Fermilab Library
0 1160 0023146 8

KONS-RGKU-93-02

Relativity Group • Konstanz University

⟨ Theory of Gravity | Quantum Theory ⟩

Preprint KONS-RGKU-93-02

16

Influence of external potentials on optical Ramsey fringes in atomic interferometry [1]

Jürgen Audretsch and Karl-Peter Marzlin

Fakultät für Physik
der Universität Konstanz
Postfach 5560
D-7750 Konstanz, Germany [2]

Konstanz U.

Abstract

Interaction of an atomic beam with four traveling laser beams leads to optically induced Ramsey fringes in the atomic beam. This experimental setup has been used as an optical frequency standard as well as a matter-wave interferometer. We calculate for 2-level atoms the shift of the fringes due to the influence of fairly general external forces in lowest order of a perturbation approach. A comparison with the phase shift found in a corresponding WKB treatment of the interaction geometry is made. The consequences of beam reversal and cooling of the atomic beam are studied. As application we treat the linear acceleration and rotation of the interferometer and discuss the influence of spacetime curvature.

[1] To appear in Physical Review A.
[2] e-mail: peter@spock.physik.uni-konstanz.de

1 Introduction

It has been pointed out by Bordé et al. [1] that the interaction of two level atoms or molecules or ions passing successively two counterpropagating pairs of separated copropagating traveling laser waves leads to optical Ramsey fringes in the atomic beam. The respective times of influence are shown in fig.1. The laser beams influence the atoms only in the *laser zones* during the time τ. Between the laser zones the atoms travel for the times T and T' in the *dark zones*. In the four laser zones there

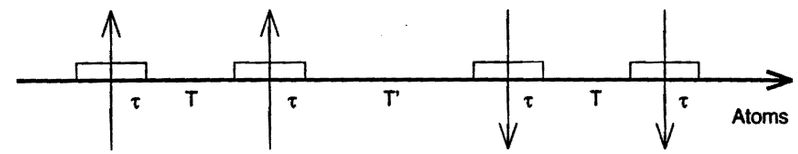

Figure 1: An atomic beam passing successively two counterpropagating pairs of copropagating traveling laser waves. The interaction time with the laser beam is τ. The flight time in the dark zones are T and T'.

is, because of absorption and emission processes, an exchange of energy and related with this an exchange of momentum. Accordingly in each laser zone a matter wave is coherently split into two components with discrete values of the momentum (comp. fig. 2). Denoting the internal energy states by $|a>$ and $|b>$ and the states of external momentum by $|\vec{p}>$, the laser with wave vector $\vec{k} = k\vec{e}_3$ splits in the first laser zone the atomic beam $|\vec{p}> \otimes |a>$ entering the setup in the x_1 direction into the two beams $|\vec{p}> \otimes |a>$ and $|\vec{p}+\vec{k}> \otimes |b>$. As far as it proves to be important in the following, the second zone and the rest of the interaction geometry can be read off from fig. 2, where $|\vec{p}+n\vec{k}> \otimes |a>$ is denoted by a_n (or b_n for the internal state $|b>$, respectively) and the extensions of the laser zones are neglected. In this paper we will use natural units ($\hbar = c = 1$) if not otherwise stated. After having passed the last laser zone, the population in the excited states, the fluorescent decay of which is detected, oscillates with laser detuning and atomic beam velocity. These oscillations are registered and called the *Ramsey fringes*. They result from interference of matter waves which have propagated along two different paths. For early experimental realizations of the setup see Bordé et al. [2] and Helmcke et al. [3].

The case that the atoms propagate freely in the dark zones has theoretically been treated in detail in ref. [1]. One finds after a final integration over the distribution of the initial velocity components $v_3^{(D)}$ in the direction x_3, that in fact only the two interferometer geometries of fig. 3 contribute to the Ramsey fringes. Each of these geometries corresponds thereby to one recoil component of the Ramsey fringes. Apart from a background, the resulting Ramsey fringes can thereby be described by a bilaterally damped cosine

$$\cos[2(\Delta\omega \pm \delta)D/v_1 + \Delta\varphi + \phi] \tag{1}$$

$\Delta\omega = \omega - \omega_0$ is the detuning with the atomic transition frequency $\omega_0 = E_b - E_a$ and the laser frequency ω. $\delta = k^2/(2M)$ denotes the recoil shift caused by the exchanged momentum $\vec{k}$ with M being the mass of the atom. The sign of δ refers to the two different interferometer geometries of fig.

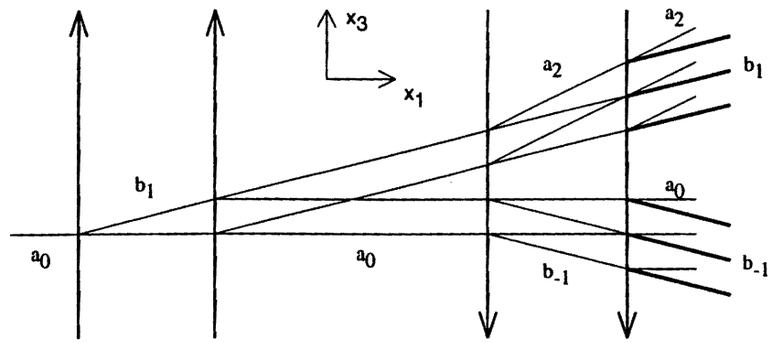

Figure 2: The interaction with the laser beams $\pm\vec{k}$ lead to coherent splits of the beam of two-level atoms ($|a>$, $|b>$). a_n and b_n denote energy and momentum of the atoms: $a_n \leftrightarrow |\vec{p}+n\vec{k}> \otimes |a>$, $b_n \leftrightarrow |\vec{p}+n\vec{k}> \otimes |b>$.

3. D is the distance between the two copropagating laser beams, v_1 the atom velocity. The flight time in the respective dark zone is $T = D/v_1$. The phase alignment $\Delta\varphi = \varphi_4 - \varphi_3 + \varphi_2 - \varphi_1$ is the difference of the phases φ_r in the four laser zones. In the result (1) the 2nd order Doppler effect ($\sim v^2/c^2$) has been neglected. If apart from the four laser beams acting in the four laser zones there is strictly no influence on the atomic beam, the phase contribution ϕ included in eqn. (1) vanishes. It is the aim of the article to show in detail how the influence of external potentials leads to specific nonvanishing phase shifts ϕ.

If one may disregard the recoil splitting and assume perfect phase alignment ($\Delta\varphi = 0$) as well as absence of external disturbances ($\phi = 0$), the maximum of the damped cosine (1) is given by the transition frequency ω_0. This demonstrates the first important application of this scheme of Ramsey excitations: It represents a *frequency standard* operating in the optical range. For example, using the intercombination transition $^3P_1 - {}^1S_0$ of ^{40}Ca, the respective wavelength is $\lambda = 657.46nm$. See Morinaga et al. [4] for details and further literature on frequency stabilization by means of Ramsey excitation. For this application, an additional phase shift ϕ in (1) would represent unwanted errors. It is important to know their theoretical structure in order to eliminate them.

