitrary functions of the world-sheet coordinates, due to reparameterization gauge invariance.

There exists however, a choice of world-sheet parametrization, i.e. of gauge-fixing, in which the equations of motion simply reduce to those of massless two-dimensional scalar fields. Indeed, locally on the world-sheet it is always possible to choose a parameterization such that

$$\gamma_{01} = 0 \quad , \quad \gamma_{00} + \gamma_{11} = 0; \tag{2.19a}$$

$$\text{or } (\dot{x} \pm x')^2 = 0. \tag{2.19b}$$

Geometrically, these conditions mean that the tangent vectors in the σ and τ directions are orthonormal (with respect to the space-time Minkowski metric) up to a local scale factor. (Strictly speaking, such a choice of parametrization always exist for a world sheet metric of Euclidean signature, but otherwise requires some additional technicalities in the case of a Minkowski signature[20]. This point is not essential in these lectures). These gauge-fixing conditions define the conformal or orthonormal gauge (the meaning of conformal will become clear shortly).

Using these conditions in the definition for P^a_μ, one finds:

$$P^0_\mu = \frac{\dot{x}_\mu}{2\pi\alpha'}, \quad P^1_\mu = \frac{-x'_\mu}{2\pi\alpha'}, \tag{2.20}$$

so that the equations of motion reduce to

$$(\partial_\tau^2 - \partial_\sigma^2)x^\mu(\sigma, \tau) = 0. \tag{2.21}$$

These are indeed the Klein-Gordon equations for D massless scalar fields in two dimensions. The general solution is of the form

$$x^\mu(\sigma, \tau) = x^\mu_{(1)}(\tau - \sigma) + x^\mu_{(2)}(\tau + \sigma), \tag{2.22}$$

thus clearly showing the separation into right- and left-moving modes in the conformal gauge. Imposing the boundary conditions (2.15a) for open strings, one finds that the functions $x^\mu_{(1)}$ and $x^\mu_{(2)}$ are essentially equal and 2π-periodic (more precisely these are the properties of their derivatives). This is the expression of the fact that modes are reflected back at the end points of open strings. In the case of closed strings, one finds that the two functions $x^\mu_{(1)}$ and $x^\mu_{(2)}$ are completely independent. Thus, for closed string theories, right- and left-moving modes are entirely decoupled and "transparent" to each other. This is the basic property which allows for the vast richness in string theory constructions, since one may then use completely different structures in each moving sector.

Although in the conformal gauge the equations of motion have reduced to those of free massless scalar fields, the system has not really simplified to such a set of fields. Any solution to the equations of motion has to satisfy the gauge-fixing conditions (2.19), thus leading to constraints on the integration constants defining the solution.

In the case of the open string, the general solution to (2.21) and the corresponding boundary conditions in σ may be parametrized as:

$$x^\mu(\sigma, \tau) = \sqrt{2\alpha'}(q^\mu + \alpha^\mu_0\tau + i\sum_n{}' \frac{1}{n}\alpha^\mu_n e^{-in\tau} cos n\sigma), \tag{2.23a}$$

$$\text{where} \quad \alpha^\mu_n = \alpha^\mu_{-n}, \tag{2.23b}$$

$$\alpha^\mu_0 = \sqrt{2\alpha'}P^\mu. \tag{2.23c}$$

($\sum'_n$ means summing over all positive and negative integers, but excluding $n = 0$). The gauge-fixing conditions are then equivalent to the constraints

$$L^{(\alpha)}_n = 0, \tag{2.24}$$

where the bosonic Virasoro generators are given by

$$L^{(\alpha)}_n = \frac{1}{2}\sum_m \alpha^\mu_{n-m}\alpha_{m\mu}.$$

In particular, the zero-mode constraint $L^{(\alpha)}_0 = 0$ leads to the classical mass formula

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha^\mu_{-n}\alpha_{n\mu} = N. \tag{2.25}$$

In the case of closed strings, we have

$$x^\mu(\sigma, \tau) = \sqrt{2\alpha'}\{q^\mu + (\alpha^\mu_0 + \overline{\alpha}^\mu_0)\tau + \frac{1}{2}i\sum_n{}' \frac{1}{n}(\alpha^\mu_n e^{-2in(\tau-\sigma)} + \overline{\alpha}^\mu_n e^{-2in(\tau+\sigma)})\}, \tag{2.26a}$$

where

$$\alpha^{\mu *}_n = \alpha^\mu_{-n}, \overline{\alpha}^{\mu *}_n = \overline{\alpha}^\mu_{-n}, \tag{2.26b}$$

$$\alpha^\mu_0 = \frac{1}{2}\sqrt{2\pi\alpha'}P^\mu = \overline{\alpha}^\mu_0. \tag{2.26c}$$

The gauge-fixing conditions then reduce to

$$L_n^{(a)} = 0 \quad , \quad \overline{L}_n^{(a)} = 0, \tag{2.27a}$$

where

$$L_n^{(a)} = \frac{1}{2}\sum_m \alpha^\mu_{n-m}\alpha_{m\mu} \quad , \quad \overline{L}_n^{(a)} = \frac{1}{2}\sum_m \overline{\alpha}^\mu_{n-m}\overline{\alpha}_{m\mu}. \tag{2.27b}$$

The zero-mode constraints $L_0^{(a)} = 0, \overline{L}_0^{(a)} = 0$ lead to the classical mass formulae:

$$\frac{1}{2}\alpha' M^2 = N + \overline{N}, \tag{2.28a}$$

$$N = \overline{N}, \tag{2.28b}$$

with

$$N = \sum_{n=1}^{\infty} \alpha^\mu_{-n}\alpha_{n\mu} \quad , \quad \overline{N} = \sum_{n=1}^{\infty} \overline{\alpha}^\mu_{-n}\overline{\alpha}_{n\mu}. \tag{2.28c}$$

Note that except for the bosonic zero-modes q^μ, α_0^μ, and $\overline{\alpha}_0^\mu$ giving the space-time position of the center-of-mass of the string, we have twice as many non-zero modes α_n^μ and $\overline{\alpha}_n^\mu$ for the closed string as compared to the open string, corresponding to each moving sector. As said above, such a factorization in characteristic of all closed string theories and is the essential property which allows for a large variety of constructions in the conformal gauge.

As the reader has been suspecting since the beginning of this section, the conformal gauge-fixing conditions (2.19) actually do not completely fix the world-sheet parametrization. This may be seen in many ways.

We made it plausible that the constraints (2.16a) are the generators of the local reparametrization gauge invariance of the system. In the conformal gauge, these constraints precisely reduce to the gauge-fixing conditions, whose Fourier modes in σ are the Virasoro generators. The fact that these latter quantities must vanish for physical solutions thus points to the existence of a remaining gauge invariance in the conformal gauge, generated by the Virasoro generators.

As was noted above, the common scale of the tangent vectors $\dot{x}^\mu$ and x'^μ is not fixed by the conformal gauge conditions (2.19). Actually, there exist local reparametrizations $\tilde{\xi}^a(\xi)$(i.e. non-singular and orientation preserving transformations) preserving the conformal gauge-fixing conditions. Such transformations must satisfy the equations

$$\frac{\partial\tilde{\tau}}{\partial\tau} = \frac{\partial\tilde{\sigma}}{\partial\sigma}, \frac{\partial\tilde{\tau}}{\partial\sigma} = \frac{\partial\tilde{\sigma}}{\partial\tau}. \tag{2.29}$$

For a world-sheet metric of Euclidean signature, the second of these relations would involve a minus sign. These relations would then correspond to the Cauchy conditions for analytic or conformal transformations on the complex plane. The condition (2.29) thus corresponds to (pseudo)-conformal transformations, the remaining local gauge invariance of the system in the conformal gauge (hence the name).

The general solution to (2.29) is of the form

$$\tilde{\tau} + \tilde{\sigma} = f_+(\tau + \sigma), \tilde{\tau} - \tilde{\sigma} = f_-(\tau - \sigma). \tag{2.30}$$

It may be checked that the world-sheet metric $\gamma_{\alpha\beta}$(or $g_{\alpha\beta}$) is then indeed rescaled by a local factor.

Depending on the world-sheet topology, the functions f_+ and f_- above have to satisfy some boundary conditions in σ. These functions are associated to the existence of conformal Killing vectors on the world-sheet. In the case of free closed and open strings, such conformal transformations may be shown to exist. Hence, the solutions given above in the conformal gauge are not the most general solutions to the two-dimensional massless Klein-Gordon equation. The general solution is obtained as in (2.23) and (2.26), with σ and τ given through two functions f_+ and f_- satisfying the appropriate boundary conditions, as in (2.30). Of course, all these solutions correspond to the same physical configuration of the system.

In conclusion, in the conformal gauge, the system has reduced to a set of conformal invariant "free" massless scalar fields on the world-sheet. The existence of this conformal symmetry follows from a remaining reparametrization gauge invariance of the system in the conformal gauge. This is also the reason why physical solutions must satisfy the Virasoro conditions $L_n^{(a)} = 0, \overline{L}_n^{(a)} = 0$. Indeed, as will be shown, these quantities are precisely the generators of conformal transformations in the conformal gauge.

2.1.3 The light-cone gauge

It is possible to completely fix the local reparametrization invariance of the system in the conformal gauge, by introducing an additional condition fixing the local scale of the world-sheet metric. However, as this amounts to solving explicitly the Virasoro constraints, such a complete gauge fixing cannot be achieved in an explicitly space-time covariant manner.

Since conformal reparametrizations (2.29) also satisfy the massless Klein-Gordon equation, it may be possible to fix this local scale by relating σ or τ to some linear combination of $x^\mu(\sigma,\tau)$ in the conformal gauge. Due to the boundary conditions in σ, both for the open and for the closed string, this may actually be done only for the time evolution parameter τ. Moreover, to maintain explicit space-time convariance as much as possible, let us introduce some constant space-time vector n^μ to define the additional gauge-fixing condition.

Thus finally, complete gauge-fixing of reparameterization invariance of the bosonic string is obtained through the three conditions,

$$(\dot{x} \pm x')^2 = 0, \quad n_\mu x^\mu(\sigma,\tau) = 2\alpha' n_\mu p^\mu \tau, \tag{2.31}$$

the first two being the conformal gauge conditions, the third fixing the scale of the world-sheet metric. The proportionality constant in this last condition is determined by relating it to the total momentum of the string. Let us also remark that use was made of the invariance under constant shifts in τ (these are indeed conformal transformations) to remove any constant which could appear in the last condition.

That the conditions (2.31) completely fix the world-sheet parametrization may best be seen from their geometrical meaning. A condition of the form $\eta_\mu x^\mu$ = constant defines a $(D-1)$-hyperplane. Hence, the third gauge fixing condition determines for each value of τ, a hyperplane intersecting the world-sheet. This then determines the parametrization in σ for constant τ, up to a constant shift in σ for the closed string. By the conformal gauge-fixing conditions, the parametrization in τ is then also determined.

The additional non-covariant gauge-fixing condition explicitly breaks the space-time Poincaré group down to the little group of the vector n_μ (for a résumé, see Ref. 17), which is still an explicit space-time symmetry of the gauge-fixed system. A priori, any vector n^μ may be used[29]. The most convenient choice turns out to be a light-like vector[30]

$n^\mu, n^2 = 0$, in which case the little group is isomorphic to the $(D-2)$-dimensional Euclidean group, having as subgroup the group $SO(D-2)_n$ of space rotations transverse to the vector n^μ. Such a choice corresponds to the light-cone gauge.

By convention, one takes for the light-like vector:

$$n^\mu = (\frac{1}{\sqrt{2}}, 0, \ldots, 0, \frac{-1}{\sqrt{2}}), \tag{2.32}$$

and one introduces the corresponding light-cone coordinates for any vectors u^μ, v^μ:

$$u^\pm = \frac{1}{\sqrt{2}}(u^0 \pm u^{D-1}), \quad u^i \qquad i = 1, \ldots D-2, \tag{2.33}$$

with

$$u.v = -u^+v^- - u^-v^+ + \sum_{i=1}^{D-2} u^i v^i. \tag{2.34}$$

The generators of the connected part of the little group of n^μ are then M^{+i}, M^{ij}, with M^{ij} being the generators of $SO(D-2)_n$. The remaining generators of the Lorentz group, M^{-i} and $M^{0(D-1)}$, are not explicit symmetries in the light-cone gauge. Their algebra needs to be checked explicitly, especially at the quantum level where anomalies may arise.

With the use of light-cone gauge-fixing conditions, it is possible to explicitly solve the constraints of reparametrization invariance and express the gauge degrees of freedom in terms of the physical ones[30]. One then finds that the latter degrees of freedom are the transverse coordinates $x^i(\sigma,\tau)$ (and their associated conjugated momenta $P^{oi}(\sigma,\tau)$), whereas the gauge degrees of freedom are the light-cone components given by:

$$x^+(\sigma,\tau) = 2\alpha' P^+ \tau, P^{o+}(\sigma,\tau) = \frac{1}{\pi} P^+, \tag{2.35}$$

and

$$P^{o-}(\sigma,\tau) = \frac{\pi}{2P^+}[(P^{oi})^2 + (\frac{x'^i}{2\pi\alpha'})^2], \tag{2.36}$$

$$x'^-(\sigma,\tau) = \frac{\pi}{P^+} P^{oi}(\sigma,\tau) x'^i(\sigma,\tau). \tag{2.37}$$

The solution to (2.37) is

$$x^-(\sigma,\tau) = \sqrt{2\alpha'} q^- + 2\alpha' P^- \tau + [1 - \frac{1}{\pi}\int_0^\pi d\sigma]. \int_0^\sigma d\sigma' \frac{\pi}{P^+} P^{oi}(\sigma',\tau) \frac{\partial x^i}{\partial \sigma'}(\sigma',\tau), \tag{2.38}$$

with q^- an integration constant, and P^- given by

$$P^- = \int_0^\pi d\sigma P^{o-}(\sigma,\tau). \tag{2.39}$$

Hence, given the physical degrees of freedom x^i, P^{oi}, q^- and P^+, the gauge degrees of freedom $x^\pm, P^{o\pm}$ are uniquely determined and so is the trajectory of the bosonic string in space-time.

The relations above may also be expressed through mode expansions as in the conformal gauge. For open strings, this gives

$$x^i(\sigma,\tau) = \sqrt{2\alpha'}(q^i + \alpha_0^i \tau + i \sum_n \frac{1}{n} \alpha_n^i e^{-in\tau} \cos n\sigma), \tag{2.40a}$$

with

$$\alpha_0^i = \sqrt{2\alpha'} P^i. \tag{2.40b}$$

For the light-cone components, one obtains:

$$q^+ = 0, \quad \alpha_n^+ = \sqrt{2\alpha'} P^+ \delta_{n,0}, \tag{2.41a}$$

$$q^- = q^-, \quad \alpha_n^- = \frac{1}{\sqrt{2\alpha'} P^+} L_n^\perp, \tag{2.41b}$$

with the transverse Virasoro generators

$$L_n^\perp = \frac{1}{2} \sum_m \alpha_{n-m}^i \alpha_m^i. \tag{2.42}$$

This leads to the mass formula:

$$\alpha' M^2 = N^\perp \equiv \sum_{n=1}^\infty \alpha_{-n}^i \alpha_n^i, \tag{2.43}$$

thus showing that the classical mass spectrum is indeed positive definite, a property not so obvious in the conformal gauge.

For the closed string, the corresponding results are:

$$x^i(\sigma,\tau) = \sqrt{2\alpha'}[q^i + (\alpha_0^i + \overline{\alpha}_0^i).\tau + \frac{1}{2} i \sum_n \frac{1}{n}(\alpha_n^i e^{-2in(\tau-\sigma)} + \overline{\alpha}_n^i e^{-2in(\tau-\sigma)})], \tag{2.44a}$$

with

$$\alpha_0^i = \frac{1}{2}\sqrt{2\alpha'} P^i = \overline{\alpha}_0^i, \tag{2.44b}$$

and

$$q^+ = 0 \quad , \quad \alpha_n^+ = \frac{1}{2}\sqrt{2\alpha'} P^+ \delta_{n,o} = \overline{\alpha}_n^+, \tag{2.45a}$$

$$q^- = q^- \quad , \quad \alpha_n^- = \frac{2}{\sqrt{2\alpha' p^+}} L_n^\perp, \ \overline{\alpha}_n^- = \frac{2}{\sqrt{2\alpha' P^+}} \overline{L}_n^\perp, \tag{2.45b}$$

with

$$\alpha_0^- = \frac{1}{2}\sqrt{2\alpha'} P^- = \overline{\alpha}_0^-, \tag{2.45c}$$

$$L_n^\perp = \frac{1}{2}\sum_m \alpha_{n-m}^i \alpha_m^i, \ \overline{L}_n^\perp = \frac{1}{2}\sum_m \overline{\alpha}_{n-m}^i \overline{\alpha}_m^i. \tag{2.46}$$

The mass formulae are then:

$$\frac{1}{2}\alpha' M^2 = N^\perp + \overline{N}^\perp, \tag{2.47a}$$

$$N^\perp = \overline{N}^\perp, \tag{2.47b}$$

with

$$N^\perp = \sum_{n=1}^\infty \alpha_{-n}^i \alpha_n^i, \overline{N}^\perp = \sum_{n=1}^\infty \overline{\alpha}_{-n}^i \overline{\alpha}_n^i. \tag{2.47c}$$

As we shall see, the constraints $\alpha_0^- = \overline{\alpha}_0^-$, or equivalently $N^\perp = \overline{N}^\perp$, is the consequence of the fact that in the case of closed strings, the light-cone gauge-fixing conditions fix the parametrization in σ only up to a constant.

2.2 Hamiltonian Quantization

As is well-known, two different but essentially equivalent methods are available for quantizing a given system. One is the operatorial or canonical quantization approach. The other is the path-integral approach. In these lectures, we shall only consider the former approach for the quantization of string theories.

For regular system, i.e. systems without constraints, canonical quantization consists in the following procedure. Given a description of the system through an action principle, with a Lagrange function depending on coordinates and velocities, the Hamiltonian formulation is derived through the usual Legendre transform. The system is then described by a phase-space, i.e. the space of coordinates and their conjugate momenta, with time-evolution determined by a Hamiltonian function through its Poisson brackets. These Poisson brackets define a symplectic structure on phase-space. Canonical quantization of the system then proceeds by the correspondence principle. Phase-space degrees of freedom now correspond to quantum operators acting on a space of states. The algebra of these operators is given from the Poisson brackets : the value of the commutator (or anticommutator for Grassmannian quantities) of two operators is obtained as $(i\hbar)$ times the result of the corresponding Poisson bracket (for example $\{q,p\} = 1$ leads to $[q,p] = i\hbar$, for commuting degrees of freedom q and p). In addition, the space of states is equipped with an inner product such that the relevant adjointness properties of the quantum operators are satisfied. Finally, time-evolution on the space of states is determined by the Schrödinger equation, with the quantum Hamiltonian operator given by the classical Hamiltonian function, through a specifically chosen ordering of quantum operators.

It is then also possible to obtain a phase-space path-integral representation for any quantum matrix element, which is unambiguously defined in so far that the quantum system itself is uniquely determined.

If the integral over conjugate momenta may be completed, one then obtains the configuration space path-integral quantization approach for the same quantum system.

This is the approach which we shall adopt here. The presence of constraints however, renders the analysis more involved. The general Hamiltonian formulation of constrained systems has been given by Dirac[31] (for a résumé see Refs. 17,32) but we shall not need to consider this discussion in full generality here. Within that formalism, there are two possible ways of quantizing a constrained system.

The first, where the constraints are not solved for but are imposed on quantum physical states, typically leads to an explicitly covariant quantization of the system with states of negative norm. In the case of QED, this corresponds to the Gupta-Bleuler quantization procedure.

In the second procedure, one first solves for the constraints by introducing gauge-fixing conditions. The ensuing reduced phase-space[33] then leads to the gauge-fixed quantized system, in the manner explained above. Typically, the corresponding space of states has a positive-definite inner product, but explicit covariance is lost. For example, this is the case of QED quantized in the Coulomb gauge.

2.2.1 The Hamiltonian formalism[17-20].

In the case of the bosonic string in the Nambu-Goto description, associated to the coordinates $x^\mu(\sigma,\tau)$, we have the conjugate momenta defined by

$$\pi_\mu(\sigma,\tau) = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu(\sigma,\tau)} = P^0_\mu(\sigma,\tau). \tag{2.48}$$

The corresponding phase-space is then equipped with a symplectic structure defined through the Poisson brackets (we always only give the non-vanishing ones):

$$\{x^\mu(\sigma,\tau), \pi_\nu(\sigma',\tau)\} = \delta^\mu_\nu \delta(\sigma-\sigma'). \tag{2.49}$$

All these phase-space degrees of freedom are not independent however. They must satisfy the primary[31] constraints:

$$\phi^{(\alpha)}_+ = 0 \quad , \quad \phi^{(\alpha)}_- = 0, \tag{2.50a}$$

where

$$\phi^{(\alpha)}_\pm = \frac{1}{2}\pi\alpha'\phi^\mu_\pm\phi_{\pm\mu} \quad , \quad \phi^\mu_\pm = \pi^\mu \pm \frac{\partial_\sigma x^\mu}{2\pi\alpha'}. \tag{2.50b}$$

They satisfy the following relations:

$$\{x^\mu(\sigma,\tau), \phi^{(\alpha)}_\pm(\sigma',\tau)\} = \pi\alpha'\phi^\mu_\pm(\sigma',\tau)\delta(\sigma-\sigma') \tag{2.51a}$$

$$\{x^\mu(\sigma,\tau), \phi^{(\alpha)}_\pm(\sigma',\tau)\} = \pm\frac{1}{2}\phi^\mu_\pm(\sigma',\tau)\partial_\sigma\delta(\sigma-\sigma') \tag{2.51b}$$

$$\{\phi^{(\alpha)}_\eta(\sigma,\tau), \phi^{(\alpha)}_{\eta'}(\sigma',\tau)\} = \eta\delta_{\eta\eta'}[\phi^{(\alpha)}_\eta(\sigma,\tau) + \phi^{(\alpha)}_{\eta'}(\sigma',\tau)]\partial_\sigma\delta(\sigma-\sigma'), \eta,\eta' = +,- \tag{2.52}$$

Time-evolution of the system in phase-space is generated by an Hamiltonian, through its Poisson brackets. The canonical Hamiltonian density

$$\mathcal{H}_0 = \dot{x}^\mu\pi_\mu - \mathcal{L} = 0 \tag{2.53}$$

vanishes identically, as a consequence of reparametrization invariance. However, one may also add to it an arbitrary linear combination of the primary constraints[31], and thus generate time-evolution (or translations in τ) with the Hamiltonian

$$H = \int_0^\mu d\sigma[\lambda^+\phi^{(\alpha)}_+ + \lambda^-\phi^{(\alpha)}_-], \tag{2.54}$$

where λ^+, λ^- are functions of the world-sheet coordinates. These functions should be such that the constraints $\phi^{(\alpha)}_\pm = 0$ are consistently preserved through time-evolution[31], namely

$$\dot{\phi}^{(\alpha)}_\pm = \{\phi^{(\alpha)}_\pm, H\} = 0 \text{ for } \phi^{(\alpha)}_\pm = 0 \tag{2.55}$$

From (2.52), one sees that this is always the case. Hence, the functions λ^+ and λ^- may be chosen arbitrarily, and the full set of constraints consists of the primary and first-class[31] constraints $\phi^{(\alpha)}_+$ and $\phi^{(\alpha)}_-$, whose algebra is given in (2.52).

The corresponding equations of motion are:

$$\dot{x}^\mu = \{x^\mu, H\} = \pi\alpha'[\lambda^+\phi^\mu_+ + \lambda^-\phi^\mu_-], \tag{2.56a}$$

$$\dot{\pi}^\mu = \{\pi^\mu, H\} = \frac{1}{2}\partial_\sigma[\lambda^+\phi^\mu_+ - \lambda^-\phi^\mu_-]. \tag{2.56b}$$

Together with the constraint, they also follow from the first-order action on phase-space:

$$S[x^\mu, \pi_\mu; \lambda^+, \lambda^-] = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma [\dot{x}^\mu \pi_\mu - \lambda^+ \phi_+^{(\alpha)} - \lambda^- \phi_-^{(\alpha)}]. \tag{2.57}$$

Hence, λ^+, λ^- are the Lagrange multipliers for the constraints.

Solving for π^μ from (2.56a), the action (2.57) reduces to the linear action (2.5) , with $g_{\alpha\beta}$ give by

$$g_{\alpha\beta} = e^\phi \frac{2}{\lambda^+ + \lambda^-} \begin{bmatrix} -\lambda^+\lambda^- & \frac{1}{2}(\lambda^+ - \lambda^-) \\ \frac{1}{2}(\lambda^+ - \lambda^-) & 1 \end{bmatrix}. \tag{2.58}$$

Here, ϕ is an arbitrary function of ξ^α, which does not appear in the linear action due to its Weyl invariance.

