Relativity Group • Konstanz University

⟨ Theory of Gravity | Quantum Theory ⟩

Preprint KONS-RGKU-93-04

A New Constructive Axiomatic Scheme for the Geometry of Space-Time

Jürgen Audretsch and Claus Lämmerzahl

Fakultät für Physik der Universität Konstanz
Postfach 5560, D-7750 Konstanz, Germany
(January 1993)

Abstract

A new scheme for a constructive axiomatics for the space-time geometry is proposed. By treating quantum objects described by classical fields, as primitive objects it is shown that a deterministic, linear and local evolution leads to a system of partial differential equations. Founded on basic quantum mechanical experiences, the conformal structure and the paths of the classical limit are determined leading to a Riemannian space-time. A spin motion enables the introduction of space-time torsion. Therefore, *if spacetime is the entity which prescribes the behavior of characteristics, free matter waves, and spin states, then space-time is a Riemann-Cartan space-time with axial torsion.*

To appear in: U. Majer and H.-J. Schmidt (edts.): *Proceedings of the Bielefeld symposium on "Semantic Aspects of Space-Time Theories"* held in August 1991.

1 The physical and conceptual frame

1.1 Gravitation, geometry of space-time and the quantum domain

The (pseudo)-Riemannian manifold of General Ralativity is today generally accepted as the best mathematical framework for the description of space-time geometry, geometrised gravitation, and the influence of inertia. Space-time torsion and other fields can be introduced in addition and different types of field equations for geometrical quantities may describe different theories of gravity. The objects of classical physics like fluids, electromagnetic fields, and so on are influenced by the gravi-inertial interaction and can act as their source. The same is the case for quantum objects like elementary particles. Today a large part of the studies in General Relativity is devoted to the role of the unquantised geometrised gravito-inertial interaction in the quantum domain, whereby the quantum objects are treated either on the level of first quantisation or with reference to the field quantisation of matter (quantum field theory in curved space-time).

It is an important new development that these theoretical studies are now complemented by increasing efforts concerning the *empirical foundation* of the interaction between quantum objects and gravity. Today, matter wave interferometry with electrons, neutrons, and atoms provides an ever increasing number of experiments in which the influence of classical gravity and inertia on quantum objects can be studied in the laboratory in a very direct and precise way [1]. On cosmological scales, on the other hand, the effects originating from the influence of unquantised space-time curvature on processes of high energy physics, i.e. on quantum field theoretically described matter, dominate the physics of the extreme hot very early universe and have observable consequences today. To sum up, one may say that 'gravity has invaded the quantum domain'. By *quantum domain* we mean all those physical effects which reflect the fact that matter (and not gravitation or geometry) must be quantised. The respective 'particles' like neutrons, electrons, etc., are called *quantum objects.*

To investigate the role of geometrised classical gravity and inertia in quantum effects we usually assume a particular structure of space-time in the quantum domain. It is common practice to essentially rely for this on the postulate "Space-time in the quantum domain obeys a Riemann geometry". On the other hand for the domain of classical physics it has convincingly been disputed that it is reasonable to put such type of axiom on top of a theory of space-time. Why should therefore the Riemann geometry be adequate for the quantum domain?

It has successfully been demonstrated for classical physics that a physical axiomatics of space-time can be build up which does not postulate the particular geometrical structure from the beginning, but makes it a derived concept (see below). For the quantum domain such a constructive approach must be based on some typical quantum mechanical experiences which are regarded to be fundamental. The aim of this article is to sketch a succession of physical arguments and their mathematical formulation, which may be further elaborated to *build* a constructive axiomatics of space-time for the quantum domain. The space-time geometry of classical physics will be thereby contained in a limiting case. Before turning to this approach, we will make in the rest of this chapter some general remarks concerning the different types of axiomatic schemes.

1.2 Different approaches to space-time geometry

As mentioned above, there is a very simple way to introduce the geometrical structure of space-time. It is sufficient to postulate that the mathematical model for the physical space-time is a Riemannian manifold. Characteristic for such a *deductive* approach is that it starts from a mathematical axiom which contains an abstact structural constraint. It can be physically interpreted only after considerable elaboration of the scheme and will then be related to complex and involved laws. The physical and epimistological basis, the heuristic motivation, and the possibility of empirically testing such an axiom remain hidden. It is unclear why it should be physically plausible to assume this mathematical structure from the outset.

In contrast to this the alternative *chronogeometric axiomatic approach* of Synge [2] is based on particles and standard clocks as basic tools. One of the main objections against this approach (comp. [3]) is that the real clocks of physicists and astronomers are atom clocks. These are physically highly complicated systems which can only be understood on the basis of quantum mechanics. Because one can in principle construct ideal clocks based on light rays and freely falling particles [4, 5], the chronogeometric axiom either appears to be redundant or should be reduced from a theory of the gravitational influence on quantum matter and confirmed experimentally. Following [6] we add another argument: Within the domain of classical physics nearly all tests of General Relativity refer to astrophysical situations. The appropriate objects which reveal the structure of space-time near a binary pulsar for instance are freely falling masses and light rays. Clocks, not to speak about rigid rulers, are in this context of no use. We come back to this fundamental demand that one should choose the objects which shall indicate by their behaviour the geometry of space-time, to be adequate to the part of physical reality to which this geometry is going to be attributed.

An independent approach to space-time geometry which is located somewhere between the purely deductive schemes and the consequently constructive axiomatic schemes, which will be described below, is the gauge approach to gravity. It takes the local validity of Minkowski space with all its mathematical structure as starting point. Following Einstein, the structure and form of the gravitational potentials are read off in flat space-time from the inertial forces arising in non-inertial frames. Following Cartan, in a second step, arbitrary non-inertial reference frames are identified with a field of orthonormal (anholonomic) tetrads. In discussing the Dirac field it can be demonstrated that one only need to know the behaviour of the Lagrangian in a non-inertial reference frame, to determine the coupling to gravity. Torsion comes out as a natural consequence and space-time has a Riemann-Cartan geometry. Note that the reference to quantum mechanical matter fields and accordingly to matter wave experiments is central. For details and the literature see [1].

We now turn to *ab initio* approaches and begin with the description of the characteristic structures of constructive space-time axiomatics.

1.3 Characteristic structures of constructive axiomatic schemes for space-time geometry

The *constructive axiomatic approach* is opposed to deductive schemes. In this approach, which is in the tradition of Helmholtz, one takes serious the statement of Reichenbach [7] that in physics the observable facts are at the beginning and the abstract concepts are at the end. Axioms should therefore directly be related to experiments which can be performed without refering to the theory which is to be constructed (Reichenbach [8]). A limited number of directly observable phenomena which are regarded as fundamental, is taken as *basic experiences* and stated as prepositions which are idealisations and mathematical formulations of empirical facts. Accordingly the respective axioms may be directly confronted with particular experiments and tested separately. In this way the physical basis of the different mathematically formulated physical structures of an axiomatic scheme becomes evident. The physical entities with which the basic experiences can be made, are called *primitive objects*. They are implicitely defined by the axioms.

A constructive axiomatics for the space-time geometry is therefore based on the following scheme[1]: Discover and describe by means of the behaviour of selected primitive objects in particular basic experiences the geometrical structures of space-time. Thereby it must be possible to formulate the axioms and to perform the underlying experiments without making reference to geometry. Space-time geometry is then obtained as a derived concept. The EPS scheme sketched below is an example. It is based on the behaviour of free point particles and light rays. In contrast to this, the alternative axiomatic scheme which we are going to describe in more details is based on free quantum objects and basic experiences demonstrating their behaviour as matter waves.

The procedure always is to reveal the structure of space-time. It is discovered through the behaviour of the primitive objects in the basic experiences. Accordingly, these experiences are taken as indications for the existence of a geometrical structure which is responsible for certain physical effects, as well as we take other experiences for example with charged particles as indications for the existence and particular nature of a field called the electromagnetic field. It is typical for this procedure that at the same time we only fix or define what we are going to mean by the concept space-time geometry. What we finally will obtain is therefore a theorem of the type: "If space-time geometry is what prescribes to the particular primitive objects their typical behaviour as seen in the basic experiences, then space-time geometry is mathematically described by ... (maybe a Riemann-Cartan space)".