Since the first atomic interferometer has been built by Carnal and Mlynek [5], several alternative setups have been developed [6, 7, 8]. In this context a second important domain of application of the experimental setup described above is found. As has been pointed out by Bordé [9], the two closed arrays of coherent matter beams (comp. fig. 3) can be considered as two *atomic beam interferometers*. The laser zones represent *beam splitters* which provide coherent beam splits and reflections. Together with changes in the internal degrees of freedom they are able to separate the beams not only in momentum space but also in the physical space of the atoms. These beam splitters are based on the physically rather 'pure' effect of the atomic recoil in a traveling laser beam. This fact has the

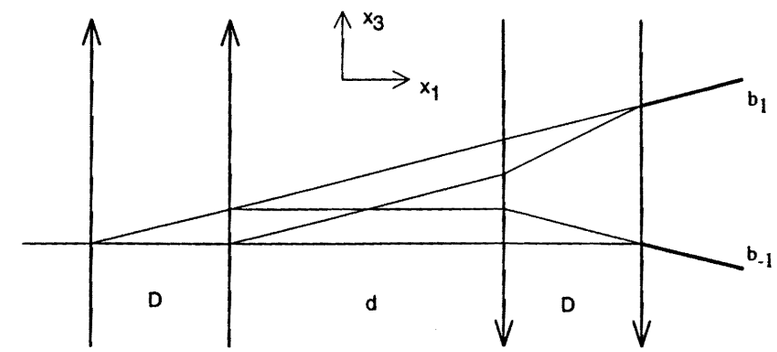

Figure 3: The interaction geometry represents two matter wave interferometers

big advantage that the respective dynamics in the beam splitter is well defined in detail and can be treated theoretically in a transparent way (comp. chapt. 3 below). Influences of external potentials on the two atomic beams of the interferometer lead in general to a nonvanishing additional phase ϕ in (1) and accordingly to a shift of the Ramsey fringes, by which the interaction of the respective forces with matter waves may be detected. An example is the Sagnac effect for matter waves, caused by the rotation of the reference system, which has been measured in this way by Riehle et al. [6]. Among other reasons, atomic beam interferometry is of importance because it offers new possibilities for the experimental verification of the influence of gravity and inertia on quantum matter. This is fundamental for general relativity (comp. Audretsch et al. [10], [11]).

The description of the experimental setup given above shows that the investigation of disturbing influences causing an unwanted shift in the frequency standard on one hand and the study of induced phase shifts in matter wave interferometry on the other represent only two aspects of the same physical problem: Study of the shift $\phi \neq 0$ of the Ramsey fringes caused by the influence of external potentials. Both aspects can be treated in the same calculation. It is the aim of this article to do this for a fairly general type of external forces. The results for linearly accelerated frames (or equivalently homogeneous gravitational fields) and for rotating frames have already been obtained by Bordé [12], compare also ref. [9].

The article is organized as follows: As starting point the general dynamics of the matter beams is described in chapt. 2. The discussion of the influence of external potentials is based on a perturbation scheme. Details of the scheme including an idealization of the beam splitters are given in chapt. 3. The resulting shifts $\phi^{(1)}$ of the Ramsey fringes are worked out in chapt. 4. In chapt. 5 a comparison with a WKB approach to matter wave interferometers is made. Finally, applications are discussed in chapt. 6.

2 Dynamics of the matter field

We follow ref. [1] and model the atomic beam by 2-level atoms with internal energy states $|a>$ and $|b>$ whereby $E_a < E_b$. The dynamic is specified by the Hamiltonian

$$H = H^{(0)} + H^{(1)} = \frac{\vec{P}^2}{2M} + H_{2lev} + \vec{d}\cdot\vec{E} - \frac{i}{2}\Gamma + H^{(1)} . \tag{2}$$

$\vec{P}$ is the momentum operator acting on the cms coordinates of the atom. H_{2lev} describes the internal degrees of freedom, i.e. $H_{2lev}|a>= E_a|a>$, $H_{2lev}|b>= E_b|b>$. The coupling to the electromagnetic field in dipole approximation in the laser zones is given by $\vec{d}\cdot\vec{E}$, $\vec{d}$ is the (internal) dipole operator with $<a|\vec{d}|a>=<b|\vec{d}|b>= 0 \neq <a|\vec{d}|b>=<b|\vec{d}|a>$. Γ describes the decay of the states: $\Gamma|a>=\gamma_a|a>$, $\Gamma|b>=\gamma_b|b>$. We take for the electric fields laser beams running along the $\pm x_3$ direction with polarization such that $<a|\vec{d}\cdot\vec{E}|b>= 2\Omega_{ab}\cos(\omega t \mp k(x_3 + v_3^{(D)}t) + \varphi_r)$ where Ω_{ab} is the Rabi frequency and ω is the laser frequency. φ_r are the phases of the laser beams. The term proportional to $v_3^{(D)}$ makes sense only in a frame of reference where the atom is at rest with respect to the x_3 direction. In such a frame it produces the first order Doppler shift of the laser fields as seen by the atoms. Throughout the paper we neglect all special relativistic effects.

The additional term $H^{(1)}$ in the Hamiltonian of eqn. (2) represents either the external disturbances of the frequency standard or the external potentials to be registered by matter wave interferometry. In the following we will study in a perturbation approach influences of the fairly general type

$$H^{(1)} = \hat{H}(\vec{P})\,(x_1)^{N_1}(x_2)^{N_2}(x_3)^{N_3} \tag{3}$$

with integer N_i. The atomic dipole moment $\vec{d}$ and the energy levels are thereby assumed to remain unchanged. The laser field $\vec{E}$ is regarded to be fixed. The Hamiltonian $H^{(1)}$ of eqn. (3) has the following property: It does not affect the internal degrees of freedom of the atoms, for example it can not cause transitions ($<a|H^{(1)}|b>= 0$). Note that it is not excluded that for charged matter like ion beams $H^{(1)}$ may depend on electromagnetic fields including the laser beams influencing the center of mass motion only. Since we will use first order perturbation theory with regard to $H^{(1)}$, our method holds also for linear combinations of different perturbations of the type (3).

To obtain the exact Schrödinger equation in momentum space, we decompose the matter state with regard to momentum eigenstates $|\vec{p}>$.

$$|\psi>= \int d^3p'\{a_{\vec{p}'}(t)|\vec{p}'>\otimes|a> + b_{\vec{p}'}(t)|\vec{p}'>\otimes|b>\} \tag{4}$$

Assuming the detuning $\Delta = \omega - \omega_0$ to be small compared with $\omega_0 = E_b - E_a$, we get after projecting with $<a,\vec{p}|$ and $<b,\vec{p}|$ for the Schrödinger equation in the momentum representation

$$\begin{aligned}
\dot{a}_{\vec{p}} &= \{-iE_a - i\frac{\vec{p}^2}{2M} - \frac{\gamma_a}{2}\}a_{\vec{p}} - i\Omega_{ab}e^{i(\omega t \mp k v_3^{(D)} t + \varphi)}b_{\vec{p}\pm\vec{k}} \\
&\quad -i\int d^3p'\, a_{\vec{p}'} <a,\vec{p}|H^{(1)}|\vec{p}',a> \\
\dot{b}_{\vec{p}} &= \{-iE_b - i\frac{\vec{p}^2}{2M} - \frac{\gamma_b}{2}\}b_{\vec{p}} - i\Omega_{ab}e^{-i(\omega t \mp k v_3^{(D)} t + \varphi)}a_{\vec{p}\mp\vec{k}} \\
&\quad -i\int d^3p'\, b_{\vec{p}'} <b,\vec{p}|H^{(1)}|\vec{p}',b>
\end{aligned} \tag{5}$$

whereby we have made a rotating wave approximation in neglecting terms containing $\exp(\pm i(2\omega + \Delta)t)$ as compared to those containing $\exp(\pm i\Delta t)$. As can be seen in figs. 2 and 3, we assume that the matter beam enters the experimental set up traveling in the x_1 direction ($\vec{p}^{(in)} = p_1^{(in)}\vec{e}_1$) so that initially only the $a_{\vec{p}^{(in)}}$ is different from zero.