From the transformations generated by the first-class constraints to be discussed below, one may actually show[20] that this matrix indeed transforms as a world-sheet metric tensor under world-sheet reparametrizations. Hence, the Lagrange multipliers λ^+ and λ^- are also components of an intrinsic world-sheet metric, with ϕ being the conformal mode. We thus recover our previous result, that the equations of motion of the world-sheet metric field are the constraints $\phi_\pm^{(\alpha)} = 0$ of local reparametrization invariance. This should convince the reader that the same Hamiltonian analysis applied to the linear action reduces exactly[20] to the one considered here, which follows from the non-linear action.

The first-order action (2.57) is invariant under infinitesimal transformations generated by the first-class constraints. Let us define the quantity

$$\phi_\epsilon^{(\alpha)} = \int_0^\pi d\sigma [\epsilon^+ \phi_+^{(\alpha)} + \epsilon^- \phi_-^{(\alpha)}], \tag{2.59}$$

where ϵ^+, ϵ^- are two arbitrary (infinitesimal) functions of (σ, τ), satisfying appropriate boundary conditions for open and for closed strings (for details, see Ref.20). Then, the following variations leave the action (2.57) invariant (up to surface terms which vanish for the appropriate boundary conditions):

$$\delta_\epsilon x^\mu = \{x^\mu, \phi_\epsilon^{(\alpha)}\} = \pi\alpha'[\epsilon^+ \phi_+^\mu + \epsilon^- \phi_-^\mu], \tag{2.60a}$$

$$\delta_\epsilon \pi^\mu = \{\pi^\mu, \phi_\epsilon^{(\alpha)}\} = \frac{1}{2}\partial_\sigma[\epsilon^+ \phi_+^\mu - \epsilon^- \phi_-^\mu], \tag{2.60b}$$

and

$$\delta_\epsilon \lambda^+ = \partial_\tau \epsilon^+ - \lambda^+ \partial_\sigma \epsilon^+ + \partial_\sigma \lambda^+ \epsilon^+, \tag{2.61a}$$

$$\delta_\epsilon \lambda^- = \partial_\tau \epsilon^- + \lambda^- \partial_\sigma \epsilon^- - \partial_\sigma \lambda^- \epsilon^-. \tag{2.61b}$$

Defining η^α by

$$\epsilon^\pm = \lambda^\pm \eta^0 \pm \eta^1, \tag{2.62}$$

and solving for π^μ, it may easily be seen that the variation $\delta_\epsilon x^\mu$ precisely reduces to the variation $\delta_\eta x^\mu = \eta^\alpha \partial_\alpha x^\mu$ induced in the Lagrangian formalism by the infinitesimal reparametrization (2.17). The matrix (2.58) may then also be seen to transform as a world-sheet metric tensor.

Thus, the constraints $\phi_+^{(\alpha)}$ and $\phi_-^{(\alpha)}$ are the generators of local connected gauge transformations, in the Hamiltonian formalism. Their Poisson brackets (2.52) define the algebra of two-dimensional diffeomorphisms.

The previous discussion shows that the arbitrariness in the choice of Lagrange multipliers λ^+, λ^- precisely corresponds to the arbitrariness in the choice of world-sheet parametrization in the Lagrangian formalism. Solutions to the Hamiltonian equations of motion depend on the two arbitrary functions λ^+ and λ^-. Hence, gauge-fixing of the system requires at least some choice for λ^+ and λ^-.

The conformal gauge corresponds to the choice

$$\lambda^+ = 1, \quad \lambda^- = 1. \tag{2.63}$$

Indeed, the equations of motion then reduce to:

$$\dot{x}^\mu = 2\pi\alpha' \pi^\mu, \quad \dot{\pi}^\mu = \frac{x''^\mu}{2\pi\alpha'}, \tag{2.64}$$

and the constraints $\phi_\pm^{(\alpha)} = 0$ then become equivalent to $(\dot{x} \pm x')^2 = 0$. The same solutions as in the Lagrangian approach are thus recovered. The Hamiltonian in the conformal gauge is

$$H = \int_0^\pi d\sigma [\phi_+^{(\alpha)} + \phi_-^{(\alpha)}].$$

The conformal gauge is not a complete gauge-fixing of the system however. There are transformations generated by $\phi_\pm^{(\alpha)}$ which leave the conformal gauge conditions (2.62) invariant but induce a transformation in phase-space. Such transformations correspond to zero-modes of the equation

$$(\partial_\tau \mp \partial_\sigma)\epsilon^\pm(\sigma, \tau) = 0. \tag{2.65}$$

With the use of (2.62) in the conformal gauge, we precisely recognize the infinitesimal (pseudo)conformal reparametrizations discussed previously, corresponding to the remaining gauge invariance of the system in the conformal gauge. The associated quantities ϕ_ϵ are thus the generators of conformal transformations in this gauge.

In the case of open strings, we have the boundary condition[17-20]:

$$(\epsilon^+ - \epsilon^-)(\sigma = 0, \pi, \tau) = 0, \tag{2.66}$$

so that the solutions to (2.65) are

$$\epsilon^+(\sigma, \tau) = \epsilon(\tau + \sigma), \epsilon^-(\sigma, \tau) = \epsilon(\tau - \sigma), \tag{2.67a}$$

with

$$\epsilon(\tau + 2\pi) = \epsilon(\tau). \tag{2.67b}$$

Using the mode expansion

$$\epsilon(\tau) = \sum_n \epsilon_n e^{in\tau}, \tag{2.68}$$

we then have

$$\phi_\epsilon^{(\alpha)} = \sum_n \epsilon_n e^{in\tau}, \tag{2.69a}$$

where

$$L_n^{(\alpha)} = \int_0^\pi d\sigma [e^{in(\tau+\sigma)} \phi_+^{(\alpha)} + e^{in(\tau-\sigma)} \phi_-^{(\alpha)}]. \tag{2.69b}$$

These quantities are precisely the Virasoro generators introduce above. They satisfy the conformal algebra

$$\{L_n^{(\alpha)}, L_m^{(\alpha)}\} = -i(n-m)L_{n+m}^{(\alpha)}. \tag{2.70}$$

Note that the generator of translations in τ, namely the Hamiltonian, is the Virasoro zero-mode $L_0^{(\alpha)}$.

In the case of closed strings, we have the boundary conditions[17-20]

$$\epsilon^{\pm}(\sigma=\pi,\tau)=\epsilon^{\pm}(\sigma=0,\tau), \tag{2.71}$$

so that the solutions to (2.65) are

$$\epsilon^{\pm}(\sigma,\tau) = \epsilon_{\pm}(\tau\pm\sigma), \tag{2.72a}$$

with

$$\epsilon_{\pm}(\tau+\pi) = \epsilon_{\pm}(\tau). \tag{2.72b}$$

Using the mode expansion

$$\epsilon_{\pm}(\tau)=\sum_n \epsilon_n^{\pm} e^{2in\tau}, \tag{2.73}$$

we then have

$$\phi_{\epsilon}^{(\alpha)} = 2\sum_n[\epsilon_n^{+}\overline{L}_n^{(\alpha)}+\epsilon_n^{-}L_n^{(\alpha)}] \tag{2.74a}$$

where

$$\overline{L}_n^{(\alpha)} = \frac{1}{2}\int_0^{\pi} d\sigma e^{2in(\tau+\sigma)}\phi_+^{(\alpha)}, L_n^{(\alpha)}=\frac{1}{2}\int_0^{\pi} d\sigma e^{2in(\tau-\sigma)}\phi_-^{(\alpha)}. \tag{2.74b}$$

These Virasoro generators, precisely the quantities introduced above, satisfy the two-dimensional conformal algebra

$$\{L_n^{(\alpha)},L_m^{(\alpha)}\} = -i(n-m)L_{n+m}^{(\alpha)}, \tag{2.75a}$$

$$\{\overline{L}_n^{(\alpha)},\overline{L}_m^{(\alpha)}\} = -i(n-m)\overline{L}_{n+m}^{(\alpha)}, \tag{2.75b}$$

$$\{L_n^{(\alpha)},\overline{L}_n^{(\alpha)}\} = 0. \tag{2.75c}$$

Note that the generator of translations in τ is now given by $2(L_0^{(\alpha)}+\overline{L}_0^{(\alpha)})$. The generator of translations in σ is easily seen to be $2(L_0^{(\alpha)}-\overline{L}_0^{(\alpha)})$. Hence, the constraint $L_0^{(\alpha)}=\overline{L}_0^{(\alpha)}$, or equivalently $N=\overline{N}$, is consequence of the fact that the origin in σ has no physical meaning for physical states of closed strings.

These results show, as was claimed previously, that the Virasoro generators $L_n^{(\alpha)},\overline{L}_n^{(\alpha)}$ are indeed the generators of conformal invariance, the remaining gauge invariance of the system in the conformal gauge. Thus, they must vanish for any physical solution, since otherwise the solution would depend on which conformal parametrization is used to describe it. Requiring that this local conformal invariance is realized at the quantum level leads to restrictions on the system, as we shall discuss.

To obtain the Hamiltonian formalism in reduced phase-space, two additional gauge-fixing conditions must be introduced, in order to completely fix the remaining gauge invariance of the conformal gauge. Corresponding to the light-cone gauge, these two conditions are[34] :

$$\Omega_1(\sigma,\tau) \equiv x^+(\sigma,\tau)-2\alpha' P^+\tau=0, \tag{2.76a}$$

$$\Omega_2(\sigma,\tau) \equiv \pi^+(\sigma,\tau)-\frac{1}{\pi}P^+=0, \tag{2.76b}$$

with P^+ being some integration constant.

The full set of constraints $\phi_+^{(\alpha)},\phi_-^{(\alpha)},\Omega_1,\Omega_2$ then becomes second-class[31], so that by the use of Dirac brackets[31], one may explicitly solve for these constraints. The gauge

degrees of freedom $x^{\pm}(\sigma,\tau),\pi^{\pm}(\sigma,\tau)$ are then determined in terms of the physical degrees of freedom $x^i(\sigma,\tau),q^-$ and their conjugate momenta $\pi^i(\sigma,\tau)$, P^+,as was explained previously. The symplectic structure defined on this reduced phase-space , which is consistent with the solutions to the constraints, is given by the Dirac brackets:

$$\{\sqrt{2\alpha'}q^-,P^+\}_D = -1, \tag{2.77a}$$

$$\{x^i(\sigma,\tau),\pi^j(\sigma',\tau)\}_D = \delta^{ij}\delta(\sigma-\sigma'). \tag{2.77b}$$

Time-evolution is then generated through these brackets by the light-cone gauge Hamiltonian:

$$H_{l.c.}=\pi\alpha'\int_0^{\pi} d\sigma[(\pi^i)^2+(\frac{x'^i}{2\pi\alpha'})^2]=2\alpha' P^+P^-. \tag{2.78}$$

2.2.2 Old covariant quantization

The so-called "old covariant quantization" corresponds to canonical quantization in the conformal gauge. By the correspondence principle, the phase-space degrees of freedom become quantum operators, whose commutation relations are given from their Poisson brackets by

$$[x^\mu(\sigma,\tau),\pi_\nu(\sigma',\tau)]=i\hbar\delta^\mu_\nu\delta(\sigma-\sigma'). \tag{2.79}$$

In terms of the mode expansions in the conformal gauge, we have equivalently ($\hbar=1$):

- open string:

$$[\sqrt{2\alpha'}q^\mu,P^\nu]=i\eta^{\mu\nu},\ [\alpha_n^\mu,\alpha_m^\nu]=n\eta^{\mu\nu}\delta_{n+m,0}, \tag{2.80}$$

- closed string:

$$[\sqrt{2\alpha'}q^\mu,P^\nu]=i\eta^{\mu\nu},\ [\alpha_n^\mu,\alpha_m^\nu]=n\eta^{\mu\nu}\delta_{n+m,0}=[\overline{\alpha}_n^\mu,\overline{\alpha}_m^\nu], \tag{2.81}$$

with the adjointness properties:

$$(q^\mu)^+=q^\mu,(P^\mu)^+=P^\mu,(\alpha_n^\mu)^+=\alpha_{-n}^\mu,(\overline{\alpha}_n^\mu)^+=\overline{\alpha}_{-n}^\mu. \tag{2.82}$$

These algebras are very familiar. They are tensor product algebras of:

- position-momentum algebras for q^μ and P^μ, for each value of μ,
- harmonic oscillator algebras with creation operator α_{-n}^μ or $\overline{\alpha}_{-n}^\mu$ and annihilation operator $\alpha_n^\mu,\overline{\alpha}_n^\mu$ (and the wrong sign of the commutator for time components), for each value of μ and $n\geq 1$.

Note that for the closed string, the algebra of non-zero modes is the tensor product of that of the open string with itself. This tensor product structure extends to the space of states, and is characteristic of closed string theories.

The corresponding spaces of states are abstract linear representation spaces of these algebras, equipped with an inner product consistent with the properties (2.82). Hence, the space of states is obtained by acting with the creation operators α_{-n}^μ (for open strings) or α_{-n}^μ and $\overline{\alpha}_{-n}^\mu$ (for closed strings) on Fock vacua or ground states. These ground states are, say, eigenstates of the momentum operator P^μ, and are annihilated by the operators $\alpha_n^\mu,\overline{\alpha}_n^\mu(n\geq 1)$, namely:

$$P^\mu|\Omega;p> = p^\mu|\Omega;p>, \tag{2.83a}$$

$$\alpha_n^\mu|\Omega;p> = 0,\overline{\alpha}_n^\mu|\Omega;p>=0,n\geq 1. \tag{2.83b}$$

They are normalized by

$$<\Omega;p|\Omega;p'> = \delta^{(D)}(p-p'). \tag{2.83c}$$

Due to the opposite sign for the time components of the oscillator algebra, many states actually are of negative norm. For example, the states $\alpha^{\mu}_{-n}|\Omega; p> (n \geq 1)$ are all of negative norm for $\mu = 0$. The presence of such states is the price to pay for having explicit space-time covariance in the quantized theory. To avoid any problems with unitarity and causality, none of these negative-norm states should be a physical state.

To define physical states, we have to consider Virasoro generators, which are composite operators. Hence, we first need to specify a normal ordering for the fundamental operators. In the conformal gauge this is easy, since we are only dealing with free fields. As usual, the ordering chosen is defined by bringing all the position and creation operators to the left of all momentum and annihilation operators, which is denoted by double dots. From now on, all quantum operators are understood to be normal-ordered, so that for example:

$$L^{(\alpha)}_n = \frac{1}{2}\sum_n : \alpha^{\mu}_{n-m}\alpha_{m\mu} : . \tag{2.84}$$

Note that among Virasoro generators, normal ordering may only affect the zero-mode operators $L^{(\alpha)}_0, \overline{L}^{(\alpha)}_0$.

Since only matrix elements of the Virasoro generators need to vanish between physical states, it is enough to define quantum physical states by the following Virasoro constraints:

- open strings:

$$[L^{(\alpha)}_0 - a]|\psi>= 0, \;\; L^{(\alpha)}_n|\psi>= 0, n \geq 1, \tag{2.85}$$

- closed string:

$$[L^{(\alpha)}_0 + \overline{L}^{(\alpha)}_0 - 2a]\psi = 0, \;\; [L^{(\alpha)}_0 - \overline{L}^{(\alpha)}_0]|\psi>= 0, \tag{2.86a}$$

$$L^{(\alpha)}_n|\psi>= 0, \;\; \overline{L}^{(\alpha)}_n|\psi>= 0, n \geq 1 \tag{2.86b}$$

Here, "a" is a subtraction constant, so far not specified, following from normal ordering. From the constraint involving the zero-modes, we obtain the quantum mass formulae:

- open string:

$$\alpha' M^2 = N - a, \qquad N = \sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n\mu}, \tag{2.87}$$

- closed string:

$$\frac{1}{2}\alpha' M^2 = N + \overline{N} - 2a, \;\; N = \overline{N}, \tag{2.88a}$$

$$N = \sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n\mu}, \;\; \overline{N} = \sum_{n=1}^{\infty} \overline{\alpha}^{\mu}_{-n}\overline{\alpha}_{n\mu}. \tag{2.88b}$$

Note that Fock vacua are always physical states of positive norm, but that they are tachyonic, both for open and closed strings, when $a > 0$.

Generally, normal ordering would also affect the conformal algebra, as follows

$$[L^{(\alpha)}_n, L^{(\alpha)}_m] = (n-m)L^{(\alpha)}_{n+m} + \frac{1}{12}c(n)\delta_{n+m,0}, \tag{2.89a}$$

$$[\overline{L}^{(\alpha)}_n, \overline{L}^{(\alpha)}_m] = (n-m)\overline{L}^{(\alpha)}_{n+m} + \frac{1}{12}c(n)\delta_{n+m,0}, \tag{2.89b}$$

$$[L^{(\alpha)}_n, \overline{L}^{(\alpha)}_m] = 0. \tag{2.89c}$$

Here, $c(n)$ is a *central extension* due to normal ordering, or short distance singularities in the two dimensional field theory describing the string theory in the conformal gauge. This central extension corresponds to a conformal anomaly[9,22], or a Schwinger term, in the algebra of the world-sheet energy-momentum tensor of the system in the conformal gauge. By the Jacobi identity, the central extension is always of the form[35]

$$c(n) = cn^3 + \beta n \tag{2.90}$$

where "c" is the *central charge* of the Virasoro algebra[9] (2.89a) and β may be modified by constant shifts in $L^{(\alpha)}_0$.

Normal ordering however, does not affect explicit space-time covariance. Indeed, it may be checked that the Poincaré algebra is satisfied by the quantum operator P^{μ} and $M^{\mu\nu}$. This implies that the space of states transforms as an infinite set of irreducible representations of the Poincaré group. Moreover, since the Virasoro generators and the excitation level operators N and $\overline{N}$ commute with P^{μ} and $M^{\mu\nu}$, all physical states at each excitation level fall into such irreducible representations.

Hence, physical states may be classified according to their mass and to their "spin". By "spin" , we mean here the representation of the rotation subgroup of their little group[17]. For massless states, "spin" is thus given by a representation of $SO(D-2)$, and for massive states (with $m^2 > 0$) by a representation of $SO(D-1)$. For example in four dimensions, we then have representations of $SO(2)$ (helicity) and $SO(3)$(spin) respectively.

This discussion completes the characterization of the quantum system and its physical states. We have to make certain however, that none of the physical states is of negative norm, as otherwise unitarity would be spoiled in amplitudes where such states contribute as intermediate states. The analysis of this question requires first of all an analysis of the Virasoro algebra (2.89), which is most efficiently done by using techniques of conformal field theory in two dimensions (see for example Refs. 9,15,16,17,19).

The result of such an analysis is that the Virasoro algebra for the bosonic string is given by:

$$[L^{(\alpha)}_n, L^{(\alpha)}_m] = (n-m)L^{(\alpha)}_{n+m} + \frac{1}{12}Dn(n^2-1)\delta_{n+m,0}, \tag{2.91}$$

or

$$c(n) = Dn^3 - Dn. \tag{2.92}$$

Hence, each scalar field $x^{\mu}(\sigma,\tau)$ in two dimensions contributes a unit $(+1)$ to the central charge of the Virasoro algebra.

Considering then the problem of negative norm physical states, the no-ghost theorem[15,36] asserts that:
the necessary and sufficient conditions such that none of the physical states is of negative norm are:

- $a \leq 1$
- if $a = 1 : D \leq 26$,
- if $a < 1 : D < 26$ i.e. $D \leq 25$.

The interested reader may check (but not prove) these statement by solving the problem for the first few excitation levels (see Ref.17).

Although these conditions are necessary for quantum consistency, they may not be sufficient. Indeed, when studying 1-loop amplitudes, one finds[37] (see for example Refs.

15,38) that the correct pole and cut singularities due to intermediate states can only be obtained for $D = 26$. Thus, the necessary conditions for the absence of negative-norm physical states and 1-loop unitarity require that

$$D = 26, \quad a = 1 \tag{2.93}$$

(as we shall comment upon later on, these are still not sufficient conditions for quantum consistency).

Under the conditions (2.93), the statement of the no-ghost theorem is actually more detailed[15,36]. For the open string for example, we then have that any physical state $|\psi>$ is necessarily of the form

$$|\psi>=|\phi_0>+L_{-1}|\chi_1>+(L_{-2}+\frac{3}{2}L_{-1}^2)|\chi_2>, \tag{2.94}$$

where $|\psi_0>$ is a positive-norm transverse state (in 1-to-1 correspondence with a state in the light-cone gauge, to be discussed below), and $L_{-1}|\chi_1>,(L_{-2}+\frac{3}{2}L_{-1}^2)|\chi_2>$ are zero-norm physical states, such that

$$L_0|\chi_1>=0,(L_0+1)|\chi_2>=0,\ L_n|\chi_{1,2}>=0, n\geq 1. \tag{2.95}$$

Unphysical and zero-norm states are necessary for explicit space-time covariance. Although zero-norm states are physical by the definitions above, they actually (must) decouple in physical amplitudes (this point will be discussed further in chapter 4). The situation is analogous to the one of QED in the Lorentz gauge, where the longitudinal photon is precisely such a zero-norm physical state which decouples from amplitudes.

With the critical values (2.93) , let us describe the first few excitation levels of bosonic strings, beginning with the open string.

The physical ground states are states $|\Omega;p>$, with $N=0$, such that $\alpha' m^2=-1$. These are space-time scalar tachyonic states, thus spelling disaster for the quantum theory which is then not unitary.

At the next excitation level, we have states $\epsilon_\mu\alpha_{-1}^\mu|\Omega;p>$ with $N=1$, hence $\alpha' m^2=0$. These massless states must satisfy the transversality condition $\epsilon_\mu p^\mu=0$, which has two types of solutions. One, consisting of the 24 linearly independent vectors ϵ_μ such that $\epsilon^2=1,\epsilon p=0$, corresponds to a positive-norm massless transverse state transforming as the vector representation $\square_{SO(24)}$. The other, with $\epsilon^\mu=p^\mu$, corresponds to a zero-norm massless scalar state, transforming as a singlet under $SO(24)$, namely the longitudinal component of the massless vector. Note that this state is given as $L_{-1}|\Omega;p>$ (see (2.94)).

At the second excitation level $N=2$, one finds three states; a positive-norm 2-index symmetric traceless transverse massive state, transforming as $\square\square_{SO(25)}$ and having 324 components, a zero-norm transverse massive vector state transforming as $\square_{SO(25)}$, and a zero-norm massive scalar state transforming as a singlet under $SO(25)$. These states are respectively given by $\frac{1}{\sqrt{2}}\lambda_{\mu\nu}\alpha_{-1}^\mu\alpha_{-1}^\nu|\Omega;p>$, $L_{-1}\epsilon_\mu\alpha_{-1}^\mu|\Omega;p>$ and $(L_{-2}+\frac{3}{2}L_{-1}^2)|\Omega;p>$, where the polarization tensors satisfy $\lambda_{\mu\nu}\lambda^{\mu\nu}=1,\lambda_{\mu\nu}-\lambda_{\nu\mu}=\lambda_\mu^\mu=p^\mu\lambda_{\mu\nu}=0$ and $\epsilon^2=1,\epsilon p=0$ (compare with (2.94)).

From the mass formula (2.87), it is clear that all states fall on "Regge trajectories" of slope α'. Indeed, if space-time were four dimensional, spin would be characterized by a single number J, and the mass formula would then lead to the linear dependence $J=\alpha' m^2+$ constant.

Similar results easily follow for closed strings as well. From the mass formulae (2.88), the slope of "Regge trajectories " is now $\frac{1}{2}\alpha'$. The physical ground states $|\Omega;p>$, with

$\frac{1}{2}\alpha' m^2=-2$, are space-time scalar tachyons in this sector of the theory as well. At the first excitation level $N=\overline{N}=1$, we have massless states, which for those of positive norm combine into a 2-index symmetric transverse traceless tensor of $SO(24)$ with 299 components, a 2-index antisymmetric transverse tensor of $SO(24)$ with 276 components, and a scalar state of $SO(24)$ with 1 component. These states correspond respectively to a massless "spin 2" state or graviton, a massless antisymmetric tensor, and a massless scalar or dilaton.

One of the remarkable features of these spectra is the appearance of massless "spin 1" and "spin 2" states. Usually, the existence of such particle is associated with some local gauge invariance, a Yang-Mills gauge invariance for "spin 1" states and a space-time reparametrization invariance for "spin 2" states. Actually, the low-energy behavior of these string states is precisely that of such gauge bosons[39].

The profound origin of such space-time local symmetries in string theories is still not really understood. Nevertheless, such a relationship between world-sheet and space-time local symmetries seems to be a generic feature of string theory constructions. Here, world-sheet reparametrization invariance is related to space-time reparametrization invariance for closed strings and space-time Yang-Mills invariance for open strings. Later, we shall see how this extends to world-sheet supersymmetry and space-time supersymmetry, and to world-sheet current algebra and space-time Yang-Mills invariance.