[1] For a more detailed discussion of such a type of axiomatics see Coleman and Korte [9, 10]. This approach is also discussed together with related epistemological and physical problems regarding the structure of space-time in contributions to the book of Audretsch and Mainzer [11].

It is evident that the choice of the primitive objects and the basic experiences will reflect the physical ideas and conceptions which are traditionally related to space-time geometry. Typically, effects which remain after all interactions and influences which can be shielded have indeed been shielded, are attributed to geometry. Geometry is what remains and accordingly it influences the description of 'free' objects[2]. 'Free' is thereby implicitely fixed (in the sense of Hilbert) by the axiom itself. Important is that one can find representations of the respective primitive objects in nature. The procedure 'shielding' itself is thereby not part of the scheme. It may only be taken as a hint when actually performing all the operational procedures to carry out the basic experiments.

In order to avoid misunderstandings it is necessary to make a final remark: The constructive procedure is to detect the geometry and accordingly to 'fill up' or 'enrich' the manifold step by step with mathematical structure. Accordingly, the fact that the geometrical structures which are revealed by free point particles and light rays, represent a Weyl space without torsion, does not mean that space-time geometry may not contain torsion. It can simply not be detected this way. Other primitive objects may well be sensitive to torsion. But, as pointed out above, it is a matter of convention, whether certain objects should be taken as basis for the scheme or not.

1.4 The EPS scheme as an example

Already in 1921 Weyl distinguished between two primitive substructures of the space-time structure of General Relativity [13]. He rejected the reliance on clocks and rigid rods. The ideas of Weyl have been given a detailed, precise, and axiomatic form by Ehlers, Pirani and Schild (EPS) in 1972 [3][3]. This scheme is a paradigm for a constructive space-time axiomatics. EPS adopt as primitive elements the notions of event, light ray, particle and freely falling particle. The introduction of local radar coordinates by means of particles and light rays leads to a differentiable structure. Further basic experiences are the causal propagation of light which introduces a conformal structure. The universality of free fall and the fact that the law of inertia holds infinitesimally reveals a projective structure. The relation between these structures is established using the basic experience that a freely falling particle is always slower than a photon, but can be made to chase a photon arbitrary close. This compatibility demand leads finally to a Weyl geometry, in which the transport of time intervals will in general be path dependent. Space-time torsion cannot be introduced or detected this way.

The last step to the further restricted Riemann geometry needs an additional Riemannian axiom to exclude the second clock effect. It essentially amounts to the demand that gravitational time given by the Weyl arc length agrees with atomic time [3, 4], since the latter is

[2]The concept 'free' must not necessarily appear in the scheme. Coleman and Korte [12] have shown that there are possibilities to isolate and to characterise the influence of geometry on the path of point particles even if other interactions are present.

[3]For elaborated discussions of the scheme see [4, 14, 15, 16].

integrably transported because of the indistinguishability of quantum objects of a particular kind. The authors themselves question that the time-equality postulate is compelling and regard it as an extraneous element of their scheme. The argument that axiomatics should not rely on so complex structures as atomic clocks, which above has been used against Synge's scheme, could be repeated. To demand alternatively that the gravitational clock constructed out of light rays and freely falling particles would lead to a path-independent transport of the time unit would imply the introduction of an experience which is not technically realisable.

It is a deficiency of the EPS approach that the Riemannian axiom breaks the internal coherence of the scheme. It has been demonstrated by Audretsch [17] that the final step from a Weyl geometry to a Riemann geometry can be done by adding rudiments of quantum mechanics to the scheme. The reason for this is that quantum mechanics must include the classical mechanics of freely falling point particles as a limiting case. The self-consistency requirement that this limiting case should agree with the behaviour of classical particles as postulated in the EPS-scheme reduces Weyl space to a Riemann space[4,5].

To close the EPS scheme it was necessary to refer to some quantum mechanical arguments. This leads legitimately to the question if it is possible to base an axiomatics of the Riemann or Riemann-Cartan geometry of space-time solely on experiences with quantum matter. That this can be done will be scetched in the following chapters. Before doing so we want to point out that an approach of this type is reasonable as well as important for several reasons.

1.5 Motivation for a constructive axiomatics founded on basic quantum mechanical experiences

On the background of the worked out EPS scheme and the arguments used above to reject earlier axiomatics, several motives for the alternative scheme become evident (comp. [1]).

There is a hierarchy within the theories of matter. Quantum physics is more fundamental

[4]That it should be possible to obtain this reduction in discussing quantum mechanical wave equations has already been conjectured by Ehlers [4]. In connection with the paper mentioned above it has been stressed by Kasper [19] and Ehlers [16] that it is the appearance of a mass function in the quantum mechanical framework in Weyl space that makes it possible to introduce a Riemannian structure.

[5]Various attempts have been made to improve and to supplement the EPS scheme. Woodhouse [20] gave a rigorous derivation of the differential and causal structure based on much the same primitive elements. Schröter *et al.* [21, 22, 23] studied the basic experiences which lead to the description of space-time as a four-dimensional differentiable manifold and a reformulation of the EPS scheme. A characterisation of free fall path by a Desargues property has been given by Heilig and Pfister [24]. The free fall structure has alternatively been characterised by admitting maximal local isotropy by Ehlers and Köhler [25] and Coleman and Korte [9, 10, 26]. The compatibilty demand has been rediscussed by Coleman and Korte [27]. An axiomatics of the Newton-Cartan geometry has been treated by Ewen and Schmidt [28]. The EPS scheme can be enlarged by including additional experiences. If the particle carries in addition a polarisation direction, it has been demonstrated by Audretsch and Lämmerzahl [28] that the space-time metric can be endowed with a totally antisymmetric torsion. A first attempt to study the possibility to obtain a Riemannian space-time if in the EPS-scheme all axioms referring to freely falling particles are replaced by axioms related to free matter waves, has been made by Audretsch and Lämmerzahl [29].

than classical physics. The latter is contained in the former as a limiting case. Classically described matter, such as satellites, stones, and other candidates for point particles, is composed of quarks, leptons, and their gauge bosons. The gravitational and inertial behaviour of the complex objects should be a consequence of the behavior of the more elementary objects. It is therefore reasonable, if not compelling, to relate a theory of the structure of space-time to the more fundamental theoretical framework of quantum mechanics. This means, by the same token, to base it on the physically more primitive objects, namely on the elementary particles, and on basic experiences which are typical for the quantum domain and which can be made in a theory-free (especially geometry-free) way. In doing so, we may have to put up with the fact, that the physically more primitive objects may be less 'primitive' with regards to the technical details of the operational handling as compared with point particles.

The Weylian geometry governs the behaviour of point particles and light rays. To infer from this that it is also the appropriate geometry for the description of quantum objects would be a deductive extrapolation, an approach which has been rejected above. From the motion of point particles and light rays it is impossible to read off any physical indication suggesting that the space-time geometry underlying the quantum domain is a Weyl geometry. Point particles and light rays are typically realised by satellites and radar signals. It is evident that, for instance, the geometry in the interior of an hydrogen atom cannot be revealed with their help.

This leads to the important conclusion that constructive axiomatic schemes have ***natural domains of application*** to which they are adjusted. These domains, as parts of physical reality, are characterised by the fact that their geometry can be explored by the typical operational realisations of the respective primitive objects. The basic experiences refer to this domain. Obviously the domain of application of the EPS-scheme is restricted to classical physics. We therefore need an alternative scheme which includes the quantum domain, and we have to base it on quantum objects as primitive objects.

There is a third motivation: Quantum objects, as compared to classical point particles and light rays, are the deeper searching probes. The matter wave interference experiments demonstrate that massive fields with spin couple to gravito-inertial fields in accordance with the strong equivalence principle. The respective experimental results depend on the parameters mass and spin. Based on this richer structure of the new primitive objects, additional physical structures can be geometrized yielding more specific statements on space-time geometry. Accordingly geometry can be further specified. This has two consequences: The axiomatic scheme will not end with the Weyl geometry but will lead to Riemann geometry directly. Secondly, the torsion of space-time can be 'sensed' if space-time is explored by quantum objects with spin.