3 Perturbation scheme

To solve the Schrödinger equation, we assume $H^{(1)}$ to be small and use a perturbation scheme:

$$i\partial_t|\psi>= (H^{(0)} + H^{(1)})|\psi>\,,\ |\psi>= |\psi^{(0)}> + |\psi^{(1)}> \tag{6}$$

where $|\psi^{(1)}>$ is of the same order as $H^{(1)}$. In momentum space we decompose $a_{\vec{p}} = a_{\vec{p}}^{(0)} + a_{\vec{p}}^{(1)}$ and correspondingly $b_{\vec{p}}$. The dynamics is then given up to first order by

$$i\partial_t|\psi^{(0)}>= H^{(0)}|\psi^{(0)}> \tag{7}$$

$$i\partial_t|\psi^{(1)}>= H^{(0)}|\psi^{(1)}> + H^{(1)}|\psi^{(0)}> \tag{8}$$

The Ramsey fringes of zeroth order go back to $|\psi^{(0)}>$. They have completely be determined by Bordé et al. [1] in the form described in chapt. 1 above. We refer to these results in a twofold way: Mathematically $H^{(1)}|\psi^{(0)}>$ enters as an inhomogeneity in the otherwise unperturbed Schrödinger equation (8) for $|\psi^{(1)}>$. Physically our intention is to determine according to (1) the shift $\phi^{(1)} \neq 0$ modifying the zeroth order Ramsey fringes ($\phi = 0$).

Returning to the momentum representation (5) of the Schrödinger equation, we apply the unitary transformation

$$\begin{aligned}
a_{\vec{p}} &= \exp\Big\{-\frac{i}{2}(E_a + E_b + \frac{(\vec{p}_\perp)^2}{M})t + \frac{i}{2}\omega t + i(p_3 - p_3^{(in)})v_3^{(D)}t\Big\}\,\tilde{a}_{\vec{p}} \\
b_{\vec{p}\pm\vec{k}} &= \exp\Big\{-\frac{i}{2}(E_a + E_b + \frac{(\vec{p}_\perp)^2}{M})t - \frac{i}{2}\omega t + i(p_3 - p_3^{(in)} \pm k)v_3^{(D)}t\Big\}\,\tilde{b}_{\vec{p}\pm\vec{k}}
\end{aligned} \tag{9}$$

($\vec{p} = \vec{p}_\perp + p_3\vec{e}_3$) to obtain a dynamical equation with constant coefficients Ω_μ:

$$\begin{pmatrix} \dot{\tilde{b}}^{(1)}_{\vec{p}\pm\vec{k}} \\ \dot{\tilde{a}}^{(1)}_{\vec{p}} \end{pmatrix} = \frac{i}{2}(\Omega_0\sigma_0 + \vec{\Omega}\cdot\vec{\sigma})\begin{pmatrix} \tilde{b}^{(1)}_{\vec{p}\pm\vec{k}} \\ \tilde{a}^{(1)}_{\vec{p}} \end{pmatrix} + \begin{pmatrix} f_b(t) \\ f_a(t) \end{pmatrix} \tag{10}$$

whereby

$$\begin{aligned}
\Omega_0 &:= -\frac{1}{2M}((p_3)^2 + (p_3 \pm k)^2) - (2(p_3 - p_3^{(in)}) \pm k)v_3^{(D)} + i\gamma_{ab} \\
\Omega_1 &:= -2\Omega_{ab}\cos\varphi \quad ,\Omega_2 := -2\Omega_{ab}\sin\varphi \\
\Omega_3 &:= \Delta - \frac{1}{2M}((p_3 \pm k)^2 - (p_3)^2) \mp k v_3^{(D)} + i(\gamma_b - \gamma_a)/2
\end{aligned} \tag{11}$$

($\gamma_{ab} := (\gamma_a + \gamma_b)/2$) and

$$f_b(t) := -i\exp\Big\{\frac{i}{2}(E_a + E_b + \frac{(\vec{p}_\perp)^2}{M})t + \frac{i}{2}\omega t - i(p_3 - p_3^{(in)} \pm k)v_3^{(D)}t\Big\}$$

$$\cdot \int d^3p'\, b^{(0)}_{\vec{p}'} < b, \vec{p} \pm \vec{k} | H^{(1)} | \vec{p}', b >$$

$$f_a(t) \;:=\; -i \exp\Big\{ \frac{i}{2}(E_a + E_b + \frac{(\vec{p}_\perp)^2}{M})t - \frac{i}{2}\omega t - i(p_3 - p_3^{(in)})v_3^{(D)} t \Big\}$$

$$\cdot \int d^3p'\, a^{(0)}_{\vec{p}'} < a, \vec{p} | H^{(1)} | \vec{p}', a > \;. \tag{12}$$

σ_i are the Pauli matrices, $\sigma_0 = 1_{2\times 2}$. Set $f_{a,b} = 0$ for the respective zeroth order equations for $a^{(0)}$ and $b^{(0)}$. To obtain (12), we have made use of the fact, that the operator $H^{(1)}$ does not act on the internal states $|a>$ and $|b>$. Ω_1 and Ω_2 vanish in the dark zones outside the beam splitters.

As can be read off from (10), the first order solution $\underline{u} = (\tilde{b}^{(1)}_{\vec{p}\pm\vec{k}}\,,\ \tilde{a}^{(1)}_{\vec{p}})$ is given by

$$\underline{u}(t) = \int_0^\infty G(t-t')\underline{f}(t')dt' \tag{13}$$

with $\underline{f} = (f_b\,,\ f_a)$ where G is the (retarded) Green function:

$$G(t-t') = e^{i\Omega_0(t-t')/2} e^{i\vec{\Omega}\cdot\vec{\sigma}(t-t')/2}\sigma_0 \Theta(t-t')\,. \tag{14}$$

In the following we simplify the index notation according to $\tilde{a}_n := \tilde{a}_{\vec{p}^{(in)}+n\vec{k}}$ and $\tilde{b}_n$ correspondingly.