2.2.3 Light-cone gauge quantization

Quantization in the light-cone gauge[30] proceeds through the correspondence principle from the Dirac brackets in the reduced phase-space Hamiltonian formulation in that gauge. Hence, we have the commutation relations

$$[\sqrt{2\alpha'}q^-,P^+]=-i\hbar,\ \ [x^i(\sigma,\tau),\pi^j(\sigma',\tau)]=i\hbar\delta^{ij}\delta(\sigma-\sigma'), \tag{2.96}$$

or equivalently in terms of the mode expansion ($\hbar=1$):

- open string:

$$[\sqrt{2\alpha'}q^-,P^+]=-i,\ \ [\sqrt{2\alpha'}q^i,P^j]=i\delta^{ij}, \tag{2.97a}$$
$$[\alpha_n^i,\alpha_m^j]=n\delta^{ij}\delta_{n+m,0}, \tag{2.97b}$$

- closed string:

$$[\sqrt{2\alpha'}q^-,P^+]=-i,\ \ [\sqrt{2\alpha'}q^i,P^j]=\delta^{ij}, \tag{2.98a}$$
$$[\alpha_n^i,\alpha_m^j]=n\delta^{ij}\delta_{n+m,0}=[\overline{\alpha}_n^i,\overline{\alpha}_m^j], \tag{2.98b}$$

with the adjointness properties:

$$(q^-)^+=q^-,(P^+)^+=P^+,(q^i)^+=q^i,(P^i)^+=P^i, \tag{2.99a}$$
$$(\alpha_n^i)^+=\alpha_{-n}^i,(\overline{\alpha}_n^i)^+=\overline{\alpha}_{-n}^i. \tag{2.99b}$$

The structure of these algebras, and of the corresponding spaces of states should now be obvious. Fock vacua $|\Omega,p^i,p^+>$ are eigenstates of the momentum operators P^i and P^+, , are annihilated by the operators $\alpha_n^i,\overline{\alpha}_n^i(n\geq 1)$, and are normalized by $<\Omega,p^i,p^+|\Omega,p'^i,p'^+>=\delta^{(D-2)}(p^i-p'^i)\delta(p^+-p'^+)$. All states are obtained from these

vacua by the action of the creation operators $\alpha^i_{-n}, \overline{\alpha}^i_{-n} (n \geq 1)$. Note again the structure for closed strings as compared to that of open strings.

In contradistinction to old covariant quantization, the present space of states does not have any state of negative norm. All states are physical ones, and correspond to transverse string excitations only.

The light-cone coordinate operators are determined from the constraints. Hence, we have the operational relations:

- open string:

$$q^+ = 0, \quad \alpha^+_n = \sqrt{2\alpha'}P^+\delta_{n,0}, \tag{2.100a}$$

$$q^- = q^-, \quad \alpha^-_n = \frac{1}{\sqrt{2\alpha'}P^+}(L^\perp_n - a\delta_{n,0}), \tag{2.100b}$$

- closed string:

$$q^+ = 0, \ \alpha^+_n = \frac{1}{2}\sqrt{2\alpha'}P^+\delta_{n,0} = \overline{\alpha}^+_n, \tag{2.101a}$$

$$q^- = q^-, \alpha^-_n = \frac{2}{\sqrt{2\alpha'}P^+}[L^\perp_n - a\delta_{n,0}], \overline{\alpha}^-_n = \frac{2}{\sqrt{2\alpha'}P^+}[\overline{L}^\perp_n - a\delta_{n,0}], \tag{2.101b}$$

$$\text{with} \quad \alpha^-_0 = \overline{\alpha}^-_0 = \frac{1}{2}\sqrt{2\alpha'}P^-. \tag{2.101c}$$

It is understood that the transverse Virasoro generators $L^\perp_n, \overline{L}^\perp_n$ have been normal ordered, with the same choice of normal ordering as in the conformal gauge. Here, "a" is the ensuing normal ordering subtraction constant for the Virasoro zero-modes.

Note that the transverse Virasoro generators satisfy the Virasoro algebra with a central extension $c(n) = (D-2)n(n^2-1)$ to which only transverse fields $x^i(\sigma,\tau)$ contribute.

From the previous relations, we have the mass formulae

- open string:

$$\alpha' M^2 = N^\perp - a, \ N^\perp = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n,$$

- closed string:

$$\frac{1}{2}\alpha' M^2 = N^\perp + \overline{N}^\perp - 2a, \ N^\perp = \overline{N}^\perp, \tag{2.102a}$$

$$N^\perp = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n, \ \overline{N}^\perp = \sum_{n=1}^{\infty} \overline{\alpha}^i_{-n}\overline{\alpha}^i_n. \tag{2.102b}$$

For the closed string the level matching condition for the right-and left-moving sectors follows from the fact that the light-cone gauge fixing conditions do not fix the origin in the parametrization in σ.

Since the light-cone gauge does not preserve space-time covariance explicitly, it may happen that quantization and normal ordering are not compatible with implicit Poincaré covariance. Since the little group of the light-like vector n^μ is an explicit

symmetry in the light-cone gauge, only the algebra involving the generators M^{-i} and $M^{0(D-1)}$ needs to be checked. Actually, since only $\alpha^-_n, \overline{\alpha}^-_n$ are given by normal ordered composite operators, only the commutator $[M^{-i}, M^{-j}]$ could be anomalous, whereas it should vanish to maintain implicit space-time Poincaré covariance. An explicit and tedious calculation shows[30] that this commutator is indeed anomalous, unless we have:

$$D - 2 = 24, \qquad a = 1. \tag{2.103}$$

It is only under these conditions that physical states may be characterized by their mass and their "spin". Since in the light-cone gauge, all physical states transform explicitly in representations of the $SO(D-2)_n$ rotation subgroup of the little group of the vector n^μ, it is straightforward to identify the "spin" for massless states. For massive states however, with $m^2 > 0$, only under the conditions (2.103) do the $SO(D-2)_n$ representations combine into representations of $SO(D-1)$, which is the rotation subgroup of the corresponding little group.

This type of consideration alone actually leads to a restriction for the first excitation level only[40]. For example, in the case of the open string, we then have the states $\alpha^i_{-1}|\Omega>$ with $\alpha' m^2 = (1-a)$. These $(D-2)$ states may transform covariantly under the full Lorentz group, and not only under $SO(D-2)_n$, only if they transform as a massless vector state in D dimensions.Hence , Lorentz covariance requires[40] $a = 1$, which implies in turn that the physical ground state is a tachyon. For states at higher levels however, the counting of physical components is always such that the corresponding states *could* be combined into irreducible representations of $SO(D-1)$. Only the explicit calculation of the full Lorentz algebra can ensure that indeed the space of states is organized into irreducible but non-linear representations of the Poincaré group, and that this occurs only if $a = 1$ and $D = 26$.

There exists however an heuristic argument[40] for the value $D = 26$, which does not require the full calculation. The value for "a" follows from normal ordering in the transverse Virasoro zero-modes, and is thus given by the formal series:

$$a = -\frac{1}{2}(D-2)\sum_{n=1}^{\infty} n. \tag{2.104}$$

Using ζ-function regularization, we have

$$\sum_{n=1}^{\infty} n = \sum_{n=1}^{\infty} n^{-s}_{|s=-1} = \zeta(-1) = \frac{-1}{12}, \tag{2.105}$$

leading to

$$a = \frac{D-2}{24}. \tag{2.106}$$

Therefore, the necessary condition for Lorentz covariance $a = 1$ implies the critical value $D = 26$. Clearly, such an argument does depend on which regularization of the series is used. Only the full calculation of the commutator $[M^{-i}, M^{-j}]$ gives an unambiguous answer. Nevertheless, ζ-function regularization happens to give the correct result, and is used as an easy check on results. Later in chapter 4, we shall justify the use of this regularization from the point of view of modular invariance.

With the critical values (2.103), let us briefly describe the first few excitation levels, first for the open string. From the mass formula , we find again the physical ground state to be a scalar tachyon. At the next excitation level, we have the massless states $\alpha^i_{-1}|\Omega; p^i, p^+>$ transforming as $\square_{SO(24)}$, i.e. as a massless space-time vector state, with

24 components. At the second excitation level $N^\perp = 2$, we have massive states with $\alpha' m^2 = 1$, corresponding to $\alpha^i_{-2}|\Omega>$ and $\alpha^i_{-1}\alpha^j_{-1}|\Omega>$ and transforming as $(\square + \square\square + \cdot)_{SO(24)}$. Clearly, they combine into a massive 2-index symmetric traceless tensor of $SO(25)$, with 324 components.

For the closed string, the physical ground-state is also a scalar tachyon with $\frac{1}{2}\alpha' m^2 = -2$. At the next level, we have the massless states $\alpha^i_{-1}\overline{\alpha}^j_{-1}|\Omega>$ transforming as $(\square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix} + \cdot)_{SO(24)}$, and thus corresponding to the graviton, the antisymmetric tensor state and the dilaton, with respectively 299, 274 and 1 components.

Clearly, positive norm physical states in the conformal and in the light-cone gauge quantizations are in 1-to-1 correspondence, as guaranteed by the no-ghost theorem (see (2.94)).

It is possible to introduce a function which counts the number of positive norm physical states at each excitation level, i.e. a generating function for level degeneracies. Let us first consider the open bosonic string.

In the light-cone gauge, we have the mass formula

$$\alpha' M^2 = N^\perp - 1. \tag{2.107}$$

Let us then introduce the generating function for the degeneracies d_n at the excitation level $N^\perp = n$:

$$f_0(q) = \sum_{n=0}^{\infty} d_n q^{2(n-1)} = \sum_{n=0}^{\infty} d_n q^{2\alpha' m_n^2}, \tag{2.108}$$

where q is a complex or real number such that $|q| < 1$. We have:

$$f_0(q) = Tr q^{2(N^\perp - 1)} \tag{2.109}$$

where the trace is taken only on the non-zero modes $\alpha^i_n (n \neq 0)$. The calculation of the trace is straightforward[17] (the reader may like to first do the calculation for a single harmonic oscillator), and one finds:

$$f_0(q) = \frac{1}{q^2}\left[\prod_{n=1}^{\infty}\frac{1}{(1-q^{2n})}\right]^{D-2} = q^{\frac{D-24}{12}}\eta^{2-D}(\tau), \tag{2.110}$$

with $\eta(\tau)$ being the Dedekind η-function of number theory and the theory of modular forms:

$$\eta(\tau) = q^{\frac{1}{12}}\prod_{n=1}^{\infty}(1-q^{2n}), \qquad q = e^{i\pi\tau}. \tag{2.111}$$

Expanding (2.110) in powers of q for $D = 26$, one obtains

$$f_0(q) = \eta^{-24}(\tau) = \frac{1}{q^2}[1 + 24q^2 + 324q^4 + 3200q^6 + 25650q^8 + \ldots], \tag{2.112}$$

thus giving the degeneracies at each excitation level. These numbers may of course be checked by explicit construction of the associated physical states. It may be shown[15] that the degeneracy numbers increase exponentially as the level increases.

For the closed bosonic string, the degeneracy number at excitation level $N^\perp = \overline{N}^\perp = n$ is simply given by d_n^2. The corresponding generating function is thus:

$$fx(|q|) = \sum_{n=0}^{\infty} d_n^2 |q|^{4(n-1)} = \sum_{n=0}^{\infty} d_n^2 |q|^{\alpha' m_n^2}, \tag{2.113}$$

where $|q|$ is a real number such that $|q| < 1$. For convenience, let us introduce the complex number

$$q = |q|e^{i\theta/2}, \qquad -\pi \leq \theta \leq \pi. \tag{2.114}$$

The generating function may then be expressed in terms of the corresponding function for the open string, as:

$$f_c(|q|) = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}|f_0(q)|^2 = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}|q^{\frac{D-24}{12}}\eta^{2-D}(\tau)|^2. \tag{2.115}$$

Hence, for $D = 26$, we have

$$f_c(|q|) = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}|\eta(\tau)|^{-48}. \tag{2.116}$$

We shall make use of these results when discussing modular invariance of 1-loop partition functions in chapter 4.

2.3 BRST Quantization

As was discussed extensively, conformal symmetry is the remaining reparametrization gauge invariance of the system in the conformal gauge. Any breakdown of conformal symmetry would thus imply breakdown of reparametrization invariance, and lead to a depence on the world-sheet parametrization used to compute physical quantities.

It seems that such a breakdown of conformal invariance is indeed obtained, since the central extension, or conformal anomaly, of the Virasoro algebra does not vanish, not even for the critical values $D = 26$ and $a = 1$. In other words, reparametrization invariance, which is the fundamental symmetry leading to the constraints $\phi_\pm^{(\alpha)} = 0$, which guarantee at the quantum level the decoupling of unphysical and zero-norm states from amplitudes, seems to be broken by quantum anomalies! Does the theory still make sense?

Since the days of Feynman, de Witt, Faddeev and Popov[41], we know that such a question can be meaningfully addressed only when taking into account the ghost sector[41] following from gauge fixing. This is usually done by considering the configuration space path-integral and its gauge-fixing, and using the Faddeev-Popov trick[41]. Such an approach however, does not always give a meaningful answer. In the general case of open algebras of constraints[42], i.e. when the structure coefficients of that algebra are functions on phase-space, one has to resort to more general Hamiltonian[43] or Lagrangian[44] methods developed by Batalin, Fradkin, Vilkovisky and others.

In the spirit of the Hamiltonian approach adopted in these lectures, we shall apply the general Hamiltonian methods leading to BRST quantization[45] of gauge invariant systems, namely the Batalin-Fradkin-Vilkovisky (BFV) formalism[32,42,43].

Actually, although we shall not need this method in its full glory, let us briefly discuss it independently of string theory, not only to give a flavor of its fundamental aspects but also because it provides today the most general approach to the quantization of gauge invariant systems. Moreover, such a brief discussion should help to demystify BRST quantization.

2.3.1 The BFV Hamiltonian formalism

For simplicity, we shall consider the case of a gauge-invariant system with bosonic phase-space degrees of freedom only, and a set of irreducible bosonic first-class constraints

ϕ_α (i.e. constraints which are linearly independent locally on phase-space, and with vanishing Poisson brackets when the constraints are imposed). The generalization to Grassmann (or anticommuting) degrees of freedom and/or to reducible constraints is rather straightforward, and follows the same general approach as the one discussed below[32,42,43].

Time-evolution on phase-space is obtained from a first-class Hamiltonian[31]

$$H_T = H + \lambda^\alpha \phi_\alpha, \tag{2.117}$$

where H is a specific first-class Hamiltonian, and λ^α are Lagrange multipliers for the constraints ϕ_α. We have the algebra of constraints

$$\{\phi_\alpha, \phi_\beta\} = C_{\alpha\beta}^{\gamma}\phi_\gamma, \tag{2.118a}$$
$$\{H, \phi_\alpha\} = V_\alpha{}^\beta \phi_\beta, \tag{2.118b}$$

where generally $C_{\alpha\beta}^{\gamma}$ and $V_\alpha{}^\beta$ are structure functions on phase-space.

The corresponding Hamiltonian equations of motion also follow from a first-order action. This action is invariant under infinitesimal variations generated by linear combinations $\epsilon^\alpha\phi_\alpha$ of the constraints. For phase-space degrees of freedom, these variations are given by the corresponding Poisson brackets, and for the Lagrange multipliers, we have[32,42]:

$$\delta_\epsilon \lambda^\alpha = \dot{\epsilon}^\alpha + \lambda^\gamma \epsilon^\beta C_{\beta\gamma}^{\alpha} - \epsilon^\beta V_\beta{}^\alpha. \tag{2.119}$$

The algebra of constraints thus characterizes, in the Hamiltonian formalism, the algebra of the local gauge invariance of the system. (The reader may like to compare this general discussion with the previous analysis for the bosonic string).

In the BFV approach, the original phase-space is extended in two steps[32,42,43]. First, to render the Lagrange multipliers λ^α dynamical degrees of freedom, one introduces bosonic conjugate momenta π_α, with the Poisson brackets:

$$\{\lambda^\alpha, \pi_\beta\} = \delta^\alpha_\beta. \tag{2.120}$$

We then have the set of constraints $G_a = (G_{\alpha_1}, G_{\alpha_2})$, with

$$G_{\alpha_1} = \pi_{\alpha_1}, \qquad G_{\alpha_2} = \phi_{\alpha_2}, \tag{2.121}$$

and the algebra:

$$\{G_a, G_b\} = C_{ab}^{c} G_c, \tag{2.122}$$
$$\{H, G_a\} = V_a{}^b G_b. \tag{2.123}$$

In order to compensate for the use of degrees of freedom which are not all independent, one then introduces a set of degrees of freedom of opposite statistics, i.e. Grassmann variables. (In the general case, for Grassmann odd constraints, the corresponding Lagrange multipliers are Grassmann odd, hence also their conjugate momenta, and the additional degrees of freedom which one introduces are then Grassmann even, i.e. bosonic). These BFV ghosts come in conjugate pairs $\eta^a, \mathcal{P}_a$, with the Poisson brackets

$$\{\eta^a, \mathcal{P}_b\} = -\delta^a_b, \tag{2.124}$$

and the properties

$$(\eta^a)^* = \eta^a, \qquad (\mathcal{P}_a)^* = -\mathcal{P}_a. \tag{2.125}$$

The BFV extended phase-space (EPS) thus consists of all these degrees of freedom, namely the original phase-space variables z_A together with $\lambda^\alpha, \pi_\alpha, \eta^a$ and $\mathcal{P}_a$. On this EPS, one also introduces an integer grading, by defining the ghost number as follows:

degree of freedom	$z_A, \lambda^\alpha, \pi_\alpha$	η^a	$\mathcal{P}_a$
ghost number	0	+1	-1

(2.126)

By counting Grassmann even and Grassmann odd degrees of freedom with plus and minus signs respectively, the reader may check that indeed the total number of locally independent degrees of freedom on EPS is the same as in a reduced phase-space approach.

One of the fundamental results of the Hamiltonian BFV formalism[42,43] is the following. Associated to the algebra of constraints ϕ_α, there exists a quantity Q_B, the BRST charge, which is uniquely determined[42], up to canonical transformations in EPS, by the properties:

- $Q_B^* = Q_B$,
- Q_B is Grassmann odd,
- Q_B has ghost number (+1),
- Q_B is nilpotent , i.e. $\{Q_B, Q_B\} = 0$.

The expression of the BRST charge is of the form

$$Q_B = \eta^a G_a - \frac{1}{2}\eta^b \eta^c C_{cb}^{a} \mathcal{P}_a + \text{"more"}, \tag{2.127}$$

where "more" stands for additional terms necessary for nilpotency when the structure coefficients are functions rather than constants.

The nilpotency property of Q_B is a very strong condition indeed, since Q_B is a Grassmann variable. This property embodies in one single equation all the information concerning the algebra of constraints and the associated Jacobi identities[42]. Moreover, it only depends on the algebra of constraints, and not on any particular choice of gauge-fixing conditions. Hence, corresponding to gauge transformations induced by the constraints on the original phase-space, we now have BRST transformations on EPS generated by the BRST charge. It is the nilpotency property which now characterizes the gauge invariance of the system.

Note also that the ghost sector and the BRST charge only depend on the algebra of constraints, and in no manner whatsoever on how this algebra is explicitly realized for a given system, i.e. for a given set of phase-space degrees of freedom z_A. Any gauge-invariant system with the same Hamiltonian gauge algebra has the same ghost sector and BRST charge, leading thus to an identical discussion of BRST quantization. This remark is fundamental in the construction of all string theories.

Time evolution on EPS is obtained from a BRST invariant effective Hamiltonian, i.e. having a vanishing Poisson bracket with the BRST charge, of the form:

$$H_{eff} = H_B - \{\Psi, Q_B\}. \tag{2.128}$$

Here, H_B is a particular BRST invariant Hamiltonian, of the form

$$H_B = H + \eta^a V_a{}^b \mathcal{P}_a + \text{"more"}, \tag{2.129}$$

where "more" stands for additional terms necessary for BRST invariance when the structure coefficients are not constants.

The function Ψ is a pure imaginary Grassmann odd function on EPS of ghost number (-1). Its role is that of gauge-fixing the system, and has thus to be chosen appropriately[46] according to the structure of the space of gauge orbits of the system[32,46]. However , the Fradkin-Vilkovisky theorem[42,43] (which is another fundamental result of the BFV approach) establishes that two functions Ψ, corresponding to gauge equivalent sections of the space of gauge orbits[46], lead to an identical physical description of the gauge invariant system (this *does not mean*[46], as is usually stated[42,43], that the description of the system is independent of Ψ).

In order to recover the same physical description as in the old covariant approach, it is necessary that the physical solutions to the equations of motion in EPS have zero ghost and BRST charges[32,42]. This requires a choice of BRST invariant and zero-ghost charge boundary conditions on EPS[32,42]. Clearly, the system may then also be described starting from some BRST invariant first-order action on EPS[32,42].

Finally, BRST quantization simply corresponds to the canonical quantization of the BFV Hamiltonian formalism. Quantum physical states are then defined as BRST invariant states of zero-ghost charge (up to possible ordering constants).

2.3.2 The BFV formalism for bosonic strings[19,20].

Let us apply this general discussion to the bosonic string. We have the phase-space degrees of freedom $x^\mu(\sigma,\tau)$ and $\pi_\mu(\sigma,\tau)$,with the Poisson brackets (2.49). Associated to the Lagrange multipliers $\lambda^\pm(\sigma,\tau)$, we now have the conjugate momenta $\pi_\pm(\sigma,\tau)$, with

$$\{\lambda^\eta(\sigma,\tau),\pi_{\eta'}(\sigma',\tau)\} = \delta^\eta_{\eta'}\delta(\sigma-\sigma'),\eta,\eta' = +,-. \tag{2.130}$$

The full set of constraints $G_a(a = 1,2,3,4)$ is thus

$$G_1 = \pi_-, G_2 = \pi_+; G_3 = \phi^{(\alpha)}_-, G_4 = \phi^{(\alpha)}_+. \tag{2.131}$$

The BFV ghost sector is then given by Grassmann odd fields $\eta^a(\sigma,\tau),\mathcal{P}_b(\sigma',\tau)$ $(a = 1,2,3,4)$, with

$$\{\eta^a(\sigma,\tau),\mathcal{P}_b(\sigma',\tau)\} = -\delta^a_b\delta(\sigma-\sigma'), \tag{2.132}$$

$$(\eta^a)^* = \eta^a, (\mathcal{P}_a)^* = -\mathcal{P}_a. \tag{2.133}$$

From the two-dimensional reparametrization algebra (2.52) and the expression (2.127), it is straightforward to obtain the BRST charge:

$$Q_B = \int_0^\pi d\sigma[\eta^1\pi_- + \eta^2\pi_+ + \eta^3\phi^{(\alpha)}_- + \eta^4\phi^{(\alpha)}_+ - \eta^3\partial_\sigma\eta^3\mathcal{P}_3 + \eta^4\partial_\sigma\eta^4\mathcal{P}_4]. \tag{2.134}$$

In particular, we then have

$$\{\mathcal{P}_3, Q_B\} = -\phi^{(T)}_-, \quad \{\mathcal{P}_4, Q_B\} = -\phi^{(T)}_+, \tag{2.135}$$

with

$$\phi^{(T)}_\pm = \phi^{(\alpha)}_\pm + \phi^{(c)}_\pm, \tag{2.136a}$$

$$\phi^{(c)}_- = -2\partial_\sigma\eta^3\mathcal{P}_3 - \eta^3\partial_\sigma\mathcal{P}_3, \tag{2.136b}$$

$$\phi^{(c)}_+ = +2\partial_\sigma\eta^4\mathcal{P}_4 + \eta^4\partial_\sigma\mathcal{P}_4. \tag{2.136c}$$

The quantities $\phi^{(c)}_\pm$ may be shown to satisfy the same algebra as the bosonic constraints $\phi^{(\alpha)}_\pm$. Hence,the total quantities $\phi^{(T)}_\pm$ are the Hamiltonian generators of world-sheet reparametrizations for all EPS degrees of freedom, with $\phi^{(\alpha)}_\pm$ generating such transformations for the bosonic coordinates x^μ and π_μ, and $\phi^{(c)}_\pm$ generating those for the BFV ghosts. In the conformal gauge, the Fourier modes of $\phi^{(T)}_\pm$ will thus also be the generators of conformal transformations, as was shown for the constrains $\phi^{(\alpha)}_\pm$.

In the present case, since the first-class Hamiltonian H vanishes identically, the BRST invariant effective Hamiltonian is simply given by

$$H_{eff} = -\int_0^\pi d\sigma\{\Psi, Q_B\}, \tag{2.137}$$

where Ψ is some function on EPS.

This is not the place to discuss the problem of the appropriate choice for Ψ[20]. Let us only mention that there does not exist any choice of Ψ such that a global and complete section of the space of gauge orbits is obtained[20]. In other words, the bosonic string suffers[20,47] from a Gribov[48] problem.