On the one hand, quantum physics contains classical physics as a limiting case. The EPS-scheme will therefore as well be contained in a limiting case. Therefore it will not be refuted but instead confirmed. On the other hand, quantum mechanical experiments rely on macrophysical objects. It must be stressed that the basic experiences refering to the quantum domain must be chosen in such a way that macro-objects are only manipulated in a geometry-free way or with

reference to the geometry which has already been introduced at the previous steps of the scheme. A double slit and the registration of an interference pattern are typical examples. It is evident that therfore only some rudiments of quantum mechanics can be related to basic experiences, because the interpretation of the majority of quantum experiments are theoryloaden and cannot be operationalised in a geometry-freee way. Accordingly we will have to base the scheme on quantum objects and on experiences out of the quantum domain which are fundamental and prior to a specific theoretical elaboration of quantum theory.

We will now turn to the details of the scheme and sketch in the following our results of [30, 31, 32, 33]. For doing so we introduce classical fields as our primitive objects representing the quantum objects we are dealing with. The basic experiences are then described in the postulates 1 - 8.

2 Derivation of the field equations

2.1 Deterministic evolution

It is difficult to specify particular field equations as, for example, the Dirac equation, operationally from basic experiences. Therefore we establish the dynamical behaviour of the considered fields. Thereby we start from the basic experience that the fields show a deterministic ***evolution***. For this a 4-dimensional differentiable manifold is assumed[6]. We consider classical fields (no second quantisation) and allow them to be vector valued complex functions: $\varphi : \mathcal{M} \to \mathbf{C}^s : x \mapsto \varphi(x)$.

The dynamical behaviour of the fields can be formulated by considering the physical phenomena with respect to some (3+1)-slicing e of the manifold $\mathcal{M}$ which consists in a 3-dimensional differentiable manifold Σ and a class of embeddings $e_t : \Sigma \to \mathcal{M}$, $t \in \mathrm{I} = [T_0, T] \subset \mathbf{R}$, so that $e : \mathrm{I} \times \Sigma \to \mathcal{M} : (t, \mathbf{x}) \mapsto x = e(t, \mathbf{x}) = e_t(\mathbf{x})$. We can define the fields $\hat{\varphi}_t := e_t^* \varphi$ as the field φ pulled back from $\Sigma_t := e_t(\Sigma)$ to Σ. The fields $\hat{\varphi}_t$ then give rise to the function space $\mathcal{D}(\Sigma, \mathbf{C}^s)$ of vector valued distributions on Σ.

To introduce a ***deterministic*** evolution there should exist at least one (3+1)-slicing with a corresponding class of 3-dimensional non-intersecting hypersurfaces Σ_t which are labeled by a monotonically increasing parameter t which may be called a 'time'-like parameter. With reference to this parameter the evolution of the field, i.e. the progression of the field from Σ_t to $\Sigma_{t+\delta t}$ and so on, takes place in a unique way[7].

Postulate 1: (Deterministic evolution)
$\exists$ $(3+1)$-slicing e so that for a set of given data $\hat{\Theta}^{(i)} := \left.\frac{d^i \hat{\varphi}_t}{dt^i}\right|_{t_0}$, $i = 0, 1, \ldots, R$ the field $\hat{\varphi}_t$ is uniquely determined for all $t > t_0$.

[6] It would be desireable to get also the differential structure of the manifold as a derived concept as has been done on the classical level in [3, 20, 22]. One may imagine that in the field theoretical context this could be achieved by considering the differentiable structure of solutions.

[7] We do not consider fields with constraints.

By unique determination we mean that there is a unique correlation, a unique mapping between the state at t_0 and the state at t [8]. This introduces a dynamical operator $U : (t, t_0, \widehat{\Theta}) \mapsto \widehat{\varphi}_t = U(t, t_0, \widehat{\Theta})$. We need more experiences for specifying this dynamical operator U.

We can now introduce the concept of a type of a field: We are dealing with a certain ***field type*** or ***quantum objects*** of a particular ***type*** if for all initial data the dynamical evolution does not decouple, that is, if there is no invariant subspace[9] of $\mathcal{D}(\Sigma, \mathbf{C}^s)$ with respect to U.

2.2 Superposition principle

Source free fields of quantum matter obey the superposition principle. For this can directly be demonstrated by interference experiments, see e.g. [35]. Other estimates [38] give even stronger limits on nonlinearities.

We will formalise this experience in demanding that the *superposition* $\alpha\varphi + \beta\psi$ of two fields φ and ψ which evolve according to U is a field which obeys with respect to the same slicing e the same dynamics for all $\alpha, \beta \in \mathbf{C}$. Formulating this demand with reference to the preparation procedure, it means that the linear combination of initial data results in the linear combination of the fields too. We therefore require for source free fields

Postulate 2: (Superposition)

$$\forall \widehat{\Theta}, \widehat{\Psi}, \forall \alpha, \beta \in \mathbf{C} : \quad \left. \begin{array}{l} \widehat{\varphi}_t = U(t, t_0, \widehat{\Theta}) \\ \widehat{\psi}_t = U(t, t_0, \widehat{\Psi}) \end{array} \right\} \quad \Rightarrow \quad \alpha\widehat{\varphi}_t + \beta\widehat{\psi}_t = U(t, t_0, \alpha\widehat{\Theta} + \beta\widehat{\Psi}).$$

From this we can conclude that there are linear operators $F^{(i)}(t, t_0)$ which act separately on the initial data:

$$\widehat{\varphi}_t = U(t, t_0, \widehat{\Theta}) = U\Big(t, t_0, \sum_{i=0}^{R} \widehat{\Theta}^{(i)}\Big) =: \sum_{i=0}^{R} F^{(i)}(t, t_0)\widehat{\Theta}^i \tag{1}$$

Differentiation with respect to t leads to an abstract Cauchy problem of higher order:

$$\frac{d^{R+1}\widehat{\varphi}_t}{dt^{R+1}} = \sum_{i=0}^{R} G_t^{(i)} \frac{d^i\widehat{\varphi}_t}{dt^i} \tag{2}$$

whereby the $G_t^{(i)} := \lim_{h \to 0} \frac{1}{h^{R+1}} \left(\sum_{l=0}^{R+1} (-1)^l \binom{R+1}{l} F^{(i)}(t + (R+1-l)h, t) \right)$ are time-dependent operators.

[8] This usually is called Hadamard's *principle of scientific determinism* [34].

[9] $\mathcal{D}(\Sigma, \mathbf{C}^s)$ admits an invariant subspace, if $B \subset \mathbf{C}^s$ is a linear subspace and $U : \mathcal{D}(\Sigma, B) \to \mathcal{D}(\Sigma, B)$, that is, initially prepared data in $\mathcal{D}(\Sigma, B)$ will remain in this subspace during their evolution.