In principle the mathematical procedure would now be to calculate the time development of the matter waves in first and second order from zone to zone ($0 \le t \le \tau$, $\tau \le t \le \tau + T$ and so on, comp. fig. 1) in order to obtain the probability for the excited states at $t = 4\tau + 2T + T'$. In practice, if there is an external influence $H^{(1)}$ on the experimental setup, this program turns out to be very cumbersome due to the complicated dependence of the zeroth order solution on the momentum in the four laser zones. The reason is the following: It can be seen from the zeroth order dynamical equation, which is obtained from eqn.(10) by setting $f_{a,b} = 0$ and writing $\tilde{a}^{(0)}_n$ and $\tilde{b}^{(0)}_n$ instead of $\tilde{a}^{(1)}_n$ and $\tilde{b}^{(1)}_n$, that the contribution of a laser zone to the evolution of the matter wave results for each laser zone in a 2×2 transformation matrix

$$e^{i\Omega_0\tau/2} e^{i\vec{\Omega}\cdot\vec{\sigma}\tau/2} = e^{i\Omega_0\tau/2}[\cos(\Omega\tau/2) + i\frac{\vec{\sigma}\cdot\vec{\Omega}}{\Omega}\sin(\Omega\tau/2)] =: e^{i\Omega_0\tau/2}\begin{pmatrix} A & Be^{-i\varphi} \\ Ce^{i\varphi} & D \end{pmatrix} \tag{15}$$

acting on the states (b, a) entering the respective laser zone, whereby $\vec{\Omega}$ and $\Omega = \sqrt{\vec{\Omega}^2}$ refer to (11). The second equation in (15) is the definition of the quantities A to D. The quantities in (15) differ from laser zone to laser zone. We have omitted the index r which denotes this. The expression (15) depends on the momenta in a complicated way.

It is therefore convenient to use an *additional approximation* in idealizing the laser zones of duration τ mathematically by beam splitters which act pointlike in time (comp. fig. 2). They can formally be obtained by taking the limit $\tau \to 0$, $\Omega_{ab} \to \infty$ in such a way that $\backslash := \tau\Omega_{ab}$ remains finite, characterizing the respective experimental setup. In this case the influence of the laser field on the atoms in a laser zone results in the simple matrix

$$\begin{pmatrix} A & Be^{-i\varphi} \\ Ce^{i\varphi} & D \end{pmatrix} = \begin{pmatrix} \cos\backslash & -i\sin\backslash\, e^{-i\varphi} \\ -i\sin\backslash\, e^{i\varphi} & \cos\backslash \end{pmatrix} \tag{16}$$

which models the beam splitters completely. We will use these *idealized beam splitters* also for the first order calculation. Their use is physically justified under the condition that the influence of the

perturbing Hamiltonian $H^{(1)}$ over the time τ in the laser zones is so small that it may be neglected for the first step of the perturbation scheme. But, if the perturbation $H^{(1)}$ itself depends on the laser electric field too, the limit $\Omega_{ab} \to \infty$ may imply $H^{(1)} \to \infty$ so that the procedure above is unreasonable. Cases like this need a particular treatment. An example is given below.

Note that the experimental setup used here allows us to model simple perfect beam splitters. This is an advantage over the case of neutron interferometry, where the complicated physical processes in the splitters can only be handled with some ad hoc assumptions.

4 Influence of the perturbation

The perturbation scheme is based on the zeroth order result ($H^{(1)} = 0$) which we have stated first. Idealizing the four beam splitters as described above, we obtain

$$\begin{pmatrix} \tilde{b}^{(0)}_1(t) \\ \tilde{a}^{(0)}_0(t) \end{pmatrix} = \begin{pmatrix} B_1 e^{-i\varphi_1}\exp[i(\Omega_0+\Omega_3)t/2] \\ D_1 \exp[i(\Omega_0-\Omega_3)t/2] \end{pmatrix} \quad \text{for } t \le T$$

$$\begin{pmatrix} \tilde{b}^{(0)}_1(t) \\ \tilde{a}^{(0)}_0(t) \end{pmatrix} = e^{i\Omega_0 t/2}\begin{pmatrix} e^{i\Omega_3(t-T)/2}[B_1A_2e^{-i\varphi_1}e^{i\Omega_3T/2} + D_1B_2e^{-i\varphi_2}e^{-i\Omega_3T/2}] \\ e^{-i\Omega_3(t-T)/2}[B_1C_2e^{i(\varphi_2-\varphi_1)}e^{i\Omega_3T/2} + D_1D_2e^{-i\Omega_3T/2}] \end{pmatrix} \quad \text{for } t \le T+T'$$

$$\begin{pmatrix} \tilde{b}^{(0)}_{-1}(t) \\ \tilde{a}^{(0)}_0(t) \end{pmatrix} = \tilde{a}^{(0)}_0(T+T')\begin{pmatrix} B_3e^{-i\varphi_3}e^{i(\hat{\Omega}_0+\hat{\Omega}_3)(t-T-T')/2} \\ D_3e^{i(\hat{\Omega}_0-\hat{\Omega}_3)(t-T-T')/2} \end{pmatrix} \quad \text{for } t > T+T'$$

$$\begin{pmatrix} \tilde{b}^{(0)}_1(t) \\ \tilde{a}^{(0)}_2(t) \end{pmatrix} = \tilde{b}^{(0)}_1(T+T')\begin{pmatrix} A_3e^{i(\hat{\Omega}_0+\hat{\Omega}_3)(t-T-T')/2} \\ C_3e^{i\varphi_3}e^{i(\hat{\Omega}_0-\hat{\Omega}_3)(t-T-T')/2} \end{pmatrix} \quad \text{for } t > T+T' \tag{17}$$

with

$$\begin{aligned}
\Omega_0 &= -\frac{1}{2M}(2(p_3)^2 + 2p_3k + k^2) - kv_3^{(D)} + i\gamma_{ab} \\
\Omega_3 &= \Delta - \frac{1}{2M}(2p_3k + k^2) - kv_3^{(D)} + i(\gamma_b - \gamma_a)/2 \\
\hat{\Omega}_0 &= -\frac{1}{2M}(2(p_3)^2 - 2p_3k + k^2) + kv_3^{(D)} + i\gamma_{ab} \\
\hat{\Omega}_3 &= \Delta - \frac{1}{2M}(-2p_3k + k^2) + kv_3^{(D)} + i(\gamma_b - \gamma_a)/2 \\
\bar{\Omega}_0 &= -\frac{1}{2M}(2(p_3)^2 + 6p_3k + 5k^2) - 3kv_3^{(D)} + i\gamma_{ab} \\
\bar{\Omega}_3 &= \Delta + \frac{1}{2M}(2p_3k + 3k^2) + kv_3^{(D)} + i(\gamma_b - \gamma_a)/2
\end{aligned} \tag{18}$$

where the last two matrix equations in (17) refer to the two different interference geometries of fig. 2, respectively. The amplitudes after the atoms have passed the fourth laser beam are

$$\tilde{b}^{(0)}_{-1}(2T+T') = \tilde{a}^{(0)}_0(T+T')[B_3A_4e^{-i\varphi_3}e^{i(\hat{\Omega}_0+\hat{\Omega}_3)T/2} + D_3B_4e^{-i\varphi_4}e^{i(\hat{\Omega}_0-\hat{\Omega}_3)T/2}] \tag{19}$$

and

$$\tilde{b}^{(0)}_1(2T+T') = \tilde{b}^{(0)}_1(T+T')[A_3A_4e^{i(\hat{\Omega}_0+\hat{\Omega}_3)T/2} + C_3B_4e^{i(\varphi_3-\varphi_4)}e^{i(\hat{\Omega}_0-\hat{\Omega}_3)T/2}] \tag{20}$$

Moduli squared of these amplitudes give the fringe signals of the Ramsey interference. The resulting expressions can be read off from equations (27) and (30) below in putting all $X_i^{(1)}$, $Y_i^{(1)}$ and $Z_i^{(1)}$ equal to zero. The result agrees with equations (16) and (17) of ref. [1] apart from the fact that in our case A, B, C, D refer to the idealized beam splitters and do therefore not depend on the detuning Δ. Only terms in square brackets in (27) and (30) cause the Ramsey fringes which are of the type (1) with $\phi = 0$ and $T = d/v_1$.