Actually, we are only interested here in gauge-fixing the system to the conformal gauge, which is done as follows[20]. Let us consider for the function Ψ:

$$\Psi = \frac{1}{\beta}\mathcal{P}_1(\lambda^- - 1) + \frac{1}{\beta}\mathcal{P}_2(\lambda^+ - 1) + \mathcal{P}_3\lambda^- + \mathcal{P}_4\lambda^+. \tag{2.138}$$

The corresponding effective Hamiltonian and first-order action are then easily obtained. In these expressions, redefine then $\pi_\pm, \mathcal{P}_1$ and $\mathcal{P}_2$ by $\beta\pi_\pm, \beta\mathcal{P}_1$ and $\beta\mathcal{P}_2$ respectively. Only then take the limit $\beta \to 0$.

Through this series of manipulations, the degrees of freedom $\lambda^\pm, \pi_\pm, \eta^{1,2}$ and $\mathcal{P}_{3,4}$ become auxiliary fields, which are determined by the dynamical degrees of freedom $x^\mu, \pi_\mu, \eta^3, \eta^4$ and $\mathcal{P}_1, \mathcal{P}_2$[19,20]. In particular, we have

$$\lambda^- = 1, \qquad \lambda^+ = 1, \tag{2.139}$$

showing that the system has indeed been gauge-fixed to the conformal gauge.

Usually, the choice $\Psi = \mathcal{P}_3 + \mathcal{P}_4$ is believed[49] to correspond to the conformal gauge.This however is not correct[32]. Indeed, it does not lead to the constraints $L^{(\alpha)}_n = 0$ for classical solutions[32]. Only the procedure described above[20,32] leads to conformal gauge-fixing within the BFV formalism.

On the other hand, the procedure just presented is actually singular, since it "squeezes out" part of EPS by taking the limit $\beta \to 0$. It may be shown however[20,32,46], also at the quantum level, that all the associated singularities in some sense "factor out" from the system, and may be ignored. Of course, one looses by this procedure any local information concerning the integration measure over the space of gauge orbits[20,32,46].

In order to make contact with the usual notation[15,16] , let us redefine the dynamical fields as follows:

$$c^- = \eta^3, c^+ = \eta^4, b_{--} = -2i\pi\mathcal{P}_1, b_{++} = -2i\pi\mathcal{P}_2. \tag{2.140}$$

With the procedure outlined above, we then obtain for the first-order action:

$$S_{eff} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma[\dot{x}^\mu\pi_\mu - (\phi^{(\alpha)}_+ + \phi^{(\alpha)}_-) + \frac{i}{2\pi}b_{--}(\partial_\tau + \partial_\sigma)c^- + \frac{i}{2\pi}b_{++}(\partial_\tau - \partial_\sigma)c^+], \tag{2.141}$$

for the BRST charge:

$$Q_B = \int_0^\pi d\sigma[c^-\phi_-^{(\alpha)} + c^+\phi_+^{(\alpha)} + \frac{i}{2\pi}c^-\partial_\sigma c^- b_{--} - \frac{i}{2\pi}c^+\partial_\sigma c^+ b_{++}], \tag{2.142}$$

and for the ghost charge:

$$Q_c = \frac{1}{2\pi}\int_0^\pi d\sigma[c^- b_{--} + c^+ b_{++}]. \tag{2.143}$$

For the Poisson brackets, we now have:

$$\{x^\mu(\sigma,\tau), \pi_\nu(\sigma',\tau)\} = \delta^\mu_\nu \delta(\sigma-\sigma'), \tag{2.144}$$

$$\{c^-(\sigma,\tau), b_{--}(\sigma',\tau)\} = -2i\pi\delta(\sigma-\sigma') = \{c^+(\sigma,\tau), b_{++}(\sigma',\tau)\}, \tag{2.145}$$

with

$$\{b_{\pm\pm}, Q_B\} = -2i\pi\phi_\pm^{(T)}, \tag{2.146}$$

and

$$\phi_\pm^{(c)} = \frac{\mp i}{2\pi}[2\partial_\sigma c^\pm b_{\pm\pm} + c^\pm\partial_\sigma b_{\pm\pm}]. \tag{2.147}$$

The corresponding BRST invariant boundary conditions in σ are[20]:

- open strings:

$$\partial_\sigma x^\mu(\sigma=0,\pi;\tau) = 0, \tag{2.148a}$$
$$(c^- - c^+)(\sigma=0,\pi;\tau) = 0 = (b_{--} - b_{++})(\sigma=0,\pi;\tau), \tag{2.148b}$$

- closed strings:

$$\text{periodicity in } \sigma \to \sigma + \pi. \tag{2.149}$$

Note that the expression for the BRST charge now explicitly depends on the choice of the conformal gauge. Actually, when solving for the conjugate momenta π_μ, the action (2.141) and the BRST charge (2.124) precisely correspond to the Lagrangian BRST formulation of the theory, first discussed in Ref. 26, and also derived by the usual Faddeev-Popov path-integral approach[1,16]. The (b,c) ghost system is the corresponding Faddeev-Popov ghost sector due to reparametrization invariance. As emphasized previously, this ghost system always appears for any two dimensional reparametrization invariant system which is gauge-fixed in the conformal gauge.

To conclude this classical Hamiltonian BRST approach, let us give the solutions to the equations of motion in the conformal gauge. Obviously, the solutions in the bosonic sector are those given previously. Only the ghost sector needs to be considered.

For open strings, we have

$$c^\pm(\sigma,\tau) = \sum_n c_n e^{-in(\tau\pm\sigma)}, \tag{2.150a}$$

$$b_{\pm\pm}(\sigma,\tau) = \sum_n b_n e^{-in(\tau\pm\sigma)}. \tag{2.150b}$$

Then

$$\phi_\pm^{(k)} = \frac{1}{2\pi}\sum_n L_n^{(k)} e^{-in(\tau\pm\sigma)}, k = T,\alpha,c, \tag{2.151}$$

with $L_n^{(\alpha)}$ given previously, and

$$L_n^{(c)} = \sum_m (n-m) b_{n+m} c_{-m}, L_n^{(T)} = L_n^{(\alpha)} + L_n^{(c)}. \tag{2.152}$$

One also obtains:

$$Q_B = \sum_n [L_n^{(\alpha)} + \frac{1}{2}L_n^{(c)}]c_{-n}, \tag{2.153}$$

$$Q_c = \sum_n c_{-n} b_n, \tag{2.154}$$

and the corresponding Poisson brackets lead to

$$\{c_n, b_m\} = -i\delta_{n+m,0}, \tag{2.155}$$

$$\{b_n, Q_B\} = -iL_n^{(T)}. \tag{2.156}$$

Similarly, for closed strings, we have

$$c^-(\sigma,\tau) = \frac{1}{2}\sum_n c_n e^{-2in(\tau-\sigma)}, c^+(\sigma,\tau) = \frac{1}{2}\sum_n \bar{c}_n e^{-2in(\tau+\sigma)}, \tag{2.157a}$$

$$b_{--}(\sigma,\tau) = 4\sum_n b_n e^{-2in(\tau-\sigma)}, b_{++}(\sigma,\tau) = 4\sum_n \bar{b}_n e^{-2in(\tau+\sigma)}. \tag{2.157b}$$

One also obtains

$$\phi_-^{(k)} = \frac{2}{\pi}\sum_n L_n^{(k)} e^{-2in(\tau-\sigma)}, \qquad k = T,\alpha,c, \tag{2.158a}$$

$$\phi_+^{(k)} = \frac{2}{\pi}\sum_n \bar{L}_n^{(k)} e^{-2in(\tau+\sigma)}, \qquad k = T,\alpha,c, \tag{2.158b}$$

with the same expressions for $L_n^{(k)}, \bar{L}_n^{(k)}$ as for the open string, in terms of right-and left-modes. Finally, we have:

$$Q_B = \sum\{[L_n^{(\alpha)} + \frac{1}{2}L_n^{(c)}]c_{-n} + [\bar{L}_n^{(\alpha)} + \frac{1}{2}\bar{L}_n^{(c)}]\bar{c}_{-n}\}, \tag{2.159}$$

$$Q_c = \sum_n [c_{-n} b_n + \bar{c}_{-n}\bar{b}_n], \tag{2.160}$$

and

$$\{c_n, b_m\} = -i\delta_{n+m,0} = \{\bar{c}_n, \bar{b}_m\}, \tag{2.161}$$

$$\{b_n, Q_B\} = -iL_n^{(T)}, \{\bar{b}_n, Q_B\} = -i\bar{L}_n^{(T)}. \tag{2.162}$$

Hence, the separation into left-and right-modes obtained for the non-zero mode bosonic sector, with each moving sector identical to the structure which appears for the open string, extends to the ghost sector as well.

Finally, it should be clear that the Virasoro generator $L_n^{(k)}, \bar{L}_n^{(k)}$ satisfy the usual algebras (2.70) and (2.75) of conformal transformations.

2.3.3 BRST quantization

BRST quantization in the conformal gauge simply follows by canonical quantization from the previous classical BRST formulation. In the bosonic sector, this leads of course to the same quantum algebra of commutation relations for $q^\mu, l^\mu, \alpha^\mu_n, \bar{\alpha}^\mu_n$ as in the old covariant quantization. In this ghost sector, we now have the anticommutation relations:

$$\{c_n, b_m\} = \delta_{n+m,0}, \quad \{\bar{c}_n, \bar{b}_m\} = \delta_{n+m,0}, \tag{2.163}$$

with

$$c_n^+ = c_{-n} \quad , \quad \bar{c}_n^+ = \bar{c}_{-n}, \tag{2.164a}$$
$$b_n^+ = b_{-n} \quad , \quad \bar{b}_n^+ = \bar{b}_{-n}. \tag{2.164b}$$

Hence, except for the bosonic zero-mode sector, the quantum operator algebra for the closed string theory and thus also the space of states, is essentially the tensor product of those for the open string, with however the important feature that in the ghost sector the right-and left-moving ghost modes anticommute rather than commute with one another, since

$$\{c_n, \bar{c}_m\} = 0 \quad , \quad \{c_n, \bar{b}_m\} = 0, \tag{2.165a}$$
$$\{b_n, \bar{c}_m\} = 0 \quad , \quad \{b_n, \bar{b}_m\} = 0. \tag{2.165b}$$

For the sake of definiteness, let us restrict the discussion to the open string (or one of the two sectors of the closed string).

In the bosonic sector, the ground-states of the space of states are the states $|\Omega, p>$ introduced in the old covariant quantization. The space of states is now considerably enlarged through the ghost sector, which is represented as follows.

The algebra of ghost zero-modes

$$\{c_0, b_0\} = 1, \quad c_0^2 = 0, \quad b_0^2 = 0, \quad c_0^+ = c_0, \quad b_0^+ = b_0, \tag{2.166}$$

is represented on a two dimensional space spanned by two basis vectors $|->$ and $|+>$ such that:

$$c_0|->=|+> \quad , \quad c_0|+>=0, \tag{2.167a}$$
$$b_0|->=0 \quad , \quad b_0|+>=|->, \tag{2.167b}$$
$$<-|+>=<+|->=1 \quad , \quad <-|->=<+|+>=0. \tag{2.167c}$$

For the non-zero mode ghost algebra, let us introduce a ground state also denoted $|\Omega>$, such that:

$$b_n|\Omega>=0, \quad c_n|\Omega>=0, \quad n \geq 1, \tag{2.168a}$$
$$<\Omega|\Omega>=1. \tag{2.168b}$$

Thus, the space of states is spanned by the ground states $|\Omega, p, \pm>$ and their bosonic and ghost excitations, obtained through the action of the bosonic and ghost creation operators α^μ_{-n}, c_{-n} and $b_{-n} (n \geq 1)$.

The complete definition of the quantum system also requires a choice of normal ordering. In the bosonic sector, this choice is as before. In the ghost sector, one again

chooses to bring all the creation operator $c_{-n}, b_{-n} (n \geq 1)$ to the left of all annihilation operators $c_n, b_n (n \geq 1)$. For the zero-modes, we must have

$$: c_0 b_0 := \frac{1}{2}[c_0 b_0 - b_0 c_0]. \tag{2.169}$$

This leads to

$$Q_B = \sum_n : (L^{(\alpha)}_{-n} + \frac{1}{2} L^{(c)}_{-n}) c_n : - a c_0, \tag{2.170}$$
$$Q_c = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty} [c_{-n} b_n - b_{-n} c_n], \tag{2.171}$$
$$L^{(T)}_n = \{b_n, Q_B\} = L^{(\alpha)}_n + L^{(c)}_n - a\delta_{n,0}, \tag{2.172}$$

where all composite operators $L^{(k)}_n$ are understood to be normal ordered quantities. Here again, "a" is a normal ordering constant. Obviously, similar expressions apply for the closed string, with left-moving modes then also contributing.

The question of quantum conformal invariance in the conformal gauge way now consistently be addressed, either by computing the central extension in the Virasoro algebra of the total Virasoro generator $L^{(T)}_n$, or equivalently by checking gauge invariance of the quantum system, namely by checking nilpotency of the quantum BRST charge.

Using techniques of conformal field theory (see for example Ref.19), one obtains:

$$[L^{(\alpha)}_n, L^{(\alpha)}_m] = (n-m) L^{(\alpha)}_{n+m} + \frac{1}{12} D n (n^2 - 1) \delta_{n+m,0}, \tag{2.173a}$$
$$[L^{(c)}_n, L^{(c)}_m] = (n-m) L^{(c)}_{n+m} + \frac{1}{12}[-26 n^3 + 2n] \delta_{n+m,0}, \tag{2.173b}$$
$$[L^{(T)}_n, L^{(T)}_m] = (n-m) L^{(T)}_{n+m} + \frac{1}{12}[(D-26) n^3 + (2 + 24a - D) n] \delta_{n+m,0}, \tag{2.173c}$$

and

$$\begin{aligned} \{Q_B, Q_B\} &= 2Q_B^2 = \sum_n \frac{1}{12}[(D-26) n^3 + (2 + 24a - D) n + 2(a-1)] : c_{-n} c_n : \\ &= \sum_n \{[L^{(T)}_n, L^{(T)}_m] - (n-m) L^{(T)}_{n+m}\} : c_{-n} c_n : . \end{aligned} \tag{2.174}$$

Hence, quantum conformal invariance and nilpotency of the quantum BRST charge are obtained under the same conditions, namely[26]

$$D = 26, \qquad a = 1. \tag{2.175}$$

That these two properties are equivalent may be seen from the relation (2.172), which leads to the following when $Q_B^2 = 0$:

$$[L^{(T)}_n, Q_B] = 0, \qquad [L^{(T)}_n, L^{(T)}_m] = (n-m) L^{(T)}_{n+m}. \tag{2.176}$$

Thus, nilpotency of Q_B or cancellation of the central extension in the total Virasoro algebra are one and the same expression of conformal invariance in the conformal gauge. The critical values (2.175), which we found in the old covariant approach, are precisely those which ensure reparametrization invariance of the quantum system.

We still have to define physical states in the present formalism. First of all, since Q_B implements gauge transformations, two states which differ by a state of the form $Q_B|\chi>$ must be considered as representing the same quantum state. Thus, gauge equivalent states fall into cohomology classes of the BRST charge. Moreover, physical states must be BRST or gauge invariant states, namely

$$Q_B|\Psi>=0. \tag{2.177}$$

Hence, physical states correspond to BRST invariant cohomology classes of Q_B, or in other words the coset $KerQ_B/ImQ_B$. Since the null class is always BRST invariant for a nilpotent BRST charge, we only need to consider the non-trivial BRST invariant classes. The answer to this problem is the following[50].

All states in non-trivial BRST invariant classes are of the form

$$|\Psi>=|\Psi^-_{(\alpha)};->+|\Psi^+_{(\alpha)};+>, \tag{2.178}$$

(up to a state $Q_B|\chi>$ of course), where $|\Psi^{\pm}_{(\alpha)}>$ are states with bosonic excitations only and of strictly positive norm, i.e. transverse physical states in 1-to-1 correspondence with states in the light-cone gauge.

In order to restrict the choice further, it is useful to require that physical states have ghost number (-1/2):

$$Q_c|\Psi>=(-1/2)|\Psi>, \tag{2.179}$$

so that we finally have

$$|\Psi>=|\Psi^-_{(\alpha)};->. \tag{2.180}$$

The BRST invariant condition (2.177) then becomes

$$\{c_0[L_0^{(\alpha)}-1]+\sum_{n=1}^{\infty}c_{-n}L_n^{(\alpha)}\}|\Psi^-_{(\alpha)};->=0, \tag{2.181}$$

which clearly shows that the Virasoro conditions defining physical states in the old covariant approach are indeed recovered.

In conclusion, within BRST quantization, physical states correspond to BRST invariant cohomology classes of the BRST charge of ghost number (-1/2). Such non-trivial classes are in 1-to-1 correspondence with physical states of strictly positive norm both in the old covariant approach and in the light-cone gauge approach.

3 FERMIONIC STRINGS

Our discussion so far has been very detailed, and emphasized the fundamental features of the quantization of bosonic strings in the conformal and in the light-cone gauge. The reason is that these basic properties are generic of any string theory, so that it should now be easy to understand how any of their extensions and generalizations are to be used in the construction of any string theory. In this chapter, one such extension will be considered in some detail, not only because it provides an example of a generalization

of the structures encountered so far, but also because it plays an important role in the construction of "realistic" string theories.

Bosonic strings obviously lack an important feature : their space-time spectrum does not contain any fermions. Such states can be obtained within string theory through two main approaches, where one introduces additional world-sheet degrees of freedom. In the first approach, the so-called spinning string[23] or Neveu-Schwarz-Ramond (NSR)[1,52] formulation, one introduces world-sheet Majorana spinors transforming as space-time vectors. In the second approach, the so-called superstring or Green-Schwarz formulation[53], one introduces world-sheet scalars transforming as space-time Majorana-Weyl spinors. In the latter approach, the construction of the theory explicitly realizes a space-time global supersymmetry, while in the former approach such a symmetry is only achieved by some specific projection of the spectrum[54], necessary for quantum consistency[55]. It is widely believed however, that the two approaches are actually equivalent[56], but this has not yet been proved in full generality.

By lack of space, we shall not consider here the superstring approach. Let us only mention that recent progress[57] has been made in the covariant quantization of these theories (for their quantization in the light-cone gauge, see for example Refs. 15,17), and that they have been used in the construction of four dimensional string theories[58].

As was shown in the previous chapter, reparametrization invariance and the corresponding constraints, which are the equations of motion for an intrinsic metric, are essential in obtaining a consistent quantum theory. In the conformal gauge, they are necessary in establishing the decoupling of unphysical and zero-norm states from physical amplitudes. In the light-zone gauge, they are essential for obtaining space-time covariance. Thus, by enlarging the system through the introduction of world-sheet spinors, we need at the same time to have an additional world-sheet local symmetry, leading to new constraints necessary to ensure consistency of the quantized system. The candidate for this additional symmetry is obviously a supersymmetry[23]. Hence, by extending the previous bosonic construction in the linear formulation to a two dimensional supergravity theory with local supersymmetry[23], we should expect that the necessary constraints would follow as the equations of motion of the supergravity sector.

Indeed, the action for the spinning string[23] is that of a two dimensional $N=1$ minimal supergravity theory, with D matter multiplets $(x^\mu(\xi),\lambda^\mu(\xi))$ coupled to a supergravity multiplet $(e^a_\alpha(\xi),\psi_\alpha(\xi))$, where $e^a_\alpha(\xi)$ is a world-sheet zweibein or graviton, $\psi_\alpha(\xi)$ is a world-sheet Majorana gravitino and the $N=1$ superpartners $\lambda^\mu(\xi)$ of the bosonic coordinates are world-sheet Majorana spinors. Here, $\alpha=0,1$ are the usual world-sheet coordinate indices, and $a=0,1$ are the corresponding tangent space indices. In the following, we shall use[17] the Majorana representation of the Dirac algebra in two dimensions, namely

$$\rho^0=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\ \rho^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \rho^5=-\rho^0\rho^1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{3.1}$$

with

$$\{\rho^a,\rho^b\}=2\eta^{ab}, \tag{3.2}$$

$$\eta^{ab}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}. \tag{3.3}$$

In this representation, a Majorana spinor is a real spinor.

As for the bosonic string, the spinning string action is not only invariant under local world-sheet diffeomorphisms and supersymmetry transformations, i.e. local super-reparametrizations, but also under local Weyl rescalings of the graviton supermultiplet

and the associated fermionic transformation, i.e. super-Weyl transformations (for more details, see for example Refs. 15,17). As a consequence, the corresponding Weyl modes of the graviton and gravitino do not couple in the classical action, and will not appear in the canonical quantization of the theory. They do couple in a path-integral quantization[22] unless conditions analogous to those discussed for the bosonic string are satisfied. Moreover, the graviton multiplet is only an auxiliary multiplet in two dimensions, whose equations of motion are precisely the constraints[15,17] associated to super-reparametrization and super-Weyl invariance and generating these transformations.

These local gauge invariances allow for gauge-fixing the system to the superconformal gauge, defined by

$$e^a_\alpha(\xi) = e^{\phi(\xi)}\delta^a_\alpha\,,\ \psi_\alpha(\xi) = e^a_\alpha(\xi)\rho_a\chi(\xi)\,, \tag{3.4}$$

where $\chi(\xi)$ is a Majorana spinor. In the superconformal gauge, the system still possesses a local gauge invariance under superconformal transformations. The corresponding generators are then simply the Fourier modes of the constraints, as we showed in the case of the bosonic string. The important point which makes this possible is that the superconformal gauge-fixing preserves an explicit *global* $N = 1$ world-sheet supersymmetry.

3.1 Old Covariant Quantization

3.1.1 The superconformal gauge[15,17]

In the superconformal gauge, the action of the spinning string reduces to

$$S = \frac{-1}{4\pi\alpha'}\int d^2\xi\left[\eta^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu + i\overline{\lambda}^\mu\delta^\alpha_a\rho^a\partial_\alpha\lambda_\mu\right]\,, \tag{3.5}$$

where

$$\eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{3.6}$$

$$\lambda^\mu = \begin{pmatrix} \lambda^\mu_- \\ \lambda^\mu_+ \end{pmatrix}\,, \tag{3.7}$$

and $\overline{\chi} = \chi^+\rho^0$ for a spinor χ.

This action is invariant under the $N = 1$ global supersymmetry defined by

$$\delta x^\mu = i\overline{\eta}\lambda^\mu\,,\ \delta\lambda^\mu = \delta^\alpha_a\rho^a\partial_\alpha x^\mu\eta\,, \tag{3.8}$$

where η is a constant infinitesimal Majorana spinor. For the Majorana representation given in (3.1), the action (3.5) becomes

$$S = \frac{-1}{4\pi\alpha'}\int d^2\xi\left[x^\mu(\partial_\tau^2 - \partial_\sigma^2)x\mu - i\lambda^\mu_+(\partial_\tau - \partial_\sigma)\lambda_{+\mu} - i\lambda^\mu_-(\partial_\tau + \partial_\sigma)\lambda_{-\mu}\right]\,, \tag{3.9}$$

thus leading to the equations of motion

$$(\partial_\tau^2 - \partial_\sigma^2)x^\mu = 0 \tag{3.10}$$

$$(\partial_\tau - \partial_\sigma)\lambda^\mu_+ = 0\,,\ (\partial_\tau + \partial_\sigma)\lambda^\mu_- = 0\,. \tag{3.11}$$

Hence, as for the bosonic string, the system has reduced in the superconformal gauge to a set of "free" massless fields, which have to satisfy a set of constraints to be discussed

below. Note that the separation into right- and left-movers extends to the fermionic sector, where λ^μ_+ and λ^μ_- correspond to left- and right-moving Majorana-Weyl spinors respectively.

The boundary conditions in σ in the bosonic sector are as specified previously. For the fermionic matter fields, we have for open strings :

$$\lambda^\mu_-(\sigma = 0,\tau) = \lambda^\mu_+(\sigma = 0,\tau)\,, \tag{3.12a}$$

$$\lambda^\mu_-(\sigma = \pi,\tau) = \epsilon\lambda^\mu_+(\sigma = \pi,\tau)\,, \tag{3.12b}$$

where $\epsilon = +1$ corresponds to the Ramond (R) sector[51], and $\epsilon = -1$ to the Neveu-Schwarz (NS) sector[52] of the spinning string. For closed strings, we have

$$\lambda^\mu(\sigma = \pi,\tau) = \Lambda\lambda^\mu(\sigma = 0,\tau) \tag{3.13}$$

with $\Lambda = -\mathbb{1}$, $\Lambda = \rho^5$, $\Lambda = -\rho^5$, $\Lambda = \mathbb{1}$ respectively for (N,NS), (R,NS), (N,R), (R,R) sectors, where the first entry corresponds to the right-moving modes and the second to the left-moving modes.