2.3 Locality

In the next step we specify a certain time-dependence of the fields during the evolution from the inital data. We demand that at points $\mathbf{x} \in \Sigma$ where all the data vanish, the field increases so slowly that at these points at least the $(R+1)^{\text{th}}$ time derivative of the field $\widehat{\varphi}_t$ vanishes too. This means that at these points the field should grows more slowly than given by the order of the abstract Cauchy problem. In this sense we demand

Postulate 3: (Locality)

$$\text{supp}\left(\frac{d^{R+1}\widehat{\varphi}_t}{dt^{R+1}}\Big|_{t_0} \right) \subset \bigcup_{i=0}^{R} \text{supp}\left(\frac{d^i\widehat{\varphi}_t}{dt^i}\Big|_{t_0} \right).$$

By use of the abstract Cauchy problem and by chosing arbitrary initial data postulate 3 implies that all $G_t^{(i)}$ are local operators[10]. By means of a theorem by Peetre [39] (see also [40]) each linear operator $P : C^\infty(\Sigma, \mathbf{C}^s) \to C^\infty(\Sigma, \mathbf{C}^s)$ which is local, is a differential operator with C^∞-coefficients. Therefore the operators $G_t^{(i)}$ are *differential operators* leading to the field equation in $\mathbf{R} \times \Sigma$

$$\frac{d^{R+1}\widehat{\varphi}_t}{dt^{R+1}} = \sum_{i=0}^{R} \sum_{j=0}^{N_i} g_{(i),t}^{m_1 \ldots m_j} \partial_{m_1} \cdots \partial_{m_j} \frac{d^i\widehat{\varphi}_t}{dt^i}. \tag{3}$$

Transscribing this result to the 4-dimensional manifold we finally get as *field equation* for quantum objects

$$0 = \sum_{i=0}^{r} \gamma^{\mu_1 \cdots \mu_i}(x) \left(\partial_{\mu_1} \cdots \partial_{\mu_i}\varphi\right)(x) \tag{4}$$

with $r := \max_{0 \le i \le R} \{R+1, N_i\}$ and where the $\gamma^{\mu_1 \cdots \mu_i}(x)$ are complex $s \times s$-matrices in $C^R(\mathcal{M}, \mathbf{C}^{s^2})$. These entities are related to external fields which may be the metric, the connection or, for instance, the electromagnetic potential.

All coefficients $\gamma^{\mu_1 \cdots \mu_i}$ but the one of heighest order transform inhomogeneously under coordinate transformations $x \mapsto x' = f(x)$ and under transformations of the basis of the vector space $\mathbf{C}^s$: $\varphi \mapsto \varphi' = S\varphi$ for $S \in Gl(\mathbf{C}, s)$. The $\gamma^{\mu_1 \cdots \mu_i}$ with $i < r$ play the role of covariantising coefficients.

To sum up: ***Fields showing a deterministic evolution which is linear and local, must obey a linear system of partial differential equations*** (PDE). The locality postulate can be replaced by

[10] Mathematically an operator P is called *local* if $\text{supp}(P\phi) \subset \text{supp}\phi \quad \forall \phi \in C^\infty(\Sigma, \mathbf{C}^s)$.

a postulate demanding finite propagation speed of solutions [11].

$$\left.\begin{array}{l}\text{deterministic evolution}\\ \text{superposition principle}\\ \text{locality / finite propagation speed}\end{array}\right\} \Longrightarrow \text{weak hyperbolic system of PDE}$$

3 The null cone structure

3.1 The singularities

As soon as a system of partial differential equations is given, we can analyse it by means of characteristic properties of solutions, as, for example, possible singularities in the solutions. Singularities are discontinuities in the solutions or in one of their derivatives which can occur only on certain subsets of $\mathcal{M}$ called *characteristics*. In General Relativity they are related to the notion of the light cone because for all physical theories of matter the characteristics are identical with the usual light cones describing also the causal behaviour of the fields. In our axiomatic reconstruction we base the introduction of the conformal structure on the characteristics of the field equation (4). Mathematically singularities are most appropriately described by the notion of wave front sets in the case of scalar field equations (Hörmander [41]), or by polarisation sets in the case of systems of partial differential equations (Dencker [42]).

Briefly, the polarisation set $\{(x, k_\mu, A) \mid \gamma^{\mu_1\cdots\mu_r}(x)k_{\mu_1}\cdots k_{\mu_r}A = 0\}$ is a subset of a vector bundle over $\mathcal{M}$ which describes at which points x, in which direction k_μ, and the components A in which the solution can be singular. A describes the jumps of lowest order of the solution along the hypersurface $\Phi(x) = const.$, called characteristic surface, which is defined by $k_\mu = \partial_\mu\Phi$. The solvability condition of $\gamma^{\mu_1\cdots\mu_r}(x)k_{\mu_1}\cdots k_{\mu_r}a = 0$ reads

$$H_c(x,k) := \det(\gamma^{\mu_1\cdots\mu_r}k_{\mu_1}\cdots k_{\mu_r}) =: g^{\mu_1\cdots\mu_{rs}}k_{\mu_1}\cdots k_{\mu_{rs}} = 0 \tag{5}$$

meaning that k_0 is a function of $k_{\hat\mu}$, $\hat\mu = 1,2,3$. (5) is the characteristic equation which has the form of a Hamilton-Jacobi equation. Therefore it can be uniquely solved for the function Φ. At a point x the solutions of (5) form the *null cones*.

[11] Assume that initial data with compact support Θ are posed and (locally) a cone $K := \{(t,x) \mid t \geq c|x|, c > 0\}$ is given. Then one can define in the neighbourhood of the initial hypersurface a region R by attaching to each point of Θ the above cone and taking the union of all these cones. Then the intersection of a later hyperplane Σ_t for $t > t_0$ with this region again is compact. Now we can formulate the alternative postulate:

Postulate 3b: (Finite propagation speed) *There is a solution which remains in R.*

By means of this condition we can show that (2) reduces to a linear system of partial differential equations and that the hypersurfaces of the (3+1)-slicing are non-characteristic. Partial differential equations with this property are called *weakly hyperbolic*. Therefore postulate 3b is stronger than the locality postulate.

For the introduction of the conformal structure it is sufficient to restrict oneself to a first order system of a certain field type [33]. Other fields will be treated separately in ch.4.4.

$$0 = i\gamma^\mu\partial_\mu\varphi - M\varphi. \tag{6}$$

Such first order system exists in nature and can therefore be taken as some realisation of our axiomatic scheme [33]. The polarisation set then is defined by $\gamma^\mu k_\mu a = 0$ which can be read as an equation for the eigenvalue k_0. It leads to the characteristic equation $H_c(x,k) = \det(\gamma^\mu k_\mu) = 0$.

3.2 A probability current

For being able to establish a physical interpretation of our classical field theory described by a first order system, and therefore to use it to describe measuring results of quantum mechanical phenomena, we have to introduce a real vector to represent a probability current j^μ. This has to be done with the elements already contained in our theory. The only vector available in our theory is the matrix γ^μ. In addition, the probability should be bilinear in the fields φ. Therefore j^μ must equal $\varphi^+\beta\gamma^\mu\varphi$ with some still open matrix β. The subject of the next postulate is to demand its reality:

Postulate 4: (probability current)
For the first order system there is a matrix β so that $j^\mu = \varphi^+\beta\gamma^\mu\varphi \in \mathbf{R}\ \forall\varphi$.

Since j^μ should be real for all φ we infer $(\beta\gamma^\mu)^+ = \beta\gamma^\mu$.

3.3 The conformal structure

In the following we make statements about the structure of the characteristics and about the jumps A which are solutions of $\gamma^\mu k_\mu A = 0$. Different solutions A of $\gamma^\mu k_\mu A = 0$ which belong to the same k solving $H_c(x,k) = 0$ are called *helicity amplitudes*.

To motivate our next postulate we recall that in nature more than two null cones (one future and one past cone) have never been observed in 'empty space'. This can be demonstrated with matter wave interferometry [43]. In addition, for our first order field equation, there should be no more than two helicity amplitudes:

Postulate 5: (conformal structure)

1. *There are two null cones only, that is, for each $k_{\hat\mu} \neq 0$ there are two different k_0 solving (5) which are non-vanishing and have different sign*[12].
2. *For the first order system (6) there are 2 helicity amplitudes.*

[12] The non-vanishing of k_0 means that the hypersurfaces Σ are non-characteristic. (Therefore, the Schrödinger equation is excluded at this point.) On the other hand, postulate 3b (finite propagation speed) *implies* that the Σ's are non-characteristic. Accordingly if we had stated postulate 3b instead of the locality postulate 3, we would not have had to demand this part of the axiom.

Because there are two helicity amplitudes on the null cone and the matrix $A\gamma^\mu k_\mu$ is hermitean, we know that the multiplicity of the zeros of the characteristic polynomial is two, so that the characteristic polynomial is the square of another one: $H_c(x,k) = (H_0(x,k))^2$. In addition, because there are two solutions k_0 only, the polynomial $H_0(x,k)$ must be of order two: $H_0(x,k) = \overset{c}{g}{}^{\mu\nu}(x)k_\mu k_\nu$. Because the characteristic polynomial now is of order four, we have $s = 4$ and the γ's and the M are complex-valued 4×4-matrices.