Turning now to the first order perturbation, we firstly note that after the unitary transformation the $a^{(0)}$ and $b^{(0)}$ do not depend on $\vec{p}_\perp$. To work out the integrals in (12) which contribute to the inhomogeneities in (10) we make use of the following *central lemma*

$$\partial_p^{N_1} e^{(\alpha p+\beta p^2)} = [\zeta_1^+ + \zeta_1^-(\alpha + 2\beta p)]\{\sum_{l=0}^{J_1}(4\beta)^{J_1-l}\binom{J_1}{l}(\alpha + 2\beta p)^{2l} P_{l,N_1}\}e^{\alpha p+\beta p^2} \tag{21}$$

with

$$J_1 := \text{integer part of } N_1/2 \tag{22}$$

$$\zeta_1^\pm := (1 \pm (-1)^{N_1})/2 \tag{23}$$

$$P_{l,N_1} := \prod_{m=l+1}^{J_1} (m - (-1)^{N_1}/2) \tag{24}$$

which can be proven by induction. P_{l,N_1} is set to one whenever $l + 1$ exceeds J_1. With the help of this lemma we then find

$$\begin{aligned} f_b(t) \;=\; & -i\hat{H}(\vec{p} \pm \vec{k}) i^{N_G} \left(\frac{\partial^{N_3}}{\partial p_3^{N_3}} \tilde{b}_{\pm 1}^{(0)}\right) \zeta_2^+ P_{0,N_2}[\zeta_1^+ - \frac{itp}{M}\zeta_1^-] \\ & \sum_{l_1=0}^{J_1} \binom{J_1}{l_1} P_{l_1,N_1} 2^{J_1+J_2-l_1}(-i/M)^{J_1+J_2+l_1} p_1^{2l_1} t^{J_1+J_2+l_1} \end{aligned} \tag{25}$$

with

$$N_G := N_1 + N_2 + N_3 \tag{26}$$

To obtain $f_a(t)$ we have to replace $\tilde{b}_{\pm 1}^{(0)}$ and $\hat{H}(\vec{p} \pm \vec{k})$ in (25) by $\tilde{a}_0^{(0)}$ and $\hat{H}(\vec{p})$ or $\tilde{a}_2^{(0)}$ and $\hat{H}(\vec{p} + 2\vec{k})$, respectively. From now on we denote $\vec{p}^{(in)}$ by $\vec{p}$.

The rest of the calculation is straightforward but long. To obtain the outgoing amplitudes up to first order $\tilde{b}_{-1} = \tilde{b}_{-1}^{(0)} + \tilde{b}_{-1}^{(1)}$ and $\tilde{b}_1$ at the time $2T + T'$, one essentially uses the zeroth order solution (17) and applies eqn. (13) in each of the three dark zones of fig. 2 in which the disturbance $H^{(1)}$ influences the atom beams. The calculation is done zone after zone whereby the initial conditions for each zone are given by the field at the end of the preceding zone multiplied by the respective constant laser matrix (16). For details of the calculation see ref. [13].

For the first interferometer geometry we find for the output

$$\begin{aligned} |\tilde{b}_{-1}(2T + T')|^2 = e^{-2T\gamma_{ab} - T'\gamma_a} \Big\{ & |B_1C_2B_3A_4|^2 e^{(\gamma_a - \gamma_b)T}(1 + X_1^{(1)}) + |B_1C_2D_3B_4|^2(1 + X_2^{(1)}) \\ & + |D_1D_2D_3B_4|^2 e^{(\gamma_b - \gamma_a)}T(1 + X_3^{(1)}) + |D_1D_2B_3A_4|^2(1 + X_4^{(1)}) \\ & + [B_1C_2B_3A_4(D_1D_2D_3B_4)^* e^{2i(\Delta-\delta)T + \Delta\varphi}(1 + iZ_1^{(1)}) + c.c.] + O(e^{ikv_3^{(D)}T}) \Big\} \end{aligned} \tag{27}$$

($\delta = k^2/(2M)$, $T = d/v_1$). Because $Z_1^{(1)}$, which will be given below, is of first order, the phase term in square brackets takes with $\exp i\varepsilon \approx 1 + i\varepsilon$ the form $\exp i[2(\Delta - \delta)T + \Delta\varphi + \phi_1^{(1)}]$ with

$$\phi_1^{(1)} = Re(Z_1^{(1)}) \tag{28}$$

At the end we have set $\vec{p} = p_1\vec{e}_1$ so that we are essentially in the frame of reference used in [1] and can apply their result that after integrating over $v_3^{(D)}$ (Doppler broadening) the term in square brackets is the only one which contributes to the fringes. As in the zeroth approximation it turns out that the integrals of the terms in factor of $\exp i(\cdots)$ give the fringe envelop, the remaining additive terms give the background and the Ramsey fringes have again the form of a bilaterally damped cosine of the structure (1). For $|\tilde{b}_{-1}|^2$ we have a recoil shift $-\delta$ and find an additive phase shift $\phi_1^{(1)}$ given by (28) which depends only on $Z_1^{(1)}$. The complicated real expressions $X_i^{(1)}$ do not contribute to the fringe shift and will therefore not be written down. For the general perturbation (3) the quantity $Z_1^{(1)}$ is thereby given by