Actually, R boundary conditions preserve the global supersymmetry (3.8), whereas NS ones break it. As we shall see, such a breakdown of world-sheet supersymmetry is related to the presence of space-time tachyons in the physical spectrum.

Corresponding to space-time Poincaré invariance of the system, we have locally conserved world-sheet currents whose charges are the total energy- and angular-momentum of the string. In the superconformal gauge, on finds[17] for the energy-momentum current

$$P^0_\mu = \frac{\dot{x}^\mu}{2\pi\alpha'}\,,\ P^1_\mu = \frac{-x'^\mu}{2\pi\alpha'}\,,\ \partial_\alpha P^\alpha_\mu = 0\,, \tag{3.14}$$

thus showing that the total energy-momentum of the spinning string

$$P_\mu = \int_0^\pi d\sigma\, P^0_\mu\,, \tag{3.15}$$

is entirely carried by the bosonic coordinates. On the other hand, fermionic degrees of freedom do contribute to the angular-momentum of the system. Indeed, one finds[17]

$$M^0_{\mu\nu} = \frac{1}{2\pi\alpha'}\left\{[\dot{x}_\mu x_\nu - \dot{x}_\nu x_\mu] + \frac{1}{2}i[\lambda^T_\mu\lambda_\nu - \lambda^T_\nu\lambda_\mu]\right\}\,, \tag{3.16a}$$

$$M^1_{\mu\nu} = \frac{1}{2\pi\alpha'}\left\{[-x'_\mu x_\nu + x'_\nu x_\mu] + \frac{1}{2}i[\lambda^T_\mu\rho^5\lambda_\nu - \lambda^T_\nu\rho^5\lambda_\mu]\right\}\,, \tag{3.16b}$$

with

$$\partial_\alpha M_{\mu\nu} = 0\,, \tag{3.17}$$

and

$$M_{\mu\nu} = \int_0^\pi d\sigma\, M^0_{\mu\nu}\,. \tag{3.18}$$

Note that the boundary conditions in σ for open spinning strings are such that there is no flow of energy- nor angular-momentum at the end points.

Local world-sheet reparametrizations and supersymmetry transformations are generated by the two dimensional matter energy-momentum tensor $T_{\alpha\beta}$ and supercurrent J_α respectively, which are given in the superconformal gauge by [15,17].

$$\begin{aligned} T_{\alpha\beta} &= \partial_\alpha x^\mu\partial_\beta x_\mu + \frac{1}{4}i\overline{\lambda}^\mu[\rho_\alpha\partial_\beta + \rho_\beta\partial_\alpha]\lambda_\mu \\ &\quad -\frac{1}{2}\eta_{\alpha\beta}\left[\eta^{\gamma\delta}\partial_\gamma x^\mu\partial_\delta x_\mu + \frac{1}{2}i\overline{\lambda}^\mu\rho^\gamma\partial_\gamma\lambda_\mu\right]\,, \end{aligned} \tag{3.19}$$

$$J_\alpha = \partial_\beta x_\mu\rho^\beta\rho_\alpha\lambda^\mu\,, \tag{3.20}$$

where

$$\rho^\beta = \delta^\beta_b \rho^b \,,\; \rho_\alpha = \eta_{\alpha\beta}\delta^\beta_a \rho^a \,. \tag{3.21}$$

They obey the local conservation equation :

$$\partial_\alpha T^{\alpha\beta} = 0 \,,\; \partial_\alpha J^\alpha = 0 \,. \tag{3.22}$$

In fact, as a consequence of super-reparametrization and super-Weyl invariance, the equations of motion for the graviton multiplet simply reduce to the constraint equations[15,17] :

$$T_{\alpha\beta} = 0 \,,\; J_\alpha = 0 \,, \tag{3.23}$$

which must be satisfied by any solution to the free field equations (3.10) and (3.11).

In the case of open spinning strings, we have the same mode expansion for bosonic coordinates as in (2.23), and for fermionic coordinates :

$$\lambda^\mu_\pm(\sigma,\tau) \;=\; \sqrt{\alpha'}\sum_q c^\mu_q e^{-iq(\tau\pm\sigma)} \,, \tag{3.24a}$$

$$\text{with} \qquad (c^\mu_q)^* \;=\; c^\mu_{-q} \tag{3.24b}$$

Here, the fermionic modes c^μ_q are integer-moded and denoted $d^\mu_n = c^\mu_q$ $(q = n \in \mathbb{Z})$ for a R sector, and half-integer moded and denoted $b^\mu_r = c^\mu_q$ $(q = r \in \mathbb{Z} + 1/2)$ for a NS sector.

The Fourier modes of the constraints then lead to the super Virasoro constraints

$$L^{(X)}_n = 0 \,,\; J^{(X)}_q = 0 \,, \tag{3.25}$$

where the super Virasoro generators are given by

$$L^{(X)}_n \;=\; L^{(\alpha)}_n + L^{(\lambda)}_n \,,\; L^{(\lambda)}_n = \frac{1}{2}\sum_q \left(q - \frac{1}{2}n\right) c^\mu_{n-q} c_{q\mu} \,, \tag{3.26}$$

$$J^{(X)}_q \;=\; \sum_m c^\mu_{q-m}\alpha_{m\mu} \,. \tag{3.27}$$

Here, the superscripts (X), (α) and (λ) stand for the contributions of the matter supermultiplet, of the bosonic modes and of the fermionic modes respectively. The supercurrent modes $J^{(X)}_q$ are integer moded and denoted $J^{(X)}_q = F^{(X)}_n$ for a R sector, and half-integer moded and denoted $J^{(X)}_q = G^{(X)}_r$ for a NS sector.

These quantities $L^{(X)}_n$ and $J^{(X)}_q$ are the generators of $N = 1$ superconformal transformations, which is the remaining super-reparametrization invariance of the system in the superconformal gauge.

For closed spinning strings, the mode expansion for bosonic coordinates is as in (2.26), and for fermionic coordinates we have :

$$\lambda^\mu_-(\sigma,\tau) = \sqrt{2\alpha'}\sum_q c^\mu_q e^{-2iq(\tau-\sigma)} \,,\; \lambda^\mu_+(\sigma,\tau) = \sqrt{2\alpha'}\sum_q \overline{c}^\mu_q e^{-2iq(\tau+\sigma)} \,, \tag{3.28}$$

with the following notation :

right-left sector	c^μ_q	$\overline{c}^\mu_q$	
(NS,NS)	b^μ_r	$\overline{b}^\mu_r$	$r \in \mathbb{Z}+1/2$
(NS,R)	b^μ_r	$\overline{d}^\mu_n$	$n \in \mathbb{Z}$
(R,NS)	d^μ_n	$\overline{b}^\mu_r$	
(R,R)	d^μ_n	$\overline{d}^\mu_n$	

(3.29)

The Fourier modes of the constraints (3.23) lead to the super Virasoro constraints :

$$L^{(X)}_n = 0 \quad , \quad \overline{L}^{(X)}_n = 0 \,, \tag{3.30a}$$

$$J^{(X)}_q = 0 \quad , \quad \overline{J}^{(X)}_q = 0 \,, \tag{3.30b}$$

with the same notations and definitions as in (3.25), (3.26) and (3.27), now in terms of right- and left-moving modes. Note the by now familiar fact of doubling of fermionic and non-zero bosonic modes, when comparing closed strings to open strings.

The old covariant quantization in the superconformal gauge would require the corresponding Hamiltonian formulation of the system. Compared to the bosonic string, there are essentially two new features which arise here. The world-sheet Majorana spinors lead to second-class constraints, which may be solved by using Dirac brackets for these degrees of freedom. The set of first-class constraints generating the gauge symmetries of the theory now consist of bosonic and Grassmannian constraints. The associated Lagrange multiplets are respectively bosonic and Grassmannian variables, related to each other by local supersymmetry transformation and transforming as components of a supergravity multiplet. Superconformal gauge fixing amounts to setting the bosonic Lagrange multipliers λ^+ and λ^- equal to 1, and the Grassmann Lagrange multiplies equal to zero. It should thus be clear to the reader how such an Hamiltonian formulation of the spinning string could be obtained, and we shall not give any other details here.

In the superconformal gauge, we therefore have the same commutation relations as before for the bosonic modes (see (2.80), (2.81), (2.82)), both for open and for closed strings. For the fermionic matter fields $\lambda^\mu_\pm(\sigma,\tau)$, we obtain the anticommutation relations :

$$\{c^\mu_{q_1}, c^\nu_{q_2}\} = \eta^{\mu\nu}\delta_{q_1+q_2,0} \quad , \quad \{\overline{c}^\mu_{q_1}, \overline{c}^\nu_{q_2}\} = \eta^{\mu\nu}\delta_{q_1+q_2,0} \,, \tag{3.31a}$$

$$\text{with} \qquad (c^\mu_q)^+ = c^\mu_{-q} \;,\; (\overline{c}^\mu_q)^+ = \overline{c}^\mu_{-q} \,. \tag{3.31b}$$

These results allow for the description of the corresponding spaces of states. Let us first consider the case of the open spinning string.

In the Ramond sector, the Fock vacuum is a space-time spinor $|u(p)\rangle$, with $u(p)$ being a D-dimensional spinor, such that

$$\alpha^\mu_n|u(p)\rangle \;=\; 0 \;,\; d^\mu_n|u(p)\rangle = 0 \,,\; n \geq 1 \tag{3.32a}$$

$$\alpha^\mu_0|u(p)\rangle \;=\; \sqrt{2\alpha'}p^\mu|u(p)\rangle \,, \tag{3.32b}$$

$$d^\mu_0|u(p)\rangle \;=\; |\frac{1}{\sqrt{2}}i\gamma^\mu u(p)\rangle \,, \tag{3.32c}$$

where γ^μ are the D-dimensional Dirac matrices which satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \;,\; \gamma^{\mu+} = \gamma^0\gamma^\mu\gamma^0 \,. \tag{3.33}$$

Indeed, the fermionic zero-modes d^μ_0 satisfy the algebra

$$\{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu} \;,\; \{d^\mu_0, d^\nu_n\} = 0 \;,\; n \neq 0 \,, \tag{3.34}$$

so that they are represented by

$$d^\mu_0 = \frac{i}{\sqrt{2}}(-1)^{\sum_{n=1}^\infty d^\mu_{-n}d_{n\mu}}\gamma^\mu \,. \tag{3.35}$$

All other states are obtained by the action of the creation operators α^μ_{-n}, d^μ_{-n} ($n \geq 1$) on these Fock vacua. Clearly, all states in the Ramond sector are space-time fermions. In particular, the ground-state spinor transforms under Lorentz transformations as :

$$M^{\mu\nu}|u(p)\rangle = \sqrt{2\alpha'}[p^\mu q^\nu - p^\nu q^\mu]|u(p)\rangle + |\Sigma^{\mu\nu}u(p)\rangle\,, \tag{3.36a}$$

where

$$\Sigma^{\mu\nu} = -\tfrac{1}{4}i[\gamma^\mu, \gamma^\nu] \tag{3.36b}$$

is the generator of Lorentz transformations in the spinor representation. The first term in (3.36a) corresponds to the orbital contribution.

In the Neveu-Schwarz sector, since there are no fermionic zero-modes, the Fock vacua are momentum eigenstates $|\Omega;p\rangle$ annihilated by α^μ_n, b^μ_r ($n \geq 1, r \geq 1/2$), and transforming as space-time scalars. Hence, all states in the NS sector are space-time bosons.

In the case of closed spinning strings, except for the bosonic zero-mode sector, the space of states is essentially obtained as the tensor product of spaces of states of the open string, with the subtlety that the right- and left-moving fermionic modes anticommute. We thus have for the Fock vacua and the space-time spectra in each sector :

sector	Fock vacua	space-time spectrum
(NS, NS)	$\lvert\Omega, p\rangle$	space-time bosons
(R, NS)	$\lvert u(p)\rangle$	space-time fermions
(NS, R)	$\lvert\overline{u}(p)\rangle$	space-time fermions
(R, R)	$\lvert u(p), \overline{u}(p)\rangle$	space-time bosons

(3.37)

where $u(p)$ and $\overline{u}(p)$ are independent D-dimensional spinors associated to the right- and left-moving sectors respectively.

Let us remark that all these spaces of states, both for open and for closed strings, have states of negative norm, due to the fact that $\eta^{00} = -1$.

The definition of the quantum system is complete only when a normal ordering of quantum operators has been specified. As usual, all position and creation operators are brought to the left of all momenta and annihilation operators. For the fermionic zero-modes in the R sector, we must choose :

$$: d^\mu_0 d^\nu_0 := \frac{1}{2}(d^\mu_0 d^\nu_0 - d^\nu_0 d^\mu_0)\,. \tag{3.38}$$

Defining then the super Virasoro generators by the same expressions as given before, but now understood to be normally ordered, it is possible to compute the corresponding $N = 1$ super-Virasoro algebras, both in R and NS sectors, using techniques of conformal field theory (see for example Refs. 17,19). We only give here the results for the open string or for the right-moving sector of the closed string. Similar results obviously hold for the left-moving modes.

<u>Ramond sector</u>

$$\{F^{(X)}_n, F^{(X)}_m\} = 2L^{(X)}_{n+m} + \frac{1}{2}Dn^2\delta_{n+m,0}\,, \tag{3.39a}$$

$$[L^{(X)}_n, F^{(X)}_m] = \left(\frac{1}{2}n - m\right)F^{(X)}_{n+m}\,, \tag{3.39b}$$

$$[L^{(X)}_n, L^{(X)}_m] = (n-m)L^{(X)}_{n+m} + \frac{1}{8}Dn^3\delta_{n+m,0}\,, \tag{3.39c}$$

where

$$[L^{(\lambda)}_n, L^{(\lambda)}_m] = (n-m)L^{(\lambda)}_{n+m} + \frac{1}{12}D\left[\frac{1}{2}n^3 + n\right]\delta_{n+m,0}\,. \tag{3.40}$$

<u>Neveu-Schwarz sector</u>

$$\{G^{(X)}_r, G^{(X)}_s\} = 2L^{(X)}_{r+s} + \frac{1}{2}D(r^2 - \frac{1}{4})\delta_{r+s,0}\,, \tag{3.41a}$$

$$[L^{(X)}_n, G^{(X)}_r] = \left(\frac{1}{2}n - r\right)G^{(X)}_{n+r}\,, \tag{3.41b}$$

$$[L^{(X)}_n, L^{(X)}_m] = (n-m)L^{(X)}_{n+m} + \frac{1}{8}Dn(n^2-1)\delta_{n+m,0}\,, \tag{3.41c}$$

where

$$[L^{(\lambda)}_n, L^{(\lambda)}_m] = (n-m)L^{(\lambda)}_{n+m} + \frac{1}{24}Dn(n^2-1)\delta_{n+m,0}\,. \tag{3.42}$$

As for the bosonic string, we obtain central extensions to the $N = 1$ superconformal algebra, thus showing that short distance quantum effects in the matter sector of the theory (seem to) break the super-reparametrization invariance of the spinning string. Note that each Majorana-Weyl spinor contributes a value $(+1/2)$ to be central charge of the total Virasoro algebra.

It should be obvious that the quantization in the superconformal gauge explicitly preserves space-time Poincaré covariance. Indeed, one may show that the Poincaré algebra is obtained at the quantum level, and that it commutes with the super Virasoro algebra. Hence, all physical states may be characterized by their mass and by their "spin", as was explained for the bosonic string.

In a R sector, physical states are defined by the super-Virasoro constraints :

$$F^{(X)}_n|\psi\rangle = 0\,,\ L^{(X)}_n|\psi\rangle = 0\,,\ n \geq 1\,, \tag{3.43a}$$

$$(F^{(X)}_0 - \sqrt{a_+})|\psi\rangle = 0\,,\ (L^{(X)}_0 - a_+)|\psi\rangle = 0\,. \tag{3.43b}$$

In a NS sector, we have

$$G^{(X)}_r|\psi\rangle = 0\,,\ L^{(X)}_n|\psi\rangle = 0\,,\ r \geq 1/2, n \geq 1\,, \tag{3.44a}$$

$$(L^{(X)}_0 - a_-)|\psi\rangle = 0 \tag{3.44b}$$

Here, a_+ and a_- are subtraction constants following from normal ordering in the super-Virasoro zero-modes. Note that we introduced a normal ordering constant for $F^{(X)}_0$. On the one hand, this does not seem necessary since $F^{(X)}_0$ does not suffer from an ordering ambiguity. On the other hand, it seems necessary since we have $(F^{(X)}_0)^2 = L^{(X)}_0$. As it turns out, a_+ indeed vanishes (see the no-ghost theorem below). Note also that the constraint which involves $F^{(X)}_0$ generalizes Dirac's equation[51]. As usual, the constraints involving the Virasoro zero-modes lead the mass formulae. For open strings, we have:

$$\text{R sector}\ :\ \alpha' M^2 = N_R - a_+\,, \tag{3.45a}$$

where

$$N_R = \sum_{n=1}^{\infty}\alpha^\mu_{-n}\alpha_{n\mu} + \sum_{n=1}^{\infty}nd^\mu_{-n}d_{n\mu}\,. \tag{3.45b}$$

$$\text{NS sector} \quad : \quad \alpha' M^2 = N_{NS} - a_- , \tag{3.46a}$$

where

$$N_{NS} = \sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n\mu} + \sum_{r=1/2}^{\infty} r b^{\mu}_{-r} b_{r\mu} . \tag{3.46b}$$

For closed strings, we have :

$$\frac{1}{2}\alpha' M^2 = (N-a) + (\overline{N} - \overline{a}) , \tag{3.47a}$$

$$N - a = \overline{N} - \overline{a} , \tag{3.47b}$$

where $N, \overline{N}$ are N_R, N_{NS}, $\overline{N}_R$, $\overline{N}_{NS}$, and a, $\overline{a}$ are $a_+, a_-, \overline{a}_+, \overline{a}_-$ depending on whether we have a R or a NS sector in each moving sector. The level matching constraint (3.47b) has the usual geometrical meaning in terms of constant shifts in σ.

For the spinning string, the results of the no-ghost theorem[59] lead to the critical values :

$$D = 10 , \quad a_+ = 0 , \quad a_- = \frac{1}{2} . \tag{3.48}$$

These are the necessary conditions for the absence of negative norm physical states and for the correct cut and pole singularities in one-loop amplitudes. The values (3.18) may be checked by explicitly constructing the first few physical excitation levels. Let us give those of strictly positive norm.

In the Ramond sector of the open spinning string, the physical ground-state is at excitation level $N_R = 0$. It corresponds to a massless space-time spinor $|u(p)\rangle$ with $\not{p}u|p\rangle = 0$. At the next excitation level $N_R = 1$, we have the states $\alpha^{\mu}_{-1}|u(p)\rangle$ and $d^{\mu}_{-1}|u(p)\rangle$, which lead to a massive Rarita-Schwinger ("spin $\frac{3}{2}$") state. All states in this sector are space-time fermions, with a double degeneracy of the Regge "trajectories" due to the space-time chirality of the spinor $u(p)$.

In the NS sector of the open string, the physical ground state is a space-time scalar tachyon $|\Omega; p\rangle$, with $\alpha' m^2 = 1/2$. At the next excitation level $N_{NS}\frac{1}{2}$, we have the massless states $b^{\mu}_{-1/2}|\Omega; p>$, giving a massless vector state. At the second excitation level $N_{NS} = 1$, we have the massive states $\alpha^{\mu}_{-1}|\Omega; p>, b^{\mu}_{-1/2}b^{\nu}_{-1/2}|\Omega; p>$, with $\alpha' m^2 = \frac{1}{2}$, leading to a 2-index antisymmetric tensor state. All states in this sector are space-time bosons, with two types of Regge "trajectories". Those corresponding to an even number of fermionic excitations, to which the tachyon belongs, and those with an odd number of fermionic excitations, to which the massless vector belongs.

For the closed spinning string, the discussion is very similar. In the (N,NS) sector, the physical ground-state $|\Omega, p\rangle$ is a scalar tachyon with $\frac{1}{2}\alpha' m^2 = -1$. At the next excitation level, we have a massless graviton, a massless 2-index antisymmetric transverse tensor and a massless dilaton, obtained from $b^{\mu}_{-1/2}\overline{b}^{\nu}_{-1/2}|\Omega; p\rangle$.

In the (R,NS) sector, the would-be spinor tachyon $|u(p)\rangle$ is not a physical state, due to the level matching condition. The physical ground-state has $N_R = 0$, $\overline{N}_{NS} = 1/2$, and thus corresponds to the massless states $\epsilon_{\mu}\overline{b}^{\mu}_{-1/2}|u(p)\rangle$, with $\not{p}u(p) = 0$ and $\epsilon \cdot p = 0$. Since the condition $\not{\epsilon}u(p) = 0$ is not obtained, we have a massless spinor ("spin 1/2") and a massless Rarita-Schwinger ("spin 3/2") state.

The discussion in the (NS, R) sector is similar, by exchanging the right- and left-moving sectors. Finally, in the (R, R) sector, the physical ground-state $|u(p), \overline{u}(p)\rangle$ is massless, with $\not{p}u(p) = 0.$, $\not{p}\overline{u}(p) = 0$, and corresponds to all completely antisymmetric

space-time tensors, appearing in the tensor product of the two space-time spinors $u(p)$ and $\overline{u}(p)$.

Clearly, the Regge "trajectories" of the closed string also have a double "degeneracy", associated either to the chirality of the spinor in a R sector or to the parity of the fermion excitation number in a NS sector, in each of the two moving sectors.

3.1.2 Light-cone gauge quantization

Let us briefly describe how the same results are obtained in the light-cone gauge. For the spinning string, the light-cone gauge fixing conditions are[17] (in addition to the superconformal gauge conditions) :

$$x^+(\sigma, \tau) = 2\alpha' P^+ \tau , \quad \lambda^+(\sigma, \tau) = 0 . \tag{3.49}$$

The super-Virasoro constraints $L^{(X)}_n = 0$, $J^{(X)}_q = 0$ may then be solved for explicitly, by expressing the light-cone components $x^{\pm}(\sigma, \tau)$ and $\lambda^{\pm}(\sigma, \tau)$ (or equivalently $\alpha^{\pm}_n$, $\overline{\alpha}^{\pm}_n$ and $c^{\pm}_q$, $\overline{c}^{\pm}_q$) in terms of the transverse components $x^i(\sigma, \tau)$, $\lambda^i(\sigma, \tau)$, (or equivalently q^i, α^i_n, $\overline{\alpha}^i_n$, c^i_q, $\overline{c}^i_q$) and the two integration constants q^- and P^+.

From the corresponding Dirac brackets, we then have the usual commutation and anticommutation relations for the transverse modes q^i, P^i, α^i_n, $\overline{\alpha}^i_n$, c^i_q, $\overline{c}^i_q$ and q^-, P^+. Thus, the Fock vacua for a NS sector are simply states $|\Omega, p^i, p^+\rangle$ annihilated by α^i_n and b^i_r $(n \geq 1, r \geq 1/2)$. For a R sector, they are $SO(D-2)_n$ spinors $|u(p^i, p^+)\rangle$ annihilated by α^i_n and d^i_n $(n \geq 1)$, since the fermionic zero-modes d^i_0 satisfy the Clifford algebra $\{d^i_0, d^j_0\} = \delta^{ij}$. By construction, all states have strictly positive norm, and correspond to transverse excitations only.

At the quantum level, when solving for the light-cone coordinates in terms of the transverse ones, normal ordering constants appear in the expressions for the zero-modes α^-_0 and $\overline{\alpha}^-_0$. This then leads to the mass formulae. For open strings, we have :

$$\alpha' M^2 = N^{\perp} - a , \tag{3.50}$$

with

$$R \text{ sector} : \quad N^{\perp} = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{n=1}^{\infty} n d^i_{-n} d^i_n , \quad a = a_+ , \tag{3.51a}$$

$$NS \text{ sector} : \quad N^{\perp} = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r=1/2}^{\infty} r b^i_{-r} b^i_r , \quad a = a_- . \tag{3.51b}$$

For closed strings, we obtain :

$$\frac{1}{2}\alpha' M^2 = (N^{\perp} - a) + (\overline{N}^{\perp} - \overline{a}) , \tag{3.52a}$$

$$N^{\perp} - a = \overline{N}^{\perp} - \overline{a} , \tag{3.52b}$$

with an obvious meaning for the notation (as in (3.47)). Here again, the level matching condition (3.52b) follows from the fact that the light-cone gauge fixing conditions do not fix the origin in σ parametrization for closed strings.