Therefore the subsets of $\mathcal{M}$ where singularities in solutions can occur are characterised by the equivalence class $[\overset{c}{g}{}^{\mu\nu}(x)] := \{\overset{c}{g}{}'^{\mu\nu}(x) \mid \overset{c}{g}{}'^{\mu\nu}(x) = \lambda\overset{c}{g}{}^{\mu\nu}(x), \lambda\in\mathbf{R}\}$ which is defined by the k_μ via $\overset{c}{g}{}^{\mu\nu}k_\mu k_\nu = 0$ and which is called *conformal structure*. Chosing another representant of the conformal structure results in a rescaling of the metric which is called *conformal transformation*.

In virtue of the above postulates all metrical tensors are non-singular and have (according to convention) the signature +2 or −2. Therefore there is an inverse metric $\overset{c}{g}_{\mu\nu}$.

The bi-characteristics $v^\mu = \overset{c}{g}{}^{\mu\nu}k_\nu$ fulfill the geodesic equation $v^\nu D_\nu v^\mu = \alpha v^\mu$ for some function α. $D_\nu v^\mu = \partial_\nu v^\mu + \left\{{\mu \atop \nu\sigma}\right\} v^\sigma$ is the covariant derivative with the Christoffel symbol $\left\{{\mu \atop \nu\sigma}\right\} := \frac{1}{2}\overset{c}{g}{}^{\mu\rho}(\partial_\nu\overset{c}{g}_{\sigma\rho} + \partial_\sigma\overset{c}{g}_{\nu\rho} - \partial_\rho\overset{c}{g}_{\nu\sigma})$.

By means of the conformal structure we can introduce the notion of Lorentz transformations: We choose one metric $\overset{c}{g}{}^{\mu\nu}$ from the equivalence class and define corresponding tetrads $\overset{c}{e}{}^\mu_a$ by $\overset{c}{g}{}^{\mu\nu} = \eta^{ab}\overset{c}{e}{}^\mu_a\overset{c}{e}{}^\nu_b$ with $\eta = \mathrm{diag}(+---)$. All transformations $\overset{c}{e}{}^\mu_a \mapsto \overset{c}{e}{}^\mu_{a'} = L_{a'}{}^a\overset{c}{e}{}^\mu_a$ leaving the defining relation invariant are *Lorentz-transformations*. These transformations are characterised by $\eta_{ab}L_c{}^aL_d{}^b = \eta_{cd}$.

4 The Riemannian structure

4.1 The classical limit

In the following we base our axiomatic scheme [32] on basic experiments which can be made in the classical limit of wave mechanics. By *classical limit* we denote the physics of locally approximately plane wave solutions of (4). Such a solution can be decomposed approximately according to $\varphi = ae^{iS}$ into a 'slowly varying' amplitude a and a phase S, so that all terms containing at least one derivative of the amplitude or at least the second derivative of the phase can be neglected[13]. Formally this means that a solution φ of (4) is a *locally approximately plane matter wave* in $x\in\mathcal{M}$ - briefly called *plane matter wave* - if within an appropriate neighbourhood of x there is a field of $\mathbf{C}^s$-bases and a coordinate system as well as functions $S\in C^r(\mathcal{M},\mathbf{R})$ and $a\in C^r(\mathcal{M},\mathbf{C}^s)$ so that it can be represented as

$$\varphi(x) = a(x)e^{iS(x)} \tag{7}$$

[13] A more precise way to arrive at this kind of approximate solution is via Fourier Integral operators, see e.g. [40, 44].

with

$$\left\|\gamma^{\mu\dots\mu}\prod_i(\partial^{\alpha_i}S)\ \partial^\beta a\right\| \overset{*}{\ll} \|a\| \text{ if at least one } |\alpha_i|\geq 2 \text{ or } |\beta|\geq 1 \tag{8}$$

$\|a\|$ is some norm in $\mathbf{C}^s$. Equations which are valid only in this special base and coordinate systems are marked by an $*$.

Our classical limit in contrast to the WKB scheme is not an expansion with respect to the Planck constant $\hbar$, because the field equation (4) does not contain such quantity.

4.2 Approximate plane wave solutions

Inserting (7) into the field equation (4) we get in the system $*$ using (8)

$$\sum_{j=0}^{r}\gamma_{(0)}^{\mu_1\dots\mu_j}(x)p_{\mu_1}\cdots p_{\mu_j}a \overset{*}{=} 0 \tag{9}$$

where $p_\mu := -\partial_\mu S$ is called the ***momentum*** of the plane wave. The coefficients $\gamma_{(0)}^{\mu_1\dots\mu_j}(x)$ are defined to be equal to the coefficients $\gamma^{\mu_1\dots\mu_j}(x)$ of the field equation (4) in the special base and coordinate system $*$.

The transformation properties of the coefficients $\gamma_{(0)}^{\mu_1\dots\mu_j}(x)$ can be derived by means of the requirement, that after a base and coordinate transformation the transformed equation (9) must have the transformed amplitude a' as solution. It turns out that the coefficients $\gamma_{(0)}^{\mu_1\dots\mu_j}(x)$ transform homogeneously.

In regions where there is a classical limit, the amplitude a cannot vanish, so that the solvability condition

$$H(x,p) := \det\Big(\sum_{j=0}^{r}\gamma_{(0)}^{\mu_1\dots\mu_j}(x)p_{\mu_1}\cdots p_{\mu_j}\Big) = 0 \tag{10}$$

must be fulfilled. This equation, the ***Hamilton-Jacobi-equation***, is a polynomial of order rs in the momentum p_μ. (10) corresponds to the eikonal equation of geometrical optics. For given $p_{\tilde\mu}, \tilde\mu = 1,2,3$, equation (10) can be (not necessarily uniquely) solved for $p_0 = f(x,p_{\tilde\mu})$. (10) is a complex equation and is invariant against coordinate transformations and transformations of the $\mathbf{C}^s$-bases.

4.3 Local Lorentz-isotropy

By means of the orthotetrads $\overset{c}{e}{}^\mu_a$ it is possible to formulate the Hamilton-Jacobi-equation in terms of the $p_a := \overset{c}{e}{}^\mu_a p_\mu$, that is $H(x,p_\mu) = H(x,\overset{c}{e}{}^a_\mu p_a) =: \widehat{H}(x,p_a)$. This equation can be solved for $p_{(0)} := \widehat{f}(x,p_{\tilde a})$.

Having already established a conformal structure[14] on the manifold and the related equivalence class of metrics $[\overset{c}{g}_{\mu\nu}(x)]$, some elementary measurements are operationally possible. For

[14] In [32] we took over the conformal structure of EPS [3]. Note that because of ch.3 this is not necessary in the complete scheme here.

two different plane matter waves φ and φ' the ratio of the related Lorentz components $\frac{p_a}{p'_a}$ has an invariant meaning and represents a measuring quantity related in the usual way to the phase function $S(x)$ and the succession of hyperplanes of constant phase. For index $a = 0$ it corresponds to a ratio of frequencies, for $a = \hat{a}$ it corresponds to a ratio of wavelenghts and a propagation direction relative to the orthotetrad.

On the basis of this, it is possible to formulate two additional basic experiences made with a subclass of all fields containing the *plane matter waves* which are *free*. The first one is the following: In an event and its neighborhood it is possible to find a plane matter wave and to arrange the experimental setup in such a way, that *active* Lorentz transformations transport the whole arrangement including the plane wave into an equally possible arrangement. As in corresponding experiments with free point particles (as opposed to interacting particles), this may in practice need some shielding. Loosely speaking one could say that the following is demanded: If all the direction dependent external influences which can be eliminated are indeed eliminated, then that what remains as structure allows that an active Lorentz transformation of the experimental setup leads to one which can also physically be realised. This characterises the 'remaining structure' which is related to the geometry of space-time.