$$\begin{aligned} Z_1^{(1)} = & -\zeta_2^+ P_{0,N_2} i^{N_G} \sum_{l_1=0}^{J_1}\sum_{l_3=0}^{J_3} \binom{J_1}{l_1}\binom{J_3}{l_3} P_{l_1,N_1} P_{l_3,N_3} 2^{J_G-l_1-l_3}(-\frac{i}{M})^\mu p_1^{2l_1} k^{2l_3} \\ & \Big[\hat{H}(\vec{p} + \vec{k})\, T^{\mu+1}\Big\{\frac{\zeta_1^+\zeta_3^+}{\mu+1} - \frac{iT(p_1\zeta_1^-\zeta_3^+ + k\zeta_1^+\zeta_3^-)}{M(\mu+2)} - \frac{T^2 p_1 k\zeta_1^-\zeta_3^-}{M^2(\mu+3)}\Big\} \\ & + \hat{H}(\vec{p})\, T^{2l_3}\, [\zeta_3^+ - \zeta_3^- \tfrac{ikT}{M}]\Big\{\frac{\zeta_1^+}{J_G + l_1 - l_3 + 1}\big((T+T')^{J_G+l_1-l_3+1} - T^{J_G+l_1-l_3+1}\big) \\ & - \frac{ip_1\zeta_1^-}{M(J_G + l_1 - l_3 + 3)}\big((T+T')^{J_G+l_1-l_3+2} - T^{J_G+l_1-l_3+2}\big)\Big\} \\ & + \hat{H}(\vec{p} - \vec{k}) \sum_{m=0}^{2l_3}(-1)^m\binom{2l_3}{m}(2T+T')^m\Big\{\zeta_1^+ \\ & \frac{[\zeta_3^+ - ik(2T+T')\zeta_3^-/M]}{\mu+1-m}\big((2T+T')^{\mu+1-m} - (T+T')^{\mu+1-m}\big) \\ & + \frac{i\zeta_1^+\zeta_3^- k - i\zeta_1^- p_1[\zeta_3^+ - ik(2T+T')\zeta_3^-/M]}{M(\mu+2-m)}\big((2T+T')^{\mu+2-m} - (T+T')^{\mu+2-m}\big) \\ & + \frac{p_1 k\zeta_1^-\zeta_3^-}{M^2(\mu+3-m)}\big((2T+T')^{\mu+3-m} - (T+T')^{\mu+3-m}\big)\Big\} \\ & - \hat{H}^*(\vec{p})\delta_{l_3,0}\zeta_3^+(-1)^{\mu+N_G}\Big\{\zeta_1^+\frac{(2T+T')^{\mu+1}}{\mu+1} + \zeta_1^-\frac{ip_1(2T+T')^{\mu+2}}{M(\mu+2)}\Big\}\Big] \end{aligned} \tag{29}$$

For notational convenience we have introduced $J_G := J_1 + J_2 + J_3$ and $\mu := J_G + l_1 + l_3$.

Before discussing this result, we turn to the output of the second interferometer geometry. The lengthy calculation leads to

$$\begin{aligned} |\tilde{b}_1(2T + T')|^2 = e^{-2T\gamma_{ab} - T'\gamma_b} \Big\{ & |B_1A_2A_3A_4|^2 e^{(\gamma_a - \gamma_b)T}(1 + Y_1^{(1)}) + |B_1A_2C_3B_4|^2(1 + Y_2^{(1)}) \\ & + |D_1B_2C_3B_4|^2 e^{(\gamma_b - \gamma_a)}T(1 + Y_3^{(1)}) + |D_1B_2A_3A_4|^2(1 + Y_4^{(1)}) \\ & + [B_1A_2A_3A_4(D_1B_2C_3B_4)^* e^{2i(\Delta+\delta)T + \Delta\varphi}(1 + iZ_2^{(1)}) + c.c.] + O(e^{ikv_3^{(D)}T}) \Big\} \end{aligned} \tag{30}$$

Based on the same arguments as above, we obtain a shift

$$\phi_2^{(1)} = Re(Z_2^{(1)}) \tag{31}$$

of the Ramsey fringes which in this case show a recoil shift $+\delta$. The responsible quantity $Z_2^{(1)}$ is thereby

$$
\begin{aligned}
Z_2^{(1)} = & -\zeta_2^+ P_{0,N_2} i^{N_G} \sum_{l_1=0}^{J_1} \sum_{l_3=0}^{J_3} \binom{J_1}{l_1} \binom{J_3}{l_3} P_{l_1,N_1} P_{l_3,N_3} 2^{J_G-l_1-l_3} (-\frac{i}{M})^{\mu} p_1^{2l_1} k^{2l_3} \\
& \Big[\hat{H}(\vec{p}+\vec{k}) \Big\{ \frac{\zeta_1^+\zeta_3^+}{\mu+1}(2T+T')^{\mu+1} - \frac{i(p_1\zeta_1^-\zeta_3^+ + k\zeta_1^+\zeta_3^-)}{M(\mu+2)}(2T+T')^{\mu+2} \\
& - \frac{p_1 k \zeta_1^-\zeta_3^-}{M^2(\mu+3)}(2T+T')^{\mu+3} \Big\} - \hat{H}^*(\vec{p})\, \delta_{l_3,0} (-1)^{\mu+N_G} \zeta_3^+ \Big\{ \frac{\zeta_1^+}{\mu+1} T^{\mu+1} \\
& + \frac{i p_1 \zeta_1^-}{M(\mu+2)} T^{\mu+2} \Big\} - \hat{H}^*(\vec{p}+\vec{k})\, (-1)^{\mu+N_G} \sum_{m=0}^{2l_3} (-1)^m \binom{2l_3}{m} T^m \\
& \Big\{ \zeta_1^+ \frac{[\zeta_3^+ - ikT\zeta_3^-/M]}{\mu+1-m} \big((T+T')^{\mu+1-m} - T^{\mu+1-m} \big) \\
& + \frac{i\zeta_1^+\zeta_3^- k + i\zeta_1^- p_1 [\zeta_3^+ - ikT\zeta_3^-/M]}{M(\mu+2-m)} \big((T+T')^{\mu+2-m} - T^{\mu+2-m} \big) \\
& - \frac{p_1 k \zeta_1^-\zeta_3^-}{M^2(\mu+3-m)} \big((T+T')^{\mu+3-m} - T^{\mu+3-m} \big) \Big\} \\
& - \hat{H}^*(\vec{p}+2\vec{k})\, (-1)^{\mu+N_G} \sum_{m=0}^{2l_3} (-1)^m \binom{2l_3}{m} 2^{2l_3-m} (2T+T')^m \Big\{ \zeta_1^+ \\
& \frac{[\zeta_3^+ - ik(2T+T')\zeta_3^-/M]}{\mu+1-m} \big((2T+T')^{\mu+1-m} - (T+T')^{\mu+1-m} \big) \\
& + \frac{2i\zeta_1^+\zeta_3^- k + i\zeta_1^- p_1 [\zeta_3^+ - ik(2T+T')\zeta_3^-/M]}{M(\mu+2-m)} \big((2T+T')^{\mu+2-m} \\
& - (T+T')^{\mu+2-m} \big) - \frac{2p_1 k \zeta_1^-\zeta_3^-}{M^2(\mu+3-m)} \big((2T+T')^{\mu+3-m} - (T+T')^{\mu+3-m} \big) \Big\} \Big]
\end{aligned}
\tag{32}
$$

Because of the generality of the external influence given by (3), the expressions $Z_1^{(1)}$ and $Z_2^{(1)}$ seem to be clumsy. This is in fact not the case, because they reduce for all practical applications to only a very few terms. Examples will be given below. Before doing so we read off some general properties of our result (28) and (31):

It is easy to check that $\phi^{(1)}$ vanishes if $H^{(1)} = \hat{H}(P_1, P_2)(x_1)^{N_1}(x_2)^{N_2}$. This is physically evident, because in this case the two interferometer paths are disturbed in the same way.