Due to the necessary normal ordering of composite operators, the Lorentz algebra may not be realized at the quantum level due to quantum anomalies. Here again, only the commutator $[M^{-i}, M^{-j}]$ needs to be checked, and can be shown to vanish only if we have[60] :

$$D = 10 , \quad a_+ = 0 , \quad a_- = 1/2 . \tag{3.53}$$

The values for the subtraction constants are easily understood[40]. For definiteness, let us consider the open string. In the Ramond sector, the ground state $|u(p)\rangle$ has $2^{[D-2]/2}$ components and a mass $\alpha' m^2 = -a_+$. This state *may* transform covariantly under the full Lorentz group only if it is massless, i.e. only if $a_+ = 0$. In the NS sector, the same argument applies to the first excitation level $b^i_{-1/2}|\Omega; p\rangle$, which has $(D-2)$ components and a mass $\alpha' m^2 = 1/2 - a_-$. We must have $a_- = 1/2$, thus also implying that the physical ground state is a space-time scalar tachyon.

If we now use our favorite[40] ζ-function regularization for the evaluation of a_+ and a_-, we obtain :

$$-a_+ = (D-2)\left[\sum_{n=1}^{\infty}\frac{1}{2}n - \sum_{n=1}^{\infty}\frac{1}{2}n\right] = \frac{(D-2)}{2}[\zeta(-1) - \zeta(-1)] = 0, \tag{3.54}$$

$$-a_- = (D-2)\left[\sum_{n=1}^{\infty}\frac{1}{2}n - \sum_{n=1/2}^{\infty}\frac{1}{2}r\right] = \frac{(D-2)}{2}[\zeta(-1) - \frac{-1}{2}\zeta(-1)] = -\frac{D-2}{16}. \tag{3.55}$$

In each of these expressions, the first term is the contribution of the bosonic zero-point fluctuations, and the second term that of the fermionic ones. They cancel in the R sector but not in the NS sector, due to the global $N = 1$ supersymmetry which is preserved by R boundary conditions but broken by NS boundary conditions. This breakdown of supersymmetry is responsible for the appearance of a space-time tachyon in the NS sector. If we now set $a_- = 1/2$, we find indeed $D = 10$ from (3.55).

The reader is invited to check that all states of strictly positive norm obtained in the superconformal gauge, both for open and for closed strings, are indeed recovered in the light-cone gauge, and that all massive states indeed transform as irreducible representations of the covering group Spin(9) of $SO(D-1) = SO(9)$. Note also that the same types of "degeneracies" of Regge "trajectories" as in the covariant approach appear in the light-cone gauge.

3.1.3 GSO projection and space-time supersymmetry

The physical space-time spectrum of spinning strings contains bosonic and fermionic states. Actually, we have to think of the complete spectrum of these theories as the direct sum of the spectra in the different sectors.

We saw however that in the NS and (N,NS) sectors, one obtains space-time scalar tachyons, causing problems for unitarity. Moreover, the massless "spin 3/2" states also lead to problems, unless they couple to a conserved current of "spin 1/2", i.e. to a space-time supersymmetry current, in the same way that gravitons couple to a conserved energy-momentum tensor. Gliozzi, Olive and Scherk (GSO)[54] were the first to realize that by taking advantage of the double "degeneracy" of Regge "trajectories", there exists a consistent truncation of the spectrum of the theory, in which these problems are avoided, namely such that :

- tachyons are no longer physical states,
- the numbers of space-time bosons and fermions are equal at each mass level[54],
- the spectrum is space-time supersymmetric[61].

The corresponding truncation is called the GSO projection. In the next chapter, we shall find a more geometrical understanding of it, from the point of view of two dimensional reparametrization invariance. Actually, it is the only projection making the theory quantum consistent to all orders of perturbation theory[55].

To discuss the GSO projection, let us first consider the open spinning string. The quantity which distinguishes between the "degenerate" Regge "trajectories" is essentially the fermion number F, given in the R sector by

$$F = \sum_{n=1}^{\infty} d^{\mu}_{-n} d_{n\mu}, \tag{3.56}$$

and in the NS sector by

$$F = \sum_{r=1/2}^{\infty} b^{\mu}_{-r} b_{r\mu}. \tag{3.57}$$

Of course, the same definitions apply in the light-cone gauge, where one then sums only over transverse components.

Finally, let us introduce in the R sector the string chirality operator

$$\Gamma = \gamma_{11}(-1)^F, \tag{3.58}$$

where γ_{11} is the space-time chirality operator

$$\gamma_{11} = \gamma^{11} = \gamma^0\gamma^1\ldots\gamma^9. \tag{3.59}$$

(In the light-cone gauge, we would have $\Gamma = \gamma_9(-1)^F$ with $\gamma_9 = \gamma^1\ldots\gamma^8$).

GSO projection of the open spinning string is then (not yet completely) defined by the conditions[54] :

- R sector : $\Gamma|\psi\rangle = +|\psi\rangle$, (or $\Gamma = -1$), (3.60a)
- NS sector : $(-1)^F|\psi\rangle = -|\psi\rangle$. (3.60b)

The latter condition projects out in the NS sector all states having an half-integer value for $\alpha' m^2$, thus including the tachyon. Only states with integer values for $\alpha' m^2$ are retained. The physical ground state is then the massless vector state with 8 components. At the next excitation level with $\alpha' m^2 = 1$, we have the positive norm states among $b^{\mu}_{-3/2}|\Omega; p\rangle$, $b^{\mu}_{-1/2}b^{\nu}_{-1/2}b^{\rho}_{-1/2}|\Omega; p\rangle$ and $\alpha^{\mu}_{-1}b^{\nu}_{-1/2}|\Omega; p\rangle$ leading to the $SO(9)$ representations $(\boxed{}+\boxed{})_{SO(9)}$, and having $(84 + 44 = 128)$ components respectively.

In the R sector, the ground state spinor $|u(p)\rangle$ decomposes into right- and left-handed components $u_{\pm}(p) = \frac{1}{2}(1 \pm \gamma_{11})u(p)$. The GSO projections retain states at all integer values of $\alpha' m^2$, but only half the original number. At the massless level, the condition $\Gamma = +1$ (resp. $\Gamma = -1$) implies that the physical ground-state of the truncated theory is a right-handed (resp. left-handed) massless Weyl spinor, with 16 real components.

A Majorana condition on $u(p)$ would reduce this number further to 8, i.e. the same number of components as in the bosonic sector, which is a necessary condition for having space-time supersymmetry. The remarkable fact is precisely that Majorana-Weyl spinors exist in Minkowski space-times only of dimension $D = 2(\text{mod } 8)$[54], which in particular includes $D = 10$. Hence, the complete GSO*projectionconditionsaredefinedby*(3.60)*togeth*

In the R sector, the physical ground-state is then a right-handed massless Majorana-Weyl spinor, with 8 physical components. At the excitation level $\alpha' m^2 = 1$, we have the

states $\alpha^{\mu}_{-1}|u_+(p)\rangle$ and $d^{\mu}_{-1}|u_-(p)\rangle$, combining into a massive Majorana Rarita-Schwinger "spin 3/2" state, with 128 components.

That the number of bosonic and fermionic physical states (of positive norm) is the same at each mass level may best be seen by computing the corresponding degeneracy generating functions[54]. In the R sector, we have :

$$f_R(q) = \sum_{n=0}^{\infty} d_R(n) q^{2\alpha' m_n^2} = \sum_{n=0}^{\infty} d_R(n) q^{2n} = d\,Tr\, q^{2N_R^{\perp}} \tag{3.61a}$$

$$= d \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1-q^{2n}} \right)^8 \tag{3.61b}$$

$$= d[1 + 16q^2 + 144q^4 + 960q^6 + \ldots], \tag{3.61c}$$

where q is a complex number such that $|q| < 1$, d_n is the degeneracy at excitation level $N_R^{\perp} = n$, d is the number of physical components of the ground-state spinor $u(p)$ ($d = 8$ for a Majorana-Weyl spinor on-shell), and the trace in (3.61a) is taken over all non-zero bosonic and fermionic modes.

In the NS sector, let us distinguish the states with an even and an odd number of fermionic excitations. We then have :

$$f_{NS}^{\pm} = Tr \frac{1}{2}(1 \pm (-1)^F) q^{2[N_{NS}^{\perp} - 1/2]} \tag{3.62a}$$

$$= \frac{1}{2q} \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})^8} \left\{ \prod_{n=1}^{\infty} (1+q^{2n-1})^8 \pm \prod_{n=1}^{\infty} (1-q^{2n-1})^8 \right\}, \tag{3.62b}$$

leading to the series expansions :

$$f_{NS}^{+}(q) = \frac{1}{q}[1 + 36q^2 + 402q^4 + 1964q^6 + \ldots], \tag{3.63a}$$

$$f_{NS}^{-}(q) = 8[1 + 16q^2 + 144q^4 + 960q^6 + \ldots]. \tag{3.63b}$$

Comparing (3.61) and (3.63), it becomes clear that only the GSO projection as defined above may lead to a space-time supersymmetric spectrum. To show the equality of $f_R(q)$ and $f_{NS}^{-}(q)$, it is useful to express these functions in terms of Jacobi theta functions defined as follows :

$$\theta_1(z|\tau) = i \sum_n (-1)^n q^{(n-1/2)^2} e^{2i\pi(n-1/2)z}$$

$$= 2q^{1/4} \sin \pi z \prod_{n=1}^{\infty} (1-q^{2n}) \prod_{n=1}^{\infty} (1-q^{2n}e^{2i\pi z})(1-q^{2n}e^{-2i\pi z}), \tag{3.64a}$$

$$\theta_2(z|\tau) = \sum_n q^{(n-1/2)^2} e^{2i\pi(n-1/2)z}$$

$$= 2q^{1/4} \cos \pi z \prod_{n=1}^{\infty} (1-q^{2n}) \prod_{n=1}^{\infty} (1+q^{2n}e^{2i\pi z})(1+q^{2n}e^{-2i\pi z}), \tag{3.64b}$$

$$\theta_3(z|\tau) = \sum_n q^{n^2} e^{2i\pi n z} = \prod_{n=1}^{\infty} (1-q^{2n}) \prod_{n=1}^{\infty} (1+q^{2n-1}e^{2i\pi z})(1+q^{2n-1}e^{-2i\pi z}), \tag{3.64c}$$

$$\theta_4(z|\tau) = \sum_n (-1)^n q^{n^2} e^{2i\pi n z} = \prod_{n=1}^{\infty} (1-q^{2n}) \prod_{n=1}^{\infty} (1-q^{2n-1}e^{2i\pi z})(1-q^{2n-1}e^{-2i\pi z}), \tag{3.64d}$$

where as before we set $q = e^{i\pi\tau}$. In the following, we shall only need the values $\theta_i(z = 0|\tau)$ ($i = 1,2,3,4$), which will be denoted as θ_i ($i = 1,2,3,4$). Then, one finds:

$$f_R(q) = \frac{1}{16} d \frac{\theta_2^4}{\eta^{12}}, \tag{3.65a}$$

$$f_{NS}^{\pm}(q) = \frac{1}{2} \frac{1}{\eta^{12}} [\theta_3^4 \pm \theta_4^4], \tag{3.65b}$$

where $\eta(\tau)$ is the Dedekind η-function defined in (2.111).

That the GSO projection leads indeed to a space-time spectrum with the same number of bosons and fermions at each mass level now follows[54] from the Riemann identity ($\theta_1 = 0$) :

$$\theta_1^4 + \theta_3^4 = \theta_2^4 + \theta_4^4. \tag{3.66}$$

Moreover, it has been shown[61] that the spectrum then transforms under a $D = 10$ $N = 1$ supersymmetry algebra, with in particular the massless states falling into a $D = 10$ $N = 1$ Yang-Mills supermultiplet, i.e. a massless vector and a massless Majorana-Weyl spinor. From that point of view, the Riemann identity (3.66) is an expression of space-time supersymmetry.

For closed spinning strings, GSO projections are defined in a similar way[17]. Here however, we have two choices in the relative chirality of the two Majorana spinors in the two moving sectors, thus leading to two different theories. The first is the Type IIa fermionic string, defined by

- $u(p), \bar{u}(p)$: Majorana spinors
- $\Gamma = +1$ $\bar{\Gamma} = -1$ (or vice-versa) (3.67)
- $(-1)^F = -1$, $(-1)^{\bar{F}} = -1$

and the second is the Type IIb fermionic string, defined by

- $u(p), \bar{u}(p)$: Majorana spinors
- $\Gamma = +1$, $\bar{\Gamma} = +1$ (or $\Gamma = \bar{\Gamma} = -1$) (3.68)
- $(-1)^F = -1$, $(-1)^{\bar{F}} = -1$

Here, as usual, quantities with and without a bar correspond respectively to the left- and right-moving sectors.

It should be clear that in both cases, the numbers of space-time fermions and bosons are equal at each mass level, and are given by the square of the corresponding degeneracy numbers of the open spinning string. Moreover, in each theory, the space-time spectrum transforms under a space-time supersymmetry. For the Type IIa theory, this is the non-chiral $D = 10$ $N = 2a$ supersymmetry, and for the Type IIb theory, it is the chiral $D = 10$, $N = 2b$ supersymmetry, with the massless states falling into the corresponding graviton supermultiplets. In both cases, the (NS,NS) sector leads to the massless graviton, 2-index antisymmetric tensor and dilaton, transforming as $(\square + \square + \bullet)_{SO(8)}$ and having respectively $(35 + 28 + 1 = 64)$ components. In the (R,R) sector, since the two Majorana spinors $u(p)$ and $\bar{u}(p)$ have different relative chiralities in each theory, we have different bosonic states. For the Type IIa theory, we have the states $(\square + \square)_{SO(8)}$ with $(8 + 56 = 64)$ components respectively. For the Type IIb theory, we have the states $(\bullet + \square + \square \pm)_{SO(8)}$ with $(1 + 28 + 35 = 64)$ components respectively, where $\square_{\pm}$ is a self-dual or anti-selfdual tensor depending on the chirality of

the spinors $u(p)$ and $\bar{u}(p)$. From the (R,NS) and (NS,R) sectors we obtain the massless fermions. In the type IIa theory, we have a Majorana spinor and a Majorana Rarita-Schwinger "spin 3/2" gravitino, none being Weyl fermions. In the Type IIb theory, we have a left-handed Weyl spinor and a right-handed Rarita-Schwinger "spin 3/2" gravitino, none being a Majorana fermion. In both cases, the number of components are $2(8+56) = 128$, respectively. Clearly, the spectrum of the Type IIa theory is non-chiral, whereas that of the Type IIb theory is indeed chiral.

3.2 BRST quantization

So far in our discussion, we have not taken into account the necessary ghost sector associated to superconformal gauge fixing. In principle, to be faithfull to the Hamiltonian approach adopted in these lectures, we should again use the BFV Hamiltonian formalism for the BRST quantization of the spinning string.

However, as long as one is not primarily interested in that aspect of the theory, such an analysis can be avoided by taking advantage[19] of the explicit world-sheet global $N = 1$ supersymmetry remaining after superconformal gauge fixing. We simply need to find out how to supersymmetrize the BRST quantization of the bosonic string, which is most efficiently done by means of world-sheet superfields[16,19].

Indeed, matter fields (x^μ, λ^μ) correspond to such superfields, and the ghost sector now follows from a ghost superfield[16,19]. The fermionic components of the ghost superfield are the (b, c) ghost system associated to the bosonic constraints of world-sheet reparametrization invariance. Its bosonic components are the ghost system associated to the fermionic constraints of world-sheet local supersymmetry, the so-called (β, γ) system of superghosts.

Since we have in the superconformal gauge, the action (3.9) for the matter fields, and the part of the BRST invariant action associated to the bosonic constraints and its ghost system (see (2.141)), it is rather easy to guess[19] what the BRST invariant action and the BRST charge for the spinning string should be in the superconformal gauge. By lack of space, and since we do not really need the details of BRST quantization in the remainder of these notes, we shall not develop these considerations any further, despite the fact that the superghost system plays an important role in string theory[16], and is essential in one general approach to string theory constructions[62,63]. The interested reader is invited to apply the approach sketched above and work out for himself the BRST quantization of the spinning string in the superconformal gauge. Details can be found in Refs. 15, 16, 19.

For later purposes, it suffices here to state some of the results following from such an analysis. As for the bosonic string, superconformal transformations of the spinning string in this BRST formulation are generated by total super-Virasoro generators $L_n^{(T)}$ and $J_q^{(T)}$. Now, they not only include the contributions $L_n^{(X)}$ and $J_q^{(X)}$ of the matter fields considered previously, but they also include the total contributions $L_n^{(B)}$ and $J_q^{(B)}$ of the ghost sector. In particular, the total ghost Virasoro generators $L_n^{(B)}$ are given by the sum of the (b, c) contribution $L_n^{(c)}$ obtained for the bosonic string, and a contribution $L_n^{(\gamma)}$ from the (β, γ) superghost system.

The moding of the ghost supercurrent modes $J_q^{(B)}$ is the same as that of the fermions, namely integer for a Ramond sector and half-integer for a Neveu-Schwarz sector. Indeed, the matter supercurrent J_α in (3.20) satisfies the same boundary conditions as the matter fermions, hence so do the superghosts (β, γ) associated to the gauge invariance generated by J_α, and the ghost supercurrent coupling to them. Hence, the contribution

of the superghost system to the central extension of the total Virasoro algebra depends on the fermionic boundary condition. For the Virasoro algebra generated by $L_n^{(\gamma)}$, one finds[15,19] :

$$\bullet\ R \text{ sector}: \qquad c(n) = 11n^3 - 2n\,, \tag{3.69}$$

$$\bullet\ NS \text{ sector}: \qquad c(n) = 11n^3 + n\,. \tag{3.70}$$

If we now add up all contributions to the total central extension of the total Virasoro algebra, which, as for the bosonic string, includes the subtraction constants a_+ and a_- following from normal ordering, one obtains[15,19] :

$$\bullet\ R \text{ sector}: \qquad c(n) = \frac{3}{2}(D-10)n^3 + 24a_+ n\,, \tag{3.71}$$

$$\bullet\ NS \text{ sector}: \qquad c(n) = \frac{3}{2}(D-10)n^3 + \left[\frac{3}{2}(2-D) + 24a_-\right] n\,. \tag{3.72}$$

Hence, conformal invariance, and thus reparametrization invariance, is maintained at the quantum level only if we have the same critical values as obtained previously from different considerations

$$D = 10\,,\ \ a_+ = 0\,,\ \ a_- = \frac{1}{2}\,. \tag{3.73}$$

Recall that within the present quantization approach, super-Weyl invariant is explicit, since the superconformal modes of the world-sheet graviton and gravitino decouple. In Polyakov's path-integral approach[22], super-reparametrization invariance is explicitly preserved, but super-Weyl invariance is only realized if we have the conditions (3.73)[22]. Otherwise, the superconformal modes are dynamical fields.

Actually, with the critical values (3.73), not only does the central extension of the total Virasoro algebra vanish, but so do all the central extensions of the total $N = 1$ super-Virasoro algebra generated by $L_n^{(T)}$ and $J_q^{(T)}$[15,19]. Moreover, the total quantum BRST charge is then also nilpotent[64], thereby ensuring gauge invariance, namely super-reparametrization invariance, of the quantum spinning string.

That all these considerations are related and all lead to the critical values (3.73) is not surprising, considering the fact that the super-Virasoro generators $L_n^{(T)}$ and $J_q^{(T)}$ are given by anticommutators and commutators of the b and β ghosts with the BRST charge, as was the case also for the bosonic string. This is the reason why it is enough to consider for example, the total central extension of the Virasoro algebra to determine the critical values for gauge invariance.

As a last comment, let us say that physical states are again defined as BRST invariant states of definite ghost charge. Indeed, as for the bosonic string, the non-trivial BRST invariant cohomology classes of the BRST charge, both in Ramond and Neveu-Schwarz sectors, are in 1-to-1 correspondence with positive norm physical states in the old covariant or light-cone gauge quantization, with however a degeneracy due to ghost and superghost zero-modes[64,65]. In the NS sectors, this degeneracy is two-fold since only the (b, c) system has zero-modes. But in the R sector, this two-fold degeneracy is itself infinitely degenerate[16] due to the (β, γ) bosonic zero-modes. Hence the necessary restriction on the ghost number, to obtain a 1-to-1 correspondence with the physical content of the theory.

4 THE PRINCIPLES OF STRING THEORY CONSTRUCTION

Now that we have described and understood the structure, the origin and the meaning of the restrictions which appear for bosonic and spinning strings, we may actually infer a general description of the requirements that any string theory should meet. So far, we have concentrated on the restrictions following from local gauge invariance, i.e. invariance under infinitesimal gauge transformations. As we shall discuss, additional requirement also follow from global gauge invariance, i.e. invariance under gauge transformations not connected to the identity transformation.

The formulation of the basic rules for string theory construction that we are about to present, actually grew out from the construction of ten dimensional heterotic strings with space-time supersymmetry[66], and their toroïdal compactifications[67].

4.1 Quantum Consistency Constraints

4.1.1 Local world-sheet invariances

As is now clear, string theories are two dimensional field theories of quantum gravity coupled to matter fields, which are reparametrization invariant in the world-sheet. The constraints which follow from this gauge invariance are the vanishing of the generators of local diffeomorphisms, i.e. local world-sheet reparametrizations connected to the identity.

This world-sheet gauge invariance may be extended to some larger symmetry, including reparametrization invariance, as is the case for example for the spining string where we have world-sheet super-reparametrization invariance under a local $N = 1$ supersymmetry. Other extensions are also possible.

Such local world-sheet symmetries are essential in obtaining a consistent quantum string theory. In the conformal gauge, the existence of such symmetries at the quantum level guarantees that negative and zero-norm states decouple from physical states and amplitudes, as is necessary for unitarity (actually, we only established this property at the level of the physical spectrum, through the no-ghost theorem, but it may be shown to be also true for tree level amplitudes[18]). In the light-cone gauge, the existence of local world-sheet symmetries is essential for the space-time Poincaré covariance of the space of states.

Moreover, *world-sheet* local gauge invariances seem to be related in a very profound and natural way to *space-time* local symmetries. Indeed, string spectra always contain the associated massless gauge bosons, such as the graviton, Yang-Mills bosons and gravitinos, and the corresponding quantum field theories are known to be consistent only if such massless particles couple to the associated conserved space-time currents.

Hence, it is vital for quantum consistency of a string theory that its local world-sheet symmetries be preserved at the quantum level. As we know, this leads to restrictions on the number of world-sheet degrees of freedom, since the associated quantum anomalies must vanish.

Gauge invariances of string theories require gauge-fixing, with the ensuing ghost sectors. In the conformal gauge, as was discussed in detail, the ghost sector associated to world-sheet reparametrization invariance, which is a symmetry for any string theory, is the (b, c) ghost system. Its contribution to the central extension of the total Virasoro algebra is

$$c(n) = -26n^3 + 2n. \tag{4.1}$$

If in addition we have local $N = 1$ supersymmetry in the world-sheet, the corresponding (β, γ) superghost system contributes the following values to the total central extension.

$$\text{R sector: } c(n) = +11n^2 - 2n, \tag{4.2}$$

$$\text{NS sector: } c(n) = +11n^3 + n. \tag{4.3}$$

Similarly, for any other local world-sheet gauge invariance of the theory, one must take into account the corresponding contributions of the associated ghost sector to the central extension in the conformal gauge.

In this gauge, the remaining world-sheet degrees of freedom, namely the matter fields which couple to the two dimensional quantum gravity sector, define a conformal field theory, with the corresponding Virasoro algebra and central extension. Local reparametrization invariance at the quantum level, requires that the total central extension of the total Virasoro algebra, which includes the subtraction constant due to normal ordering, and the contributions from all the ghost sectors, vanishes identically. Clearly, this determines the value of the total central charge for the matter fields, and the value of the normal ordering constant. The same values should ensure that any other local world-sheet symmetries of the system are then also realized at the quantum level. This may be checked by computing the central extensions of the full symmetry algebra in the conformal gauge, or equivalently, by checking that the corresponding BRST charge is indeed nilpotent.

In order to describe a string theory in a D dimensional flat Minkowski space-time, we must have as world-sheet matter fields D scalar fields $x^\mu(\sigma, \tau)$, giving the space-time position of the string. In the conformal gauge, each of these scalar fields contributes to the central extension of the Virasoro algebra by

$$c(n) = n^3 - n. \tag{4.4}$$

For string theories with space-time fermions in the NSR formulation, we need a world-sheet local $N = 1$ supersymmetry such that the D scalar fields have as superpartners D world-sheet Majorana-Weyl spinors $\lambda^\mu_\pm(\sigma, \tau)$. In the conformal gauge, each of these spinors contributes as follows to the central extension of the Virasoro algebra:

$$\text{R sector: } \quad c(n) = \frac{1}{2}n^3 + n, \tag{4.5}$$

$$\text{NS sector: } \quad c(n) = \frac{1}{2}n^3 - \frac{1}{2}n. \tag{4.6}$$

The remaining world-sheet degrees of freedom, if any, are then considered as internal degrees of freedom, which in the conformal gauge define a conformal field theory contributing to the Virasoro algebra with the corresponding central extension:

$$c(n) = c_{int}n^3 + \beta_{int}n. \tag{4.7}$$

Hence, for a theory having only world-sheet reparametrization invariance, so called $N = 0$ supersymmetry, quantum gauge invariance requires that the central charge of the internal conformal field theory and the subtraction constant in the total Virasoro algebra be given by:

$$c_{int} = 26 - D, \tag{4.8}$$

$$a = \frac{1}{24}[(D-2) - \beta_{int}]. \tag{4.9}$$

For a theory having $N = 1$ world-sheet supersymmetry, the corresponding values are:

$$c_{int} = \frac{3}{2}(10 - D), \tag{4.10}$$

$$\text{R sector}: \quad a_+ = \frac{1}{24}[-\beta^+_{int}], \tag{4.11}$$

$$\text{NS sector}: \quad a_- = \frac{1}{24}[\frac{3}{2}(D-2) - \beta^-_{int}]. \tag{4.12}$$

Clearly, the value for the subtraction constant determines the lowest mass values of the physical spectrum. We have the following contributions to this normal ordering constant: $+\frac{1}{24}$ from a scalar degree of freedom, $-\frac{1}{24}$ from a Majorana-Weyl spinor in the Ramond sector and $+\frac{1}{48}$ from a Majorana-Weyl spinor in the Neveu-Schwarz sector. These are also the values given by ζ-function regularization of the corresponding infinite series. Note again the breakdown of $N = 1$ supersymmetry for NS boundary conditions.