Postulate 6: (isotropy)

Given an event and a free plane matter wave with momentum p_a, then the momentum $p_{a'}$ in this event obtained by an active Lorentz transformation $p_a \mapsto p_{a'} := L_a^b p_b$ belongs to an equally possible free plane matter wave:

$\widehat{H}(x, L_a^b p_b) = 0 \quad \forall L_a^b$ *fulfilling* $L_a^b L_c^d \eta_{bd} = \eta_{ac}, \forall p_b$ *fulfilling* $\widehat{H}(x, p_b) = 0$.

Therefore $\widehat{H}(x,p) = 0 \Leftrightarrow \widehat{H}(x, Lp) = 0$. For free plane matter waves the respective Hamilton functions must be invariant under Lorentz transformations. This has the important consequence that according to the fundamental theorem of vector invariants of the Lorentz group, $H(x,p)$ can only be a function of $\overset{c}{g}{}^{\mu\nu}(x)p_\mu p_\nu$. In this case the polynomial Hamilton-Jacobi-equation (10) must have the structure

$$H(x,p) = \prod_{k=1}^{\frac{mr}{2}} \left(\overset{c}{g}{}^{\mu\nu}(x)p_\mu p_\nu - V_{(k)}(x) \right) \tag{11}$$

with some complex scalar functions $V_i(x)$ which do not depend on p. Chosing another representant of $[\overset{c}{g}{}^{\mu\nu}]$, that is making a rescaling of $\overset{c}{g}{}^{\mu\nu}$, results in a rescaling of $V_{(i)}$ also. These functions depend in a complex way on the $\gamma^{\mu_1 \cdots \mu_i}(x)$ of (9). We call the $V_{(k)}(x)$ *scalar mass potentials* of the field $\varphi(x)$. Whenever locally approximately plane wave solutions of (4) are possible which are in addition free, they must fulfill (11) and these mass potentials must exist. Together with the class $[\overset{c}{g}{}^{\mu\nu}(x)]$ of metrics they determine the phase functions $S(x)$ as solutions of (10). The functions $[\overset{c}{g}{}^{\mu\nu}(x)]$ and $V_{(k)}(x)$ together characterise the geometry of the classical limit of (4).

By grouping together identical factors in (11), we can write

$$H(x,p) \;=\; \prod_{i=1}^{r'} \left(H_{(i)}(x,p) \right)^{\alpha_{(i)}} \tag{12}$$

$$H_{(i)}(x,p) : \;=\; \overset{c}{g}{}^{\mu\nu}(x)p_\mu p_\nu - V_{(i)}(x) = 0 \tag{13}$$

whereby the powers have to fulfill $\sum_{i=1}^{r'} \alpha_{(i)} = \frac{rs}{2}$. Since $\overset{c}{g}{}^{\mu\nu}(x)$ and p_μ are real, $V_{(j)}(x)$ must be real too.

4.4 The conformal structure for the other field equations

Before proceeding in our axiomatics we derive the conformal structure of the general field equations (4) which are not of first order.

Firstly we observe that the heigh energy limit of the Hamilton-Jacobi equation (11) is $H(x,p) = \prod_i (\overset{c}{g}{}^{\mu\nu} p_\mu p_\nu)^{\alpha_{(i)}}$. On the other hand, the heigh energy limit of the Hamilton-Jacobi equation is proportional to the characteristic polynomial. Therefore $H_c(x,k) = \beta \prod_i (\overset{c}{g}{}^{\mu\nu} p_\mu p_\nu)^{\alpha_{(i)}}$ for some proportionality factor β. Therefore the characteristics of the general field equations (4) agree with those of the first order system of ch.3. *All field equations (4) define the same conformal structure.* Note that if we had chosen another field equation to establish the conformal structure, we would have obtained the same conformal structure. Postulate 6 guarantees the self-consistency of our scheme.

4.5 Constancy of the ratios of mass potentials and Riemann space.

Postulate 6 is not sufficient to single out 'free' waves, because only direction dependent influences are excluded. The potentials $V_{(i)}(x)$ may still contain in addition to mass parameters the contributions from isotropic external fields. To complete the characterisation 'free', we must describe the physics obtained after a successful 'shielding' of these directional independent influences too. The influence which is commonly called the gravitational one, cannot be shielded and is therefore contained in the geometry of free plane matter waves.

It is well known and has been demonstrated in the COW type experiments (for a review see e.g. [1]), that interference of plane matter waves in gravitational fields lead to mass dependent results. This is in contrast to the behaviour of free test particles on which, according to the equivalence principle, mass has no influence (which does not mean that there is no equivalence principle in the quantum domain, see [1]). *This sensitivity with regard to mass makes matter waves superior to test particles as primitive objects in a space-time axiomatics.* Returning to our axiomatic scheme this means that additional information can be extracted from interference phenomena.

Each type of quantum object φ (labeled by λ) characterised by a respective field equation (4) may lead to different scalar mass potentials $V_{(i)}^{(\lambda)}, \lambda, i = 1, 2, \ldots$. Based on matter wave interferometry we take as basic experiences with free plane waves that for the same physical set-up (the same interferometer apparatus under identical conditions) the pattern of interference fringes, upp to a constant rescaling, are identical. This means that for all λ, λ', i, i' the ratio $V_{(i)}^{(\lambda)}(x)/V_{(i')}^{(\lambda')}(x)$ of these potentials turns out to be the same in every space-time event. Accordingly we demand the universality of the mass function:

Postulate 7: *For free plane matter waves the ratio of any two scalar mass potentials proves to be constant.*

For $V_{(i)}(x) = 0$ equation (13) describes null rays. Fields with at least one $V_{(i)}(x) \neq 0$ will be called *massive*. Because of postulate 7 one can take one of the non-vanishing potentials as universal function $V_{(0)}(x)$ and write for the other potentials

$$\left|V_{(i)}(x)\right| = m_{(i)}^2 \left|V_{(0)}(x)\right| \tag{14}$$

with real positive constants $m_{(i)}$. It is the universality of (14) which guarantees the absence of external (in the sense of non-gravitational) influences. Note that negative $V_{(i)}(x)$ indicating tachyonic behaviour are not excluded. The constants $m_{(i)}$ are called *masses*. One field equation (4) can lead to several masses.

We can therefore conclude from the postulates the result: The space-time manifold $\mathcal{M}$ is endowed with a class of metrics $[\overset{c}{g}_{\mu\nu}(x)]$ and a universal mass function $V_{(0)}(x)$. Therefore we can define a unique conformally invariant *metric*

$$g_{\mu\nu}(x) := \frac{1}{V_{(0)}(x)} \overset{c}{g}_{\mu\nu}(x) \tag{15}$$

and the manifold $\mathcal{M}$ becomes a *Riemann space*[15]. This result does not mean that other geometrical fields like torsion are vanishing, they have simply not yet been established.

5 Establishing space-time torsion

In the following we restrict ourselves to massive first order systems from which the conformal structure has been derived. From these fields the propagation of the spin can be derived which in turn is used to introduce another geometric entity, namely a space-time torsion (comp. [30]).

[15] In an different approach postulate 7 can be replaced by one using the notion of paths of wave packets and their group velocity defined by $v^\mu \sim \partial H(x,p)/\partial p_\mu$, and requiring: All wave packets out of massive free plane matter waves follow the same paths. This again results in a *Riemann space*. Therefore we can alternatively state: *The geometry of bi-characteristics and wave packets is a Riemannian space-time.* More details can be found in [32].

5.1 The spin states

If we insert $\varphi(x)$ of (7) into the first order system (6) we get the following two equations for approximately plane matter waves:

$$\begin{aligned} 0 &= (\gamma^i p_i - M^{(0)})a\,, && (16)\\ i\gamma^i \partial_i a &= M^{(1)} a\,, && (17) \end{aligned}$$

for some 4×4-matrices $M^{(0)}$ and $M^{(1)}$ where $M^{(0)}$ transforms homogeneously. The solvability condition of the first equation gives a polynomial of fourth order in p, namely the Hamilton-Jacobi equation $H(x,p) = (g^{\mu\nu}p_\mu p_\nu + m_{(1)}^2)(g^{\mu\nu}p_\mu p_\nu + m_{(2)}^2)$ with real $m_{(1)}$ and $m_{(2)}$. The *spin states a* which will now become important are solutions of (16) corresponding to a particular solution p of the Hamilton-Jacobi equation.