In case the Ramsey fringes discussed above are used as basis of an optical frequency standard, it is essential that phase errors leading to a $\Delta\varphi \neq 0$ in eqn. (1) of the excitation can be compensated by laser *beam reversal*. For a discussion see ref. [4]. Above we have treated the influence of disturbances with potentials of the type (3). This leads to the corresponding question if the disturbances too leading to $\phi^{(1)} \neq 0$ in eqn. (1) can be compensated by reversal of laser or matter beams. Due to the symmetry of the experimental setup the effect of this procedure in the disturbed case can easily be deduced from (29) and (32) by inverting the sign of p_1 or k for atom beam reversal or laser beam reversal, respectively. For matter beam reversal it results in $Z_i^{(1)}$ in a factor $(-1)^{N_1}$ and the replacement $\hat{H}(\vec{p}+n\vec{k}) \rightarrow \hat{H}(-\vec{p}+n\vec{k})$. For the special case of constant $\hat{H}$ only the factor $(-1)^{N_1}$ in $\phi_i^{(1)}$ remains, so that for odd N_1 beam reversal allows an elimination of the disturbance $\phi_i^{(1)}$ while this is not possible for even N_1. For laser beam reversal on the other hand $Z_i^{(1)}$ pics up a factor $(-1)^{N_3}$ and $\hat{H}(\vec{p}+n\vec{k})$ has to be replaced by $\hat{H}(\vec{p}-n\vec{k})$. For constant $\hat{H}$ only the factor $(-1)^{N_3}$ for $\phi_i^{(1)}$ remains.

The disturbances $\phi_i^{(1)}$ can accordingly be eliminated for N_3 odd and not be eliminated for N_3 even. This shows that if both N_1 and N_3 are even and $\hat{H}$ is constant, the shifts $\phi_i^{(1)}$ do not alter sign if the laser or matter beam directions are changed so that there is no way to eliminate these unwanted shifts by any type of beam reversal.

In the context of matter wave interferometry a well known strategy to obtain optimal experimental results is to use as low atomic velocities as possible. This is also favourable in the context of frequency standards, because the uncertainty to realize the line center of the fringes decreases with decreasing atomic velocity v_1 (comp. eqn. (1)). It is therefore important to discuss how the additional shift $\phi^{(1)}$ depends on the atomic velocity.

For a general perturbation (3) one cannot say much because of the arbitrary dependence of $\hat{H}$ on $\vec{p}$, but if we assume that $\hat{H}$ is independent of p_1 one can read off in setting $p_1 = Mv_1$ that the terms in $Z_i^{(1)}$ are proportional to $v_1^{-(J_G-l_1+l_3+1+\zeta_3^-)}$. In varying l_1 and l_3 one sees that the lowest possible value of the exponent is $-(J_1+J_2+N_3+1)$ and the highest possible value is $-(J_2+J_3+1+\zeta_3^-)$. Both are smaller than zero so that the phase shifts (28) and (31) are increasing with decreasing velocity v_1. By explicit calculation we have checked the special case of rotation (see below), where $\hat{H}$ does depend on $\vec{p}$, and have found the corresponding dependence $\phi_i^{(1)} \sim 1/v_1$. These results regarding the velocity dependence of the shifts $\phi_i^{(1)}$ have two different consequences: Cooling to low atomic velocity is favourable in case the experimental setup serves as an atomic beam interferometer by which very particular physical influences on matter waves are detected. But cooling needs particular attention if the arrangement serves as a frequency standard and the $\phi_i^{(1)}$ represent unintended phase errors which are to be avoided experimentally.

5 Comparison with a WKB phase shift

The disturbing Hamiltonian $H^{(1)}$ of (3) is characterized by not affecting the internal degrees of freedom of the atoms. In our approximation it is in addition assumed that there is no influence of the disturbance in the interaction zones of the matter beams with the laser fields (i.e. 'inside' the beam splitters). This leads to the conjecture that the resulting phase shifts $\phi_i^{(1)}$ obtained above should somehow show similarities with the phase shifts obtained in matter wave interferometers with the same interaction geometry as in fig. 3, where the beam splitters described above have been replaced by purely 'optical' mirrors and splitters. By 'optical' we indicate that the internal states of the quantum objects in the matter wave are not involved in the respective scattering process. Essentially only the momentum of the matter beam is changed, compare for example neutron interferometry. Note that this is physically a different interferometer type. It is therefore not to be expected that the phase shifts (28) and (31) or even the complete interference pattern as described by (27) and (30) will agree with the respective WKB result exactly. The results above have been obtained in first order in the perturbation $H^{(1)}$ and in arbitrary order in $\hbar$. It is nevertheless useful to ask if for particular disturbances $H^{(1)}$ and in certain limits the shifts $\phi_i^{(1)}$ of the Ramsey fringes agree with the phase shifts obtained in a WKB approach for the corresponding interferometers with 'optical' beam splitters. Because the WKB calculation is mathematically simple, this will then for all practical purposes allow a quick insight into the physical structure of the phase shifts $\phi_i^{(1)}$ to be expected for certain physical influences $H^{(1)}$. The outcome of the different types of atomic interferometers can easily be compared. For neutron interferometry a specific correction of the WKB phase is caused by spin terms, see Audretsch and Lämmerzahl [14].

To obtain this WKB phase shift in question we use the steepest descent approximation of path integral formalism. It is equivalent to the usual WKB method and leads to the result that the phase shift is given by $\oint L\,dt$, where L is the Lagrangian and the integration is done over a classical path. If $(\partial L/\partial t) = 0$ the energy E is conserved and we get $\oint L\,dt = \oint(-E + \vec{p}\cdot\dot{\vec{x}})dt = \oint \vec{p}\cdot d\vec{x}$. Here $\vec{p}$ is the canonical momentum. A Hamiltonian of the form

$$H = \frac{\vec{p}^2}{2M} + H^{(1)}\,, \quad H^{(1)} = \hat{H}(\vec{p})(x_1)^{N_1}(x_2)^{N_2}(x_3)^{N_3} \tag{33}$$

with $H^{(1)}$ small can be deduced, in first order in $H^{(1)}$, from the Lagrangian

$$\frac{1}{2}M\dot{\vec{x}}^2 - \hat{H}(M\dot{\vec{x}})\,(x_1)^{N_1}(x_2)^{N_2}(x_3)^{N_3} \tag{34}$$

Working out the WKB phase shifts for the two corresponding interferometer geometries shown in fig. 3, we obtain for the lower paths related to the output $|b_{-1}|^2$

$$\begin{aligned}\phi^{(1)}_{\mathrm{WKB}_1} &= \oint \vec{p}\cdot d\vec{x} = \frac{p_1^{N_1}k^{N_3}}{M^{N_1+N_3}}\delta_{N_2,0}\Big\{ -\hat{H}(\vec{p}+\vec{k})\,\frac{T^{N_1+N_3+1}}{N_1+N_3+1} - \hat{H}(\vec{p})\,\frac{T^{N_3}}{N_1+1}\cdot \\ &\Big((T+T')^{N_1+1} - T^{N_1+1}\Big) - \hat{H}(\vec{p}-\vec{k})\sum_{m=0}^{N_3}(-1)^{N_3-m}\frac{(2T+T')^m}{N_1+N_3+1-m}\binom{N_3}{m}\cdot \\ &\Big[(2T+T')^{N_1+N_3+1-m} - (T+T')^{N_1+N_3+1-m}\Big] + \hat{H}(\vec{p})\,\delta_{N_3,0}\frac{(2T+T')^{N_1+1}}{N_1+1}\Big\}\end{aligned} \tag{35}$$