The same restrictions on c_{int} and the subtraction constant may be obtained in the light-cone gauge[68]. Indeed, the transverse string coordinates together with the internal degrees of freedom define a conformal field theory, with a central extension such that the full Lorentz algebra is realized at the quantum level, i.e. such that the commutator $[M^{-i}, M^{-j}]$ vanishes identically. For example, the total central charge of this conformal field theory must be $c = 24$ for a string theory with $N = 0$ supersymmetry[30], and $c = 12$ for a theory with $N = 1$ supersymmetry[60]. These restrictions then lead again to (4.8) and (4.10). The same applies to the normal ordering constants, which however may also be determined by considering the would-be massless physical states, as was explained for bosonic and spinning strings[40].

With the string theories discussed in the previous two chapters, we have a situation where there are no internal degrees of freedom. From (4.8) to (4.12), we then recover of course the same critical values. Historically, the values $D = 26$ and $D = 10$ were believed to correspond to critical values of the dimension of space-time for which these theories could be consistently quantized. Now, they are seen to correspond to upper bounds on the dimension of space-time. Indeed, in order that the internal conformal field theory be unitary, we must have[9,69] $c_{int} \geq 0$.

However, in the present formulation of string theory, there is no dynamical argument which could single out any particular value of D. String theories may be constructed for space-time dimensions ranging from 26 to 2. In particular, the case $D = 4$ requires $c_{int} = 22$ for $N = 0$ supersymmetry, and $c_{int} = 9$ for $N = 1$ supersymmetry. If the classification of the corresponding conformal field theories were known, one could then give the complete list of all possible four-dimensional string theories.

From the present point of view, the problem of decoupling of conformal modes of the two dimensional gravity sector becomes somewhat academic. Indeed, in the conformal gauge, when these modes are dynamical fields, they simply correspond to some conformal field theory, and as such they may be regarded as some internal degrees of freedom. The study of the corresponding conformal field theory may be nonetheless important.[27,28]

The reader may have been wondering why we restricted all our attention to $N = 0$ and $N = 1$ world-sheet supersymmetry, and did not consider local $N \geq 2$ supersymmetries. The reason is that the corresponding possible[70] dimensions of space-time are $D \leq 2$, which does not leave much room for describing strings vibrating in transverse space dimensions! Nevertheless, the corresponding conformal structures find important applications elsewhere[9], and also in string theory as we shall comment upon later.

In conclusion, the requirements discussed in this section determine the necessary and sufficient conditions for maintaining , at the quantum level, the local world-sheet gauge invariances of string theory, namely local reparametrization invariance and its extensions. They are also necessary, but not necessarily sufficient as we shall discuss in the next section, for full quantum consistency of the theory.

The corresponding world-sheet structures can be used in the construction of open and of closed string theories. For closed strings however, we have the additional feature that different such structures may be used in each moving sector. Except for the space-time bosonic zero-modes q^μ and P^μ giving the space-time position of the center-of-mass and the total space-time momentum of the string, all other modes indeed separate into right-and left-moving modes, including the internal degrees of freedom. Hence, we have the following broad classification of closed string theories[7], depending on the number of world-sheet supersymmetries in each moving sector:

	sector		
string theory	right	left	dimension
bosonic	N=0	N=0	$D \leq 26$
heterotic	N=0	N=1	$D \leq 10$
Type II	N=1	N=1	$D \leq 10$

4.1.2 Global world-sheet invariances

Global world-sheet reparametrizations, which are not continuously connected to the identity transformation, but should also be symmetries of the quantum string theory, fall into two classes: orientation preserving and orientation reversing diffeomorphisms. If only the former symmetries are imposed on the system, one is describing an oriented string theory. Otherwise, we have an unoriented string theory. Let us first discuss this latter situation.

In the case of a closed string, an orientation reversing reparametrization, such as $\sigma \to \pi - \sigma$, clearly exchanges the right-and left-moving sectors. Hence, all states which are symmetric under this transformation remain physical for unoriented closed strings. For example, the massless 2-index antisymmetric tensor is always projected out from the spectrum.

For open strings, the orientation reversing diffeomorphism ($\sigma \to \pi - \sigma$) clearly changes the phases of bosonic and fermionic modes. These phase factors may be compensated for by coupling open strings to gauge degrees of freedom, by introducing Chan-Paton factor[71]. These Chan-Paton factors may intuitively be thought to correspond to charges attached at the end points of the open string. This was originally suggested by Chan and Paton for $U(n)$ groups[71], by viewing these charges as a quark and an anti-quark in the fundamental representation of $U(n)$.

Actually, this idea is easily generalized[72] by associating to each state of the open string some representation of some algebra. However, this may not be done arbitrarily, since the correct factorization of tree-level scattering amplitudes must be obtained[72]. Indeed, the gauge quantum numbers of intermediate states must be such that they may couple to those of the external states. This tree-level consistency requirement may be solved[72,73], with the following result:

- only the Lie algebras associated to $U(n), SO(n)$ and $USp(n)$ are allowed,
- all mass levels even in $\alpha' m^2$ must be in the adjoint representation,

- all mass levels odd in $\alpha' m^2$ must be in the following representations:
  - $U(n)$: adjoint representation
  - $SO(n)$: $\square\square + \bullet$
  - $USp(n)$: $\begin{smallmatrix}\square\\\square\end{smallmatrix} + \bullet$

Moreover, $U(n)$ gauge degrees of freedom may only be introduced for oriented open strings, and $SO(n)$ or $USp(n)$ ones for unoriented open strings (this result agrees with one's intuition).

When considering string interactions, where open strings interact at their end points, it is clear that both open and closed string sectors of a same theory must be included in the description of the spectrum. Indeed, a single open string may interact at its own end points to form a closed string, or a closed string may break to form an open string. Hence, the possible situations are:

- open and closed oriented strings,
- open and closed unoriented strings,
- closed oriented strings,
- closed unoriented strings.

Therefore, there exist the following $D = 26$ bosonic theories:

- open and closed oriented strings, with gauge group $U(n)$,
- open and closed unoriented strings, with gauge group $SO(n)$ or $USp(n)$,
- closed oriented strings,
- closed unoriented strings.

The first two classes correspond to the Veneziano Model[74], the third to the Extended Shapiro-Virasoro Model[75] and the fourth to the Restricted Shapiro-Virasoro Model[75]. All these theories have space-time scalar tachyons, a feature incompatible with unitarity.

Since the open spinning string has only one space-time supersymmetry, we have the following $D = 10$ supersymmetric string theories[72]:

- Type I open and closed unoriented superstring, with gauge group $SO(n)$ or $USp(n)$,
- Type I closed unoriented superstring,
- Type IIa closed oriented superstring,
- Type IIb closed oriented superstring.

The closed string sector of the first two classes is obtained by truncation of the the type IIb superstring under the orientation reversing diffeomorphism symmetry. Indeed, the space-time supercharges associated to the two moving sectors must have the same chirality. Hence, the first two cases have $N = 1, D = 10$ space-time supersymmetry, whereas the last two cases have $N = 2a$ or $N = 2b, D = 10$ supersymmetry.

These are thus the restrictions following from the requirement of quantum invariance under orientation reversing world-sheet diffeomorphisms, by considering quantum consistency at tree-level. By considering higher loop amplitudes however, further restrictions are imposed.

In the case of open strings, this is well known[5]. Indeed, when considering 1-loop amplitudes with external open string states of zero-norm, such amplitudes must vanish identically, i.e. all null states must decouple. In the case of Type I superstrings, this question has been investigated for external massless gauge bosons[5] and gravitons[76], with the conclusion that such a decoupling occurs only for the gauge group $SO(32) = SO(2^{10/2})$. Although such a decoupling has not been established yet for all other null states, including massless and massive fermions, it will most probably also occur for the same gauge group only.

In the case of open bosonic strings, the same question has been investigated for arbitrary external open string states, with the conclusion[38,77] that such a decoupling can never occur because of the open string tachyon, but that otherwise, except for this non-vanishing tachyon contribution, decoupling would require the gauge group $SO(2^{26/2})$[78], independently of any regularization[38,77].

Hence, higher loop unitarity restricts open string theories to those of unoriented open and closed strings with gauge group $SO(2^{D/2})$, up to possible tachyon contributions[38,77], in which case the theory is ill-defined anyway.

In the case of unoriented closed strings, not much is known along these lines. In the case of oriented closed strings however, it is widely believed that modular invariance (which is discussed below) and the absence of tachyons are sufficient conditions to guarantee decoupling of all null states. This has been established explicitly for massless bosonic states in the case of superstrings, and from the point of view of the corresponding low-energy effective field theory[63]. General arguments support this conjecture.[63]

It so happens that the $SO(2^{D/2})$ unoriented open string theories can be obtained from oriented closed string theories by a $\mathbb{Z}_2$-orbifold construction in the *world-sheet*[79]. Thus, this leads to the idea[38,77] that all consistent open string theories are unoriented string theories which are obtained through such a construction from an oriented closed string theory. Moreover, quantum consistency of these open string theories, with in particular decoupling of all null states to all orders of perturbation theory, would then simply follow from modular invariance to all orders of the corresponding oriented closed string theory[38,77,79]. Since a $\mathbb{Z}_2$-orbifold structure may be defined in two ways, one may believe that one of these structures would lead to a theory of unoriented open and closed strings, and the other structure to a theory of unoriented closed strings only. In addition, more complicated theories may possibly be obtained through other abelian (and non-abelian?) orbifold structures on the world-sheet, thus corresponding to theories of strings with more than two end points[38,77].

Hence, this opens the way to the construction of new open (and more complicated?) string theories in less than 26 or 10 dimensions, including 4 dimensions[80], by considering any consistent closed oriented string theory. This is certainly an interesting possibility, in particular with regards to the possible gauge symmetries. Much however remains to be done in this direction.

In the remainder of these lectures, we shall concentrate on the construction of oriented closed strings, i.e. theories for which the world-sheet has a well-defined orientation. In string perturbation theory, scattering amplitudes are obtained by considering a world-sheet corresponding to a surface of given topology, to which the external states are attached[11]. The topology of the surface is characterized by its genus g, which counts

the number of holes in the surface, or the number of handles attached on the sphere (of genus g=0) to obtain the surface. In the case of oriented closed strings, the space of global diffeomorphisms is non-trivial when the world-sheet has a topology more complex than that of the sphere, namely when its genus is greater than zero, $g > 0$.

Let us introduce some terminology[11]. $Diff_0$ denotes the space of all world-sheet diffeomorphisms continuously connected to the identity transformations; they thus preserve the orientation. $Diff$ denotes the space of all orientation preserving diffeomorphisms. The quotient $Diff/Diff_0 = \Omega_g$ then forms a group, the group of all disconnected components of $Diff$, known as the modular (or mapping class) group. This group is defined for each genus g separately.

By imposing the constraints discussed in the previous section, we are assured that $Diff_0$ is a symmetry of the quantized theory. However, requiring that string theory is fully reparametrization invariant still imposes the additional constraint of modular invariance under Ω_g to all orders in the loop expansion.

Actually, it is enough to impose this invariance for vacuum amplitudes only. Modular invariance for scattering amplitudes should then follow by factorization. Moreover, it seems to be a fact that modular invariance of the 1-loop vacuum amplitude and the correct spin-statistics relation (namely, that space-time bosons and fermions contribute with opposite signs in loops) are sufficient conditions for modular invariance to all orders[55,81].

For this reason, we shall restrict the discussion to 1-loop vacuum amplitudes, or partition functions, for closed oriented string theories. The corresponding world-sheet has genus $g = 1$; this is the topology of a torus. By using local diffeomorphisms and Weyl rescalings, it is always possible to map this torus into a parallelogram, with a flat metric, in the upper-half complex plane $\mathbb{C}^+$, with opposite sides identified, and defined by two vectors attached at the origin and ending at 1 and $\tau = \tau_1 + i\tau_2 (Im\tau_2 > 0)$ respectively (the parameter τ should not be confused with the world-sheet coordinate τ). There is thus a torus associated to every point in $\mathbb{C}^+$. The opposite however, is not true. A same torus may correspond to different points in $\mathbb{C}^+$. Indeed, modular transformations leave the torus invariant but act non-trivially on $\mathbb{C}^+$.

The modular group of a given compact Riemann surface is generated by so-called Dehn twists[82,83]. For the genus $g = 1$ case, there are two such Dehn twists, obtained as follows. For each of the two homology cycles of the torus, cut the surface along this cycle, twist it by 2π and glue it back together. Clearly, this is a global diffeomorphism, leaving the torus invariant, but not the associated parameter τ. Indeed, cutting along the cycle corresponding to σ-parametrization, we have the Dehn twist acting on τ as:

$$T : \tau \to \tau + 1. \tag{4.13}$$

For the Dehn twist along the other cycle, we have

$$S : \tau \to -1/\tau. \tag{4.14}$$

Hence, the modular group $\Omega_g = 1$ of the torus is generated by these two transformations S and T. It is isomorphic to the group $PSL(2\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$, since it acts on τ as:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}; \quad ad - bc = 1. \tag{4.15}$$

Let us introduce here some more terminology[11,82]. For a Riemann surface of given topology g, let us consider the space of all metrics (of Euclidean or Minkowski signature) defined on this surface. The quotient of this space of metrics by the semi-direct

product $Diff_0 \times$ Weyl, with "Weyl" representing Weyl transformations, is then called Teichmüller space. Generally, the modular group Ω_g acts non-trivially on Teichmüller space, whose quotient by Ω_g is called modular space. Modular space parametrizes all possible conformal, complex or geometrical structures of the corresponding Riemann surface. In particular, we just saw that for $g = 1$, Teichmüller space is the upper-half complex plane $\mathbb{C}^+$ and that modular space is simply $\mathbb{C}^+/PSL(2\mathbb{Z})$. Let us choose the following fundamental domain of the modular group in Teichmüller space, to represented modular space:

$$F = \{\tau \in \mathbb{C} : |\tau| > 1, -\frac{1}{2} \leq Re\tau < \frac{1}{2}, Im\tau > 0\}. \tag{4.16}$$

Modular invariance thus requires that the transformations (4.13) and (4.14) leave 1-loop vacuum amplitudes invariant. To express these amplitudes, all the machinery of path-integrals[11,24,82] is not really necessary, if one recalls that the 1-loop vacuum amplitude of a single relativistic bosonic or fermionic degree of freedom in ordinary quantum field theory is given by:

$$\mp\frac{1}{2} ln\ det[-\Box + m^2] = \mp\frac{1}{2} tr ln[-\Box + m^2], \tag{4.17}$$

where the upper (resp.lower) sign applies to a boson (resp. fermion). Through a proper-time representation, this quantity is given (up to a factor) by:

$$\Lambda(m^2) = \pm\frac{1}{2}\int_0^\infty \frac{dt}{t}(4\pi t)^{-D/2} e^{-t\frac{m^2}{\mu^2}} \tag{4.18}$$

where t measures the total proper-time in the loop, and μ^2 is some mass scale. Note that in this expression, the trace over the zero-modes, namely the loop momentum, has already been taken.

For a mass distribution $\rho(m^2)$ of states of the same statistics, given as

$$\rho(m^2) = \sum_{n=0}^{\infty} d_n \delta(\frac{m^2}{\mu^2} - (\alpha n + \beta)), \tag{4.19}$$

where α, β are rational numbers, and the degeneracies d_n are defined by the function

$$f(q) = \sum_{n=0}^{\infty} d_n q^{2(\alpha n + \beta)}, q \in \mathbb{C}, |q| < 1, \tag{4.20}$$

the total contribution to the 1-loop vacuum amplitude is then:

$$\Lambda = \pm\frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2}(8\pi^2\tau_2)^{-D/2} f(e^{-\pi\tau_2}). \tag{4.21}$$

Here, we set $t = 2\pi\tau_2$.

Let us now consider the case of a closed string. Typically, for bosonic and fermionic states, we have the mass formulae:

$$\frac{1}{2}\alpha' m^2 = (\alpha n + \beta_R) + (\alpha m + \beta_L), \tag{4.22a}$$

$$\alpha n + \beta_R = \alpha m + \beta_L, \tag{4.22b}$$

where α, β_R, β_L are rational numbers, such that $\alpha^{-1}(\beta_R - \beta_L)$ is an integer, and n, m are positive integers (it is always possible to choose the factor α to be the same

in each moving sector). The values αn and αm are the eigenvalues of the excitation level operators N and $\overline{N}$ (or $N^\perp$ and $\overline{N}^\perp$) in each moving sector, namely the Virasoro zero-modes with the contribution of the bosonic zero-modes subtracted away. To each moving sector, we associate a partition function:

$$f_R(q) = Trq^{2(N^\perp+\beta_R)} = \sum_{n=0}^{\infty} d_R(n)q^{2(\alpha n+\beta_R)}, \tag{4.23a}$$

$$f_L(q) = Trq^{2(\overline{N}^\perp+\beta_R)} = \sum_{n=0}^{\infty} d_L(n)q^{2(\alpha n+\beta_L)}, \tag{4.23b}$$

with the traces taken over all non-zero modes. As was explained for the closed bosonic string after (2.113), the level matching condition (4.22b) is easily implemented in the partition function for the theory, which then has the expression:

$$f(q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} f_R(|q|e^{i\theta/2\alpha})\overline{f_L(|q|e^{i\theta/2\alpha})}. \tag{4.24}$$

When substituted in (4.21) , we thus obtain for the corresponding 1-loop vacuum amplitude:

$$\Lambda = \pm\frac{1}{2}\alpha(8\pi^2)^{-D/2}\int_0^\infty \frac{d\tau_2}{\tau_2^2}\int_{-1/2\alpha}^{1/2\alpha} d\tau_1 \frac{1}{\tau_2^{D/2-1}} f_R(e^{i\pi\tau})\overline{f_L(e^{i\pi\tau})}, \tag{4.25}$$

where

$$\tau = \tau_1 + i\tau_2, \quad \theta = 2\pi\alpha\tau_1. \tag{4.26}$$

On the other hand, we have:

$$f_R(e^{i\pi\tau})\overline{f_L(e^{i\pi\tau})} = Tre^{i\pi\tau_1 P}e^{-\pi\tau_2 H'}, \tag{4.27}$$

where

$$P = 2[(N^\perp+\beta_R)-(\overline{N}^\perp+\beta_L)] = 2[(L_0^\perp+\beta_R)-(\overline{L}_0^\perp+\beta_L)], \tag{4.28a}$$

$$H' = 2[(N^\perp+\beta_R)+(\overline{N}^\perp+\beta_L)] = 2[(L_0^\perp+\beta_R)+(\overline{L}_0^\perp+\beta_L)] - \alpha'(P^i)^2. \tag{4.28b}$$

Hence, P is the generator of translations in σ, and H' the generator of translations in τ with the bosonic zero-modes subtracted out. The physical meaning of the expression (4.25) is thus clear. As the closed string propagates freely before annihilating itself, it propagates for a total proper-time value (proportional to) $(\pi\tau_2)$ and twists on itself by an angle (proportional to) $(\pi\tau_1)$. The factor (4.27) precisely computes the corresponding partition function. We recognize in τ_1 and τ_2 respectively the real and imaginary parts of the modular parameter τ of the associated torus topology. When multiplied by the factor $\tau_2^{1-D/2}$, which corresponds to the trace over the bosonic zero-modes (P^i) , the partition function (4.27) is then integrated in (4.25) over all possible values of τ_1 and τ_2, i.e. over all possible geometries of the torus associated to this 1-loop vacuum amplitude.

We thus conclude that the 1-loop vacuum amplitude of a closed string theory is given by the expression (up to a factor):

$$\int \frac{d\tau_1 d\tau_2}{\tau_2^2}\tau_2^{1-D/2}Trq^{2(N^\perp+\beta_R)}\bar{q}^{2(\overline{N}^\perp+\beta_L)}, \tag{4.29}$$

where

$$q = e^{i\pi\tau}, \quad \tau = \tau_1 + i\tau_2, \tag{4.30}$$

and the integration is performed over some domain of Teichmüller space. Modular invariance requires on the one hand, that the integrand in (4.29) be modular invariant, namely invariant under the S and T transformations of the modular parameter, and on the other hand, that the integral in (4.29) be performed over the fundamental domain F defined in (4.16), as otherwise the same geometrical configuration would be included more than once in the amplitude.

It is easy to see that invariance under T in (4.13) only requires that

$$N + \beta_R = \overline{N} + \beta_L \quad (mod\ 1) \tag{4.31}$$

This is the constraint expressing invariance under constants shifts in $(\sigma \to \sigma + \sigma_0)$. For the transformation S in (4.14), it is easy to show that we have

$$\frac{d\tau_1 d\tau_2}{\tau_2^2} \to \frac{d\tau_1 d\tau_2}{\tau_2^2}, \tag{4.32}$$

$$\tau_2 \to \frac{\tau_2}{|\tau|^2}. \tag{4.33}$$

Hence, invariance under $\tau \to -1/\tau$ requires that the factor

$$\tau_2^{1-D/2}Trq^{2(N^\perp+\beta_R)}\bar{q}^{2(\overline{N}^\perp+\beta_L)} \tag{4.34}$$

be modular invariant.

By restricting the integral (4.29) to the fundamental domain F, the usual short-distance divergence at $\tau_2 \to 0$ is excluded. Thus, string theories have a much better short-distance behaviour than ordinary quantum field theories of particles. This restriction due to modular invariance is one of the reasons for the possible finiteness of string theories. However, the point $\tau \to 0$, a short-distance limit, is related to the point $\tau \to i\infty$, a large-distance limit, through the modular transformation $\tau \to -1/\tau$. This is the simplest example of a duality property between large distance and small distance physics in string theories.

The fact that the large distance point $\tau_2 \to i\infty$ is included in the integral (4.29) implies that its expression is ill-defined for tachyonic theories. Indeed, we have

$$q^{m^2} = e^{i\pi\tau_1 m^2}e^{-\pi\tau_2 m^2}. \tag{4.35}$$

Actually, this divergence has no physical meaning[38,77]. It is only a consequence of the fact that the proper-time representation used in (4.18) is not adequate[38,77,78] when $m^2 < 0$. It may be shown[38,77] that the correct result is obtained by inserting in (4.29) a factor $e^{-\pi\tau_2\gamma}$, with $\gamma \geq -m^2_{\text{tachyon}}$, and then taking $\gamma \to 0$ once the integral is completed. However, modular invariance is then explicitly broken. But after all, this is not so surprising. Modular invariance is believed to be essential in the quantum consistency of string theories. For example, it should guarantee that all null states decouple to all orders of perturbation theory, which is necessary for unitarity. For a tachyonic theory, unitarity is certainly not obtained, and as we discussed, null states do not decouple either. Thus, if modular invariance indeed implies unitarity, a tachyonic theory can never be modular invariant. For these reasons of unitarity and modular invariance, a tachyonic theory can never be considered to be a consistent quantum theory.

Finally, let us remark here that the factorization of the partition function (4.29) into right-and left-moving contributions generally extends to all orders of perturbation theory. This property is known as holomorphic factorization.[11]

4.2 Examples of Modular Invariance

4.2.1 The D=26 closed bosonic string

As a first example of the principles of string theory construction at work, let us reconsider the simplest of all string theories, namely the closed oriented bosonic theory studied in chapter 2. In this case, all world-sheet matter fields are bosonic coordinates $x^\mu(\sigma,\tau)$, taking their values in some Minkowski space-time of D dimensions. From our discussion above, we know that quantum local reparametrization invariance in the world-sheet requires the values (see (4.8) and (4.9)):

$$D = 26, \qquad a = \frac{D-2}{24} = 1. \tag{4.36}$$

These values immediately lead to the description of the space-time spectrum which we gave in chapter 2.