With the help of the next postulate it is possible to derive the Clifford algebra and related notions:

Postulate 8: (spin states)
For each momentum there are two spin states.

This requirement implies that $H(x,p)$ should be the square of another polynomial leading to $m_{(1)} = m_{(2)} =: m = const \neq 0$, therefore

$$H(x,p) = (g^{\mu\nu}p_\mu p_\nu - m^2)^2\,. \tag{18}$$

For convenience, in the following we chose $m = 1$.

5.2 The Clifford algebra

The right hand side of (18) is the determinant of $\gamma^\mu p_\mu - M^{(0)}$. Given some matrix, its determinant is given by multiplication of this matrix with its minor (see e.g. [45]). Therefore, there is a minor $\tilde{B}$, so that $\tilde{B}\left(\gamma^\mu p_\mu - M^{(0)}\right) = (g^{\mu\nu}p_\mu p_\nu - 1)^2$. In addition, one can show that, if the determinant has multiple zeros of a certain degree d, then the minor is proportional to the $(d-1)$st power of this zero. In our case there is another matrix B, polynomial in p_μ, with $\tilde{B} = (g^{\mu\nu}p_\mu p_\nu - 1)\, B$. Therefore we get

$$B(\gamma^\mu P_\mu - M^{(0)}) = g^{\mu\nu}p_\mu p_\nu - 1. \tag{19}$$

Since the right hand side is a polynomial of order 2 in p_μ, B must be a polynomial of order 1: $B = B(x,p) = B^\mu(x)p_\mu + B^0(x)$. Inserting this B into (19) and equating the coefficients of the respective powers of p_μ gives $B^0 = (M^{(0)})^{-1}$, $B^\mu = B^0\gamma^\mu B^0$ and therefore the Clifford algebra:

$$\frac{1}{2}(\hat{\gamma}^\mu\hat{\gamma}^\nu + \hat{\gamma}^\nu\hat{\gamma}^\mu) = g^{\mu\nu} \tag{20}$$

with $\hat{\gamma}^\mu := B^0\tilde{\gamma}^\mu$. By this the $\hat{\gamma}^\mu$ became the usual Dirac matrices. The complete set of matrices $\mathbf{1}, i\gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ (omitting the hat) is called *Dirac algebra*. In addition, we get $B = \gamma^\mu p_\mu + 1$ and the matrix β from postulate 4 can be chosen to be $\gamma^{(0)} := e_\mu^{(0)}\gamma^\mu$ whereby $e_\mu^a e_\nu^b g^{\mu\nu} = \eta^{ab}$.

5.3 The propagation of the spin states

From (17) it is possible to derive a propagation law for the spin states a of the form $v^\mu\partial_\mu a = v^\mu f_\mu(x)a$ where v^μ is the *group velocity* of the wave packet, comp. footnote 15. For doing so we introduce a spin-connection Γ_μ by the relation

$$0 = D_\nu\gamma^\mu := \partial_\nu\gamma^\mu + \{ {}^{\;\mu}_{\nu\sigma} \}\gamma^\sigma + [\Gamma_\nu, \gamma^\mu] \tag{21}$$

and a covariant derivative $D_\mu a := \partial_\mu a + \Gamma_\mu a$. (17) can then be rewritten as

$$i\gamma^\mu D_\mu a = (M^{(1)} - i\gamma^\mu\Gamma_\mu)a. \tag{22}$$

Multiplication with B gives the propagation law for the spin states

$$v^\mu D_\mu a = \frac{1}{2}g^{\mu\nu}D_\mu p_\nu a + (\gamma^\nu p_\nu + 1)Ka \tag{23}$$

where $K := \frac{1}{2}(M^{(1)} - i\gamma^\mu\Gamma_\mu)$ is some 4×4 matrix which can be expanded with respect to the Dirac algebra: $K = k^0\mathbf{1} + \bar{k}i\gamma_5 + \bar{K}_\mu\gamma^\mu + K_\mu\gamma_5\gamma^\mu + K_{\mu\nu}2iG^{\mu\nu}$. D_μ acting on vectors denotes the usual covariant derivative in Riemann space based in the Christoffel symbols.

5.4 The propagation of the spin

We define the bilinear forms

$$\rho := \bar{a}a,\quad P := \bar{a}i\gamma_5 a,\quad j^\mu := \bar{a}\gamma^\mu a,\quad S^\mu := \bar{a}\gamma_5\gamma^\mu a,\quad S^{\mu\nu} := \bar{a}i\gamma^{[\mu}\gamma^{\nu]}a \tag{24}$$

with $\bar{a} := a^+\beta$, and get from (16) as independent equations

$$P = 0,\quad \rho v^\mu = j^\mu,\quad S^{\mu\nu} = \epsilon^{\mu\nu\rho}{}_\sigma p_\rho S^\sigma. \tag{25}$$

Therefore $S^{\mu\nu}$ can be derived from P and S^σ. We can show especially that $\rho \neq 0$ for approximately plane waves.

Because of (25) the only independent entities are the group velocity v^μ, and the normalised *spin-vector* $\hat{S}^\mu := S^\mu/\rho$. With (23) we get as propagation equation for the goup velocity a geodesic equation and for the normalised spin vector

$$v^\nu D_\nu\hat{S}^\mu = v^\nu\epsilon_{\nu\rho}{}^{\mu\sigma}K_\sigma\hat{S}^\rho - (g^{\mu\nu} - v^\mu v^\nu)\,K_{[\nu\sigma]}\hat{S}^\sigma. \tag{26}$$

This shows that the dynamics of the spin vector is influenced by, and therefore sensitive to, K_μ and $K_{[\mu\nu]}$ only.

The first term on the right hand side can be identified with an axial torsion[16]. The second term is an external torque which cannot be reformaulated as part of any connection bacause a connection term must be linear in the group velocity and the spin vector.

Because of the last equation we can state the following result: A first order system characterised by means of the axioms 1 to 8, *defines a Riemann-Cartan geometry with axial torsion.*

In other words: *If spacetime is the entity which prescribes the behavior of the characteristics, of the free matter waves, and of the spin states in the way specified above, then spacetime is a Riemann-Cartan space-time with axial torsion.*

6 Concluding remarks

We have shown that it is possible to build up a constructive axiomatic scheme for the space-time geometry in the domain of quantum and of classical physics using quantum objects as primitive objects and formalising some fundamental quantum experiences. We have reconstructed in this way the Riemann-Cartan space-time which is generally regarded as the appropriate geometry. Concerning the experiences we refered to the WKB limit of quantum mechanics which is today being explored in an increasing number of matter wave experiments. Accordingly these experiments are so rudimentary and general that they would be compatible with different theoretical elaborations of quantum mechanics. The details of the procedure above show that the lasting influence of mass and spin in this limit make the quantum objects the more sensitive probes for exploring space-time geometry as compared to point particles and light rays. This advantage is structurally correlated with the disadvantage that quantum objects are less simple from the operational point of view, and that fields as primitive objects are based on something spread out which can locally not be 'touched' or 'seen'. Experiences are in this case directly related only to concepts derived from fields. It is on the other hand very satisfying that space-time axiomatics can be based on quantum objects, which are today the most fundamental and elementary entities from the physical point of view.

[16] For theories with torsion see e.g. [1] and the article of F.-W. Hehl in this volume.

References

[1] Audretsch, J.; Hehl, F.-W.; Lämmerzahl, C. (1992): Matter Wave Interferometry and why Quantum Objects are Fundamental for Establishing a Gravitational Theory, to appear in: Ehlers, J., Schäfer, G.: *Proceedings of the Bad Honnef School on Gravitation*, Springer Lecture Notes in Physics.

[2] Synge, J.L. (1956): *Relativity: The Special Theory*, North Holland, Amsterdam. (1964): *Relativity: The General Theory*, North Holland, Amsterdam.

[3] Ehlers, J., Pirani, F.A.E., Schild, A. (1972): The Geometry of Free Fall and Light Propagation, in: L. O'Raifeartaigh (ed.): *General Relativity, Papers in Honour of J.L. Synge* (Clarendon Press, Oxford).

[4] Ehlers, J. (1973): Survey of General Relativity Theory, in W. Israel (ed.): *Relativity, Astrophysics and Cosmology*, Reidel, Dordrecht, p.1.

[5] Köhler, E. (1978): Measurement of proper time in Weyl space, *Gen. Rev. Grav.* **11** 953. Perlick, V. (1987): Characterisation of Standard Clocks by Means of Light Rays and Freely Falling Particles, *Gen. Rel. Grav.* **19**, 1059.

[6] Pfister, H. (1983): Grundlegung der Allgemeinen Relativitätstheorie mittels Lichtstrahlen und freien Teilchen, *preprint*, Univ. Tübingen.

[7] Reichenbach, H. (1924): *Axiomatik der relativistischen Raum-Zeit-Lehre*, Vieweg-Verlag, Braunschweig.

[8] Reichenbach, H. (1925): Über die physikalischen Konsequenzen der relativistischen Axiomatik, *Z. Physik* **34**, 32.

[9] Coleman, R.A., Korte, H. (1980): Jet bundles and path structures, *J. Math. Phys.* **21**, 1340; Erratum: **23** 345 (1982).

[10] Coleman, R.A., Korte, H. (1984): *A Realist Field Ontology of the Causal Inertial Structure, (The Refutation of Geometric Conventionalism)*, preprint, forthcoming volume of *Synthese Library*, Reidel, Dordrecht.

[11] Audretsch, J., Mainzer, K. (edts.) (1988): *Philosophie und Physik von Raum und Zeit*, B.I. Wissenschaftsverlag, Mannheim.

[12] Coleman, R.A., Korte, H. (1987): Any physical monopole equation of motion structure uniquely determines a projective structure and an $(n-1)$-force, *J. Math. Phys.* **28**, 1492.

[13] Weyl, H. (1921): Zur Infinitesimalgeometrie: Einordnung der projektiven und konforman Auffassung, *Nachr. d. Königl. Ges. d. Wiss. Göttingen, Math.-phys. Klasse* 99.

[14] Pirani, F.A.E. (1972): Building space-time from light rays and free particles, *Symposia Mathematica*, p.67, Rom.

[15] Ehlers, J. (1973): The Nature and Structure of Space-time, in Mehre, J. (ed.): *A Physicist's conception of Nature*, Reidel, Dordrecht.

[16] Ehlers, J. (1988): Einführung der Raum-Zeit Struktur mittels Lichtstrahlen und Teilchen, in [11].

[17] Audretsch, J. (1983): The Riemannian Structure of Space-time as a Consequence of Quantum Mechanics, *Phys. Rev.* **D 27**, 2872. Audretsch, J., Gähler, F., Straumann, N. (1984): Wave Fields in Weyl-Space and Conditions for the Existence of a Preferred Pseudo-Riemannian Structure, *Comm. Math. Phys.* **95**, 41.

[19] Kasper, U. (1986): On the importance of quantum mechanics for the axiomatic approach to the theory of gravitation, in *Proceedings of the Conference on Differential Geometry and its Applications*, Brno, CSSR.

[20] Woodhouse, N.M.J. (1973): The Differential and Causal Structure of Space-time, *J. Math. Phys.* **14**, 495.

[21] Schröter, J. (1988): Zur Axiomatik der Raum-Zeit, p.121, in [11]

[22] Schröter, J. (1988): An axiomatic basis of space-time theory, part I: construction of a causal space with coordinates, *Rep. Math. Phys.* **26**, 305. Schröter, J., Schelb, U. (1992): An axiomatic basis of space-time theory, part II: construction of a C^0-manifold, will apear in *Rep. Math. Phys.*

[23] Meister, R. (1991): *Eine Neuformulierung der EPS-Axiomatik*, diploma thesis, Paderborn.

[24] Heilig, U., Pfister, H. (1991): Characterization of free fall paths by a global or local Desargues property, *JPG* **7**, 419.

[25] Ehlers, J., Köhler, E. (1977): Path Structure on Manifolds, *J. Math. Phys.* **18**, 2014.

[26] Coleman, R.A., Korte, H. (1984): Constraints on the nature of inertial motion arising from the universality of free fall and the conformal causal structure of space-time, *J. Math. Phys.* **25**, 3513.

[27] Coleman, R.A., Korte, H. (1981): Space-time G-structures and their prolongations, *J. Math. Phys.* **22**, 2598.

[28] Ewen, H., Schmidt, H.-J. (1989): Geometry of Free Fall and Simultaneously, *J. Math. Phys.* **30** 1480.

[28] Audretsch, J., Lämmerzahl, C. (1988): Constructive Axiomatic Approach to Space-time Torsion, *Class. Quantum Grav.* **5**, 1285.

[29] Audretsch J., Lämmerzahl C.(1990): Establishing space-time geometry by means of light rays and particle interferometry, *Wiss. Zeitschrift der Universitt Jena, Naturwiss. Reihe*, Heft 1.

[30] Lämmerzahl, C. (1990): The Geometry of Matter Fields, in: Audretsch, J., deSabbata, V. (eds.): *Quantum Mechanics in Curved Space-time* (Plenum Press, New York).

[31] Audretsch J., Lämmerzahl, C. (1991): Reasons for a physical field to obey partial differential equations, *J. Math. Phys.* **32**, 1354.

[32] Audretsch, J., Lämmerzahl, C. (1991): Establishing the Riemannian structure of space-time by means of light rays and free matter waves, *J. Math. Phys.* **32**, 2099.

[33] Audretsch, J., Lämmerzahl, C. (1992): The conformal structure of linear field equations, *in preparation.*

[34] Hille, E., Phillips, R.S. (1957): *Functional Analysis and Semi-Groups*, American Mathematical Society.

[35] Shimony, A. (1979): Proposed neutron interferometer test of some nonlinear variants of wave mechanics, *Phys. Rev.* **A 20**, 394. Shull, C.G., Atwood, D.K., Arthur, J., Horne. M.A. (1980): Search for a non-linear variant of the Schrödinger equation, *Phys. Rev. Lett.* **44**, 765. Rauch, H. in Kamefuchi. S. (ed.): *Proceedings of the International Symposium on Foundations of Quantum Mechanics*, Tokyo (1983).

[38] Weinberg, S. (1989): Testing Quantum Mechanics, *Ann. Phys.* **194**, 336.

[39] Peetre, J. (1960): Rectification à l'article "Une charactérisation abstraite des opérateurs différentiels", *Math. Scand.* **8**, 116.

[40] Dieudonné, J. (1971): *Élements d'Analyse*. Gauthier-Villars, Paris.

[41] Hörmander, L. (1984): *The Analysis of Linear Partial Differential Opertors*, Springer-Verlag, Berlin.

[42] Dencker, N. (1982): On the Propagation of Polarization Sets for Systems of Real Principal Type, *J. Funct. Anal.* **46**, 351.

[43] Audretsch, J., Bleyer, U., Lämmerzahl, C. (1992): Testing Lorentz invariance in matter wave interferometry, *in preparation.*

[44] Treves, F. (1980): *Introduction to Pseudodifferential Fourier Integral Operators*, Plenum Press, New York. Taylor, M. (1981): *Pseudodifferential Operators*, Princeton University Press, Princeton, New Jersey. Nicoleau, F. (1992): Approximation semi-classique du propagateur d'un système électromagnétique et phénomène de Aharonov-Bohm, *Helv. Phys. Acta* **65**, 722.

[45] Gantmakher, F.R. (1959): *Applications of the Theory of Matrices*, Interscience Publ.