Let us now consider the global constraint of modular invariance, and, as an exercise, assume that the values (4.36) are not specified. The contribution of one scalar degree of freedom $x^\mu(\sigma,\tau)$ to the partition function (4.29) in one moving sector is simply:

$$\frac{1}{q^{2\beta^0}}\prod_{n=1}^{\infty}\frac{1}{(1-q^{2n})} = \left[q^{2(\beta^0-\frac{1}{24})}\eta(\tau)\right]^{-1}, \tag{4.37}$$

where β^0 is the contribution to the ordering constant "a" from one scalar field in one moving sector, and $\eta(\tau)$ is the Dedekind η-function defined in (2.111). Hence, the 1-loop vacuum amplitude is:

$$\int \frac{d\tau_1 d\tau_2}{\tau_2^2}\tau_2^{1-D/2}\left[q^{2(\beta^0_- -\frac{1}{24})}\bar{q}^{2(\beta^0_+ -\frac{1}{24})}\right]^{(2-D)}|\eta(\tau)|^{2(2-D)}. \tag{4.38}$$

Under modular tranformations, the $\eta(\tau)$ function transforms as follows:

$$T:\tau\to\tau+1 \qquad \eta(\tau)\to e^{i\pi/12}\eta(\tau), \tag{4.39}$$

$$S:\tau\to -1/\tau \qquad \eta(\tau)\to(-i\tau)^{1/2}\eta(\tau). \tag{4.40}$$

Hence, modular invariance under $\tau\to\tau+1$ requires

$$\beta^0_- = \beta^0_+ \ (mod\ 1), \tag{4.41}$$

whereas invariance under $\tau\to -1/\tau$ requires

$$\beta^0_- = \frac{1}{24} = \beta^0_+. \tag{4.42}$$

Therefore, modular invariance requires that each scalar field contributes a value 1/24 to the subtraction constant. This is the value also following from local reparametrization invariance, and from ζ-function regularization. From this point of view, one may say that this type of regularization is compatible with modular invariance, so that it may consistently be used to obtain the correct values for the subtraction constants. Intuitively, the reason for this compatibility is that ζ-function regularization does not require any mass scale, and thus should preserve conformal , Weyl and modular invariance[24]. Note however that modular invariance does not fix the value for D; the quantity (4.38) is modular invariant for any value D, provided we have the values (4.42) for the ordering constants.It is only local reparametrization invariance, through the requirement of a vanishing total central charge of the Virasoro algebra, which fixes the value $D = 26$.

In conclusion, the 1-loop partition function for the oriented closed string in 26 dimensions is:

$$\int_F \frac{d\tau_1 d\tau_2}{\tau_2^{14}}|\eta(\tau)|^{-48}. \tag{4.43}$$

Thus, modular invariance does not impose any additional constraints than those following from local reparametrization or conformal invariance. From that point of view, the closed bosonic string is a consistent theory. However, we know that the values (4.42) lead to the existence of a tachyon, which manifests itself in (4.43) by the fact that the factor $|\eta(\tau)|^{-48}$ diverges as $(e^{4\pi\tau_2})$ when $\tau\to i\infty$. This divergencess indeed corresponds to a state with $\frac{1}{2}\alpha' m^2 = -2$. As we discussed in the last section, such a situation leads to the breakdown of modular invariance, a consequence of the fact that the theory is not unitary.

4.2.2 The D=10 closed spinning string

Let us now consider the case of the spinning string discussed in chapter 3. The matter fields are then $D\ N = 1\ d = 2$ chiral supermultiplets (x^μ,λ^μ). Local super-reparametrization invariance requires

$$D = 10,\ a_R = 0,\ a_{NS} = \frac{1}{2}. \tag{4.44}$$

In our analysis of modular invariance, we shall use these values, although the reader may like to repeat the argument with D, a_R and a_{NS} unspecified.

The contribution of a single scalar degree of freedom to the partition function in one moving sector is:

$$Trq^{2[N^\perp-\frac{1}{24}]} = \frac{1}{q^{1/12}}\prod_{n=1}^{\infty}\frac{1}{(1-q^{2n})} = \frac{1}{\eta(\tau)}. \tag{4.45}$$

Here, we used the fact that the corresponding subtraction constant is $\frac{1}{24}$. For the Majorana-Weyl degrees of freedom, we have to distinguish between Ramond and Neveu-Schwarz boundary conditions, with the corresponding subtraction constants $(-\frac{1}{24})$ and $(+\frac{1}{48})$ respectively. In the Ramond sector, we then have the following contribution to the partition function for one Majorana-Weyl degree of freedom

$$Trq^{2[N^\perp+\frac{1}{24}]} = q^{\frac{1}{12}}\prod_{n=1}^{\infty}(1+q^{2n}) = \frac{1}{\sqrt{2}}\left[\frac{\theta_2}{\eta}\right]^{1/2}. \tag{4.46}$$

In the Neveu-Schwarz sector, we obtain

$$Trq^{2[N^\perp-\frac{1}{48}]} = q^{-\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^{2n-1}) = \left[\frac{\theta_3}{\eta}\right]^{1/2}, \tag{4.47}$$

and if the contributions are weighted by the fermion number $(-1)^F$, we have

$$Tr(-1)^F q^{2[N^\perp-\frac{1}{48}]} = q^{-\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^{2n-1}) = \left[\frac{\theta_4}{\eta}\right]^{1/2}. \tag{4.48}$$

In these expressions, η is the Dedekind η-function, θ_i are the theta functions $\theta_i(0|\tau)$ defined in (3.64), and the quantities $N^\perp$ and F represent the contribution of the corresponding degrees of freedom to the respective quantum operators, with the trace taken accordingly over the associated subspace of the space of states.

It should then be clear that the corresponding partition function has the form:

$$\int \frac{d\tau_1 d\tau_2}{\tau_2^2}\frac{1}{\tau_2^4}|\eta(\tau)|^{-24}\left[\frac{1}{16}d_R\theta_2^4-\alpha_R\theta_3^4-\beta_R\theta_4^4\right]\overline{\left[\frac{1}{16}d_L\theta_2^4-\alpha_L\theta_2^4-\beta_L\theta_4^4\right]} \tag{4.49}$$

Here, d_R and d_L count the degeneracies of the ground state spinors in the respective Ramond sectors. This degeneracy is 8 for an on-shell massless Majorana-Weyl spinor, 16 for a massless Majorana or Weyl spinor and 32 for a massless Dirac spinor. The quantities $\alpha_R, \alpha_L, \beta_R$ and β_L correspond to the "projectors" $\frac{1}{2}\left[\alpha_R+\beta_L(-1)^F\right]$ and $\frac{1}{2}\left[\alpha_R+\beta_L(-1)^{\overline{F}}\right]$ in the respective Neveu-Schwarz sectors, with $F, \overline{F}$ being the fermion numbers. For $\beta_R=0$ or $\beta_L=0$, we have no projection, hence a possible tachyon state. The case $(\alpha+\beta)_{R,L}=0$ corresponds to GSO projection; the tachyon is then avoided. In order to count correctly the number of states at each mass level, we must have $|\alpha_{R,L}|+|\beta_{R,L}|=1$. Finally, the relative minus sign between R and NS contributions is due to the fact that space-time bosons and fermions should contribute with opposite signs to the 1-loop vacuum amplitude. Thus, the correct spin-statistics relation is obtained when α_R and α_L are positive (with d_R,d_L positive).

Under modular transformations, we have:

$$T:\tau\to\tau+1 \qquad \eta(\tau)\to e^{i\pi/12}\eta(\tau), \tag{4.50}$$

$$\theta_2 \to e^{i\pi/4}\theta_2, \tag{4.51a}$$
$$\theta_3 \to \theta_4, \tag{4.51b}$$
$$\theta_4 \to \theta_3. \tag{4.51c}$$

$$S:\tau\to-1/\tau \qquad \eta(\tau)\to(-i\tau)^{1/2}\eta(\tau), \tag{4.52}$$

$$\theta_2 \to (-i\tau)^{1/2}\theta_4, \tag{4.53a}$$
$$\theta_3 \to (-i\tau)^{1/2}\theta_3, \tag{4.53b}$$
$$\theta_4 \to (-i\tau)^{1/2}\theta_2. \tag{4.53c}$$

Hence, invariance of (4.49) under T requires that:

$$\alpha_R+\beta_R=0, \qquad \alpha_L+\beta_L=0. \tag{4.54}$$

This precisely corresponds to GSO projection in the respective NS sectors. With the requirement that $|\alpha_{R,L}|+|\beta_{R,L}|=1$, (4.54) implies that all $\alpha_{R,L}, \beta_{R,L}$ take the values $\pm 1/2$. The sign of these quantities is now fixed by modular invariance under S, since it requires:

$$-\beta_R=\frac{d_R}{16}, \qquad -\beta_L=\frac{d_L}{16}. \tag{4.55}$$

Therefore, modular invariance of the 1-loop partition function implies:

$$d_R=d_L=8, \qquad \alpha_R=\alpha_L=\frac{1}{2}=-\beta_R=-\beta_L. \tag{4.56}$$

These are precisely[55] the conditions imposed in the GSO projection[54]. They show that this projection of the oriented closed spinning string indeed leads to a consistent string

theory. Note that we also obtain the correct spin-statistics relation [55]. In conclusion, the 1-loop modular invariant partition function of the type II spinning string is:

$$\int \frac{d\tau_1 d\tau_2}{\tau_2^5}\left|\frac{1}{2\eta^{12}}\left[\theta_2^4-\theta_3^4+\theta_4^4\right]\right|^2. \tag{4.57}$$

Note first of all that this quantity is finite, since the limit $\tau\to i\infty$ is not singular. Indeed, the would-be tachyon in the (N_S, N_S) sector is projected out in the difference $(\theta_3^4-\theta_4^4)$. Moreover, this quantity actually vanishes exactly, due to the Riemann identity (3.66). This is a consequence of the space-time supersymmetry in each moving sector.

Note also that the discussion of modular invariance considered each moving sector independently from the other. In particular, this implies that the chiralities of the two Majorana-Weyl spinors in the R sectors are not correlated. These chiralities may be the same or different, corresponding respectively to the Type IIb and Type IIa superstrings. In lower dimensions, there exist examples of string theories where the right-and left-moving sectors are related in a non-trivial manner through modular invariance.

4.3 Spin Structures

It is convenient to express these last results concerning the spinning string in an other language, by introducing a new geometrical concept: spin structures[55]. In the quantization of world-sheet spinors, we considered two sectors associated to Ramond and Neveu-Schwarz boundary conditions, corresponding respectively to having periodic (P) or anti-periodic (A) fermions when taken around the closed string in the σ-direction on the world-sheet.

Clearly, when the world-sheet has the topology of a torus, since fermions may then be taken around the two homology cycles of the torus, we have $2^2=4$ possible choices of boundary conditions. Thus, NS (resp.R) spinors then correspond to anti-periodic (resp. periodic) fermions around the homology cycle associated to σ-parametrization, with periodic or anti-periodic boundary conditions on the other homology cycle associated to τ-parametrization. We shall denote these choices by ${}_{A,P}\square_{A,P}$, where the horizontal line corresponds to the boundary condition in the σ-cycle, and the vertical line to the boundary condition in the τ-cycle. Hence, NS fermions correspond to ${}_{A}\square_{A}$ and ${}_{P}\square_{A}$, and R fermions to ${}_{A}\square_{P}$ and ${}_{P}\square_{P}$. These four possible choices correspond to so-called spin structures of the torus[55,82,83]. In the same way that a torus is endowed with a geometry (or equivalently a complex or a conformal structure) by specifying the modular parameter τ, the torus is endowed with a spin structure, which is necessary to describe fermions on it, by specifying these boundary conditions on a basis of the homology group of the torus. Let us only remark here that this notion of spin structure extends to compact Riemann surfaces of higher genus, once a basis of the associated homology group is given[82,83].

Under modular transformations of the torus, it should be clear that spin structures are mapped into one another as follows:

$T : \tau \to \tau + 1$

$$
\begin{aligned}
&{}^{\mathrm{P}}_{\mathrm{P}}\square \to {}^{\mathrm{P}}_{\mathrm{P}}\square, \\
&{}^{\mathrm{A}}_{\mathrm{P}}\square \to {}^{\mathrm{A}}_{\mathrm{P}}\square, \\
&{}^{\mathrm{A}}_{\mathrm{A}}\square \to {}^{\mathrm{P}}_{\mathrm{A}}\square, \\
&{}^{\mathrm{P}}_{\mathrm{A}}\square \to {}^{\mathrm{A}}_{\mathrm{A}}\square,
\end{aligned}
\tag{4.58}
$$

$S : \tau \to -1/\tau$

$$
\begin{aligned}
&{}^{\mathrm{P}}_{\mathrm{P}}\square \to {}^{\mathrm{P}}_{\mathrm{P}}\square, \\
&{}^{\mathrm{A}}_{\mathrm{P}}\square \to {}^{\mathrm{P}}_{\mathrm{A}}\square, \\
&{}^{\mathrm{A}}_{\mathrm{A}}\square \to {}^{\mathrm{A}}_{\mathrm{A}}\square, \\
&{}^{\mathrm{P}}_{\mathrm{A}}\square \to {}^{\mathrm{A}}_{\mathrm{P}}\square,
\end{aligned}
\tag{4.59}
$$

Therefore, when considering modular invariant partition functions of string theories with world-sheet fermions, one must not only sum over the different possible geometries of the torus, by integrating over τ in the fundamental domain F, but one must also sum over all possible spin structures of the torus[55]. From this point of view, the GSO projection of the previous section corresponds to summing over all geometrical data of the torus topology of the world-sheet for the 1-loop vacuum amplitude. At genus $g = 1$, the ${}^{\mathrm{P}}_{\mathrm{P}}\square$ sector does not mix with the other sectors. At a higher genus, this is not the case anymore, so that it indeed participates non-trivially to the sum over spin structures necessary for modular invariance[55].

The transformations of spin structures under the modular group are very reminiscent of the associated transformations of theta functions (see (4.51) and (4.53)). The relationship can be made exact. The ordinary partition function of a Majorana-Weyl fermion in two dimensional quantum field theory corresponds to having anti-periodic boundary conditions in the time direction[82,83]. Thus, to the spin structure ${}^{\mathrm{A}}_{\mathrm{A}}\square$, we must associate the contribution (4.47) in the NS sector, namely:

$$
{}^{\mathrm{A}}_{\mathrm{A}}\square : \left[\frac{\theta_3}{\eta}\right]^{1/2}, \tag{4.60}
$$

whereas the spin structure ${}^{\mathrm{P}}_{\mathrm{A}}\square$ is associated to (4.48), since the insertion of the fermion number $(-1)^F$ precisely corresponds to changing the boundary condition in time:

$$
{}^{\mathrm{P}}_{\mathrm{A}}\square : \left[\frac{\theta_4}{\eta}\right]^{1/2}. \tag{4.61}
$$

This association is indeed compatible with the modular transformation under T of these quantities. From the transformation under S, we then have for the spin structure ${}^{\mathrm{A}}_{\mathrm{P}}\square$:

$$
{}^{\mathrm{A}}_{\mathrm{P}}\square : \left[\frac{\theta_2}{\eta}\right]^{1/2}. \tag{4.62}
$$

Finally, we may thus expect to have for the remaining spin structure

$$
{}^{\mathrm{P}}_{\mathrm{P}}\square : \left[\frac{\theta_1}{\eta}\right]^{1/2}. \tag{4.63}
$$

This is indeed correct. The first three spin structures are known as even spin structures, whereas the last one is an odd spin structure, since the corresponding partition function vanishes identically (see(3.64)). This is due to the existence of fermion zero modes for the associated boundary conditions[83].

In terms of this notion of spin structures, the partition function (4.57) of the Type II superstring may be rewritten symbolically as:

$$
\left[\frac{1}{2}\left({}^{\mathrm{A}}_{\mathrm{P}}\square + {}^{\mathrm{P}}_{\mathrm{P}}\square\right) - \frac{1}{2}\left({}^{\mathrm{A}}_{\mathrm{A}}\square - {}^{\mathrm{P}}_{\mathrm{A}}\square\right)\right]\overline{\left[\frac{1}{2}\left({}^{\mathrm{A}}_{\mathrm{P}}\square + \eta\, {}^{\mathrm{P}}_{\mathrm{P}}\square\right) - \frac{1}{2}\left({}^{\mathrm{A}}_{\mathrm{A}}\square - {}^{\mathrm{P}}_{\mathrm{A}}\square\right)\right]} \tag{4.64}
$$

where the relative sign $\eta = \pm 1$ in the Ramond sectors corresponds to the Type IIa ($\eta = -1$) or to the Type IIb ($\eta = +1$) superstring partition function. In (4.64), the notation for the boundary conditions means that all 8 Majorana-Weyl spinors in each moving sector (in the light-cone gauge) have the same boundary conditions.

Remark

To conclude, let us remark that the notion of spin structure can be extended to arbitrary twisted fermions, where the phase factors that fermions pick up when taken around homology cycles are now arbitrary rather than simply (± 1) factors. The corresponding partition functions may then be expressed in terms of general theta functions with characteristics. For more details, see Ref.83.

5 COMMENTS AND CONCLUSIONS

Unfortunately due to lack of space, we have to leave our discussion of string theory constructions here. It would be interesting to apply the general principles of the last chapter to the torus compactification of bosonic strings[67], or to the construction of all ten dimensional heterotic string theories[55,66,85], either in a bosonic or in a fermionic formulation, i.e. where the internal conformal field theory is realized in terms of free scalars or free spinors on the world-sheet. This is left as an exercise for the interested reader. He may find in Ref. 63 a useful discussion of torus compactification of bosonic strings, and in Refs. 15, 17, 82 a description of $D = 10$ heterotic strings, both in a bosonic or a fermionic formulation. For example, he may like to construct the $D = 10$ $N = 1$ supersymmetric heterotic strings[66] with gauge group $E_8 \times E_8$ or Spin $(32)/\mathbb{Z}_2$, and the $D = 10$ non-supersymmetric tachyon-free heterotic string[55,85] with gauge group $SO(16) \times SO(16)$, by considering all possible choices of spin structures for the internal world-sheet fermions and imposing modular invariance.

A new fundamental structure that these theories realize is that of a world-sheet current algebra, namely a Kac-Moody algebra[35], which leads to the existence of massless

vector states, corresponding to the gauge bosons of a space-time gauge symmetry of the theory. In a bosonic formulation, this structure is obtained through the so-called Frenkel-Kac construction of level 1 representations of Kac-Moody algebras, using string vertex operators (for more details, see Refs. 15, 17, 35, 66). In a fermionic formulation, the current algebra structure is obtained through a quark model type of construction (for more details, see Refs. 15, 35, 86). Hence, we have here another example of the general and profound relationship which exists between space-time and world-sheet local symmetries.

Despite the necessity of ending these lecture notes here, hopefully the reader will have found in the above discussion a useful introduction to and understanding of the principles of string theory construction. This should allow him to understand the literature on the subject and develop his own research. Clearly, the "name of the game" has become the following : pick your favorite dimension D of space-time (usually $D = 4$!), and use for the internal degrees of freedom some conformal field theory with central charge $c = 26 - D$ or $c = 3(10 - D)/2$, corresponding respectively to $N = 0$ or $N = 1$ local supersymmetry in the associated moving sector. This internal conformal field theory should be such that the complete 1-loop partition function of the theory is modular invariant, and consistent with the usual spin-statistics connection (and also free of tachyon-related divergences). Then, work out the physical spectrum and its space-time symmetries, and decide if it has a chance of resembling, in the massless sector, the spectrum of quarks, leptons and gauge bosons of the $SU(3) \times SU(2) \times U(1)$ Standard Model, or some Grand Unification of it. Clearly, when such a string model is found, there is still a very large amount of work to be done[3,4,6] before calling it a "realistic string theory".

Originally, since string theory became the most prominent candidate for a unified quantum theory right after the wave of popularity of Kaluza-Klein theories[87], people tended to think of the internal degrees of freedom as compactified degrees of freedom. For example, starting from a ten dimensional string theory, by compactifying six bosons and fermions on some internal compact space, such as a torus, a four dimensional theory may be obtained[1,72]. When this was considered at the level of the low-energy effective field theory in ten dimensions, it led[2] to compactifications on Calabi-Yau manifolds[2,4,15] when requiring $N = 1$ $D = 4$ global supersymmetry. But clearly, this is only one possible approach. More generally, an arbitrary construction would not have such a simple geometrical interpretation (for example, when the right- and left-moving sectors are completely different), although conformal field theory in two dimensions is so particular that such an interpretation may sometimes be possible nevertheless.

From this point of view, very large numbers of string theories in 4 dimensions have been constructed over the last years, using many different formulations for the internal degrees of freedom which do not always lead to different theories. Let us only mention here toroïdal compactifications[67], covariant lattice constructions[62,63], orbifolds[6,88], group manifolds[89], fermionic constructions[7,90], tensor product constructions[68,91,92], etc....

Clearly, the approach to string theory construction can be considered from a much more general point of view. As was one of the main leit-motives of these lectures, string theories *in the conformal gauge*, and thus *also in the light-cone gauge*, are nothing but conformal field theories, where the internal degrees of freedom define a $N = 0$ or $N = 1$ conformal field theory (CFT) with a given central charge[7,68]. Thus, if a complete list of these CFT and of their partition functions was known, irrespective of explicit realizations of those theories in terms of two dimensional fields, a complete classification of consistent string theories could be given. This is one among many other reasons for

the very rapidly expanding present research activity on the classification of conformal field theories in two dimensions[9]. Let us remark here that from such a point of view, the calculation of string scattering amplitudes to any order of perturbation theory becomes an exercise in the calculation of correlation functions in conformal field theory[9].

As should be emphasized, the present approach to string theory construction does not give any understanding as to why space-time should be four dimensional and flat. Only requirements of quantum consistency are imposed, which only lead to the upper bound $D \leq 10$. The hope is that, by classifying all possible string theories, we could not only find out if there are potential "realistic" theories, but moreover, such an approach could bring some new insight into the little understanding that we have concerning the dynamics of these theories, especially in non-perturbative regimes[7,68]. In this context, all present constructions should be viewed as the constructions of different classical string vacua of what could possibly be only a few fundamental string theories, still to be formulated. For example, all toroïdal compactifications of bosonic strings are related to one another, via expectation values of massless physical states[93]. Dynamics in the spaces of moduli (or parameters) describing these different classical vacua, induced for example by possible non-perturbative effects, could lead to some unique "ground state" string theory. This is the general present "philosophy" behind the approach discussed in these lectures.

Clearly, since the number of possibilities for constructing string theories is so large, one may be less ambitious, and only consider the classification of possible "realistic" string theories, by imposing some additional general "phenomenological" requirements. The obvious one is to choose $D = 4$. One may also require that the gauge symmetry group of the theory contains the group $SU(3) \times SU(2) \times U(1)$, with massless chiral fermions transforming as a triplet and a doublet under $SU(3) \times SU(2)$. Under very general conditions, it has been shown[94] that such a requirement excludes all Type II string constructions, leaving thus only heterotic strings as possible candidates for "realistic" theories.

One further constraint which one may like[2] to enforce is that the four dimensional heterotic string theory has one global space-time supersymmetry. It has been shown[95] that in *the conformal gauge*, such a requirement imposes the existence of a *global* $N = 2$ superconformal invariance in at least one moving sector of the theory.

Hence, from such a point of view,"realistic" string theories should be obtained through heterotic string constructions using $N = 2$ superconformal field theories with central charge $c = 9$. The simplest such theories are obtained by tensor products of the corresponding members of the unitary discrete series[9]. Indeed, they lead to consistent string theories, which share many of the properties of Calabi-Yau compactifications of ten dimensional heterotic strings, or the associated orbifold constructions[92,96]. Other constructions of $N = 2$ superconformal theories with $c = 9$ are also possible[97], leading to new types of string theories[98]. Thus, the classification of all $N = 2$ superconformal field theories (with $c = 9$) is very important in that respect, and it has indeed grown into a research field of its own over the past months[99].

Clearly, although the main research activity in string theory emphasizes nowadays more the conformal field theoretical aspects of these theories, it is a field of research which is going to be with us for many more years, and as such any young theoretician should be familiar with its basic structure and properties. Not only is two dimensional conformal field theory a research field in its own right, it also has important connections, many probably still to be discovered, with statistical mechanics in two dimensions and pure mathematics. Moreover, after more than five years since string theory is back at

the fore-front of theoretical particle physics, with so many people having worked on it from very different perspectives, not one single argument has been put forth which could make these theories less an appealing or even inconsistent framework for the unification of all interactions.

As there is no other alternative known today, string theory still remains a fascinating and promising speculative avenue for the ultimate quantum unification of all our understanding of particle physics. The beauty of the mathematics involved makes it even more rewarding.

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These references do not claim to provide a complete list of all the work done on a given subject. They are only to be considered as indicative, and their aim is to help the interested reader in finding his own way in the vast literature of string theory.

My apologies are offered here to any author who feels unjustifiably unmentioned.

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