This is to be compared with $\phi^{(1)}_1$ of (28). For the upper paths corresponding to $|b_1|^2$ we find

$$\begin{aligned}\phi^{(1)}_{\mathrm{WKB}_2} = \frac{p_1^{N_1}k^{N_3}}{M^{N_1+N_3}}\delta_{N_2,0}\Big\{ &-\hat{H}(\vec{p}+\vec{k})\,\frac{(2T+T')^{N_1+N_3+1}}{N_1+N_3+1} + \hat{H}(\vec{p})\,\delta_{N_3,0}\frac{T^{N_1+1}}{N_1+1} + \hat{H}(\vec{p}+\vec{k})\cdot \\ &\sum_{m=0}^{N_3}\frac{(-1)^m T^m}{N_1+N_3+1-m}\binom{N_3}{m}\Big[(T+T')^{N_1+N_3+1-m} - T^{N_1+N_3+1-m}\Big] + \hat{H}(\vec{p}+2\vec{k})\cdot \\ &\sum_{m=0}^{N_3}\frac{(-1)^m(2T+T')^m}{N_1+N_3+1-m}\binom{N_3}{m}2^{N_3-m}\Big[(2T+T')^{N_1+N_3+1-m} - (T+T')^{N_1+N_3+1-m}\Big]\Big\}\end{aligned} \tag{36}$$

and compare it with $\phi^{(1)}_2$ of (31). The results depend on the different powers N_i of the coordinates x_i in the Hamiltonian $H^{(1)}$ (comp. (3)):

(i) If $N_2 \neq 0$, a comparison with the WKB scheme is impossible, because the respective WKB paths are restricted to the plane $x_2 = 0$ where $H^{(1)} = 0$. Accordingly we have $\phi^{(1)}_{\mathrm{WKB}} = 0$.

(ii) If $H^{(1)}$ does not depend on x_2 (i.e. $(N_2 = 0)$ we find, regardless if $H^{(1)}$ depends on p_2 or not, that the terms in the sums in $Z^{(1)}_i$ with $l_1 = J_1$ and $l_3 = J_3$ lead exactly to $\phi^{(1)}_{\mathrm{WKB}}$. The other terms are of higher order in $\hbar$ as can be seen by reintroducing $\hbar$ in $Z^{(1)}_i$ in replacing each T, T' by $T/\hbar$, $T'/\hbar$. Accordingly we have for $N_2 = 0$ always at least an agreement to lowest order in $\hbar$.

$$N_2 = 0 \Rightarrow \phi^{(1)}_i(\text{to lowest order in } \hbar) = \phi^{(1)}_{\mathrm{WKB}_i} \tag{37}$$

(iii) The agreement between $\phi^{(1)}_i$ and $\phi^{(1)}_{\mathrm{WKB}}$ is even exact for particular combinations and if the exponents N_1, N_2, N_3 in $H^{(1)}$ are small. This is the case for $N_1 + N_2 + N_3 \leq 3$ and also for

$(N_1, N_2, N_3) \in \{(1,2,1),\ (3,0,1),\ (1,0,3)\}$ for arbitrary $\hat{H}$ due to the fact that all corrections in $Z^{(1)}_i$ are then purely imaginary. The phase shifts agree also exactly for $N_1 + N_2 + N_3 \leq 4$ if $\hat{H}$ does not depend on p_3.

(iv) On the other hand, if the conditions above are not fulfilled, there may in general be large differences between the full $\phi^{(1)}_i$ and $\phi^{(1)}_{\mathrm{WKB}_i}$. For example $N_1 = N_2 = 0$, $N_3 = 5$ or 6 and $\hat{H}$ constant is such a case.

6 Applications

Examples for an application of our results are acceleration or rotation of the Ramsey geometry and the influence of spacetime curvature. We have for acceleration $\vec{a}$

$$H^{(1)} = M\vec{a}\cdot\vec{x} \tag{38}$$

and for rotation $\vec{\omega}_{rot}$ of the reference frame

$$H^{(1)} = -\vec{\omega}_{rot}\cdot(\vec{x}\times(\vec{p} - q\vec{A})) \tag{39}$$

In a rotating frame a nonvanishing vector potential $\vec{A}$ of an external field leads to an additional term in $H^{(1)}$ if the quantum objects in the matter beam have a charge q, as would be the case for ions.

For the spacetime curvature given by the Riemann tensor $R^{\mu\nu\rho\sigma}$ we find

$$H^{(1)} = \frac{M}{2}\sum_{l,m=1}^{3} R^{0l0m}x_l x_m \tag{40}$$

We have used here the conventions of ref. [16]. The components of the curvature tensor are taken on the worldline of the observer with respect to his Fermi-tetrad, for details see [17] and references therein.

The fact that for $q \neq 0$ a rotation gives an additional coupling between ions and the laser fields in the beam splitters needs particular considerations. Taking for example $\vec{A} = \vec{E}_0\sin(\omega t \pm kx_3 + \varphi)/\omega$, we have to investigate how the inhomogeneities (12) are affected. Evaluating f_b we get

$$\begin{aligned}\int d^3p'\, b^{(0)}_{\vec{p}} < b, \vec{p}\pm\vec{k}|H^{(1)}|\vec{p}', b> = i\vec{\omega}_{rot}\cdot(\vec{p}\times\vec{\nabla}_{\vec{p}})b^{(0)}_{\vec{p}} \\ + \frac{q}{\omega}(\vec{\omega}_{rot}\times\vec{E}_0)\cdot\frac{1}{2}\left[ie^{i(\omega t+\varphi)}\vec{\nabla}_{\vec{p}\pm\vec{k}}b^{(0)}_{\vec{p}\pm\vec{k}} - ie^{-i(\omega t+\varphi)}\vec{\nabla}_{\vec{p}\mp\vec{k}}b^{(0)}_{\vec{p}\mp\vec{k}}\right]\end{aligned} \tag{41}$$

Since the second part is proportional to the laser field and therefore proportional to the Rabi frequency, it tends to infinity if we would formally take $\Omega_{ab} \to \infty$ as we have done in chapt. 3 to approximate the beam splitters. Nevertheless the procedure above remains justified also for charged matter beams: Due to the fact that only $b^{(0)}_{\pm 1}$ are not zero to lowest order, the coupling between the modes in (41) causes nonvanishing corrections only for $b^{(1)}_0$ and $b^{(1)}_{\pm 2}$. As the relevant quantity is the output of the interferometer which is proportional to

$$|<b|\psi>|^2 = \sum_n |b_n|^2 \approx \sum_n |b^{(0)}_n|^2 + (b^{(0)}_n b^{(1)*}_n + c.c.)\,, \tag{42}$$

we see that these corrections do not contribute to first order in $H^{(1)}$. The same argument holds for the a-modes so that the additional fringe shift for $q \neq 0$ is entirely of higher order in $H^{(1)}$.

By inserting these Hamiltonians into (29) and (32) we find with (28) and (31) the phase shifts

$$\phi_1^{(1)} = \phi_2^{(1)} = -ka\,T(T+T') \tag{43}$$

for acceleration ($\vec{a} = a\vec{e}_3$) and

$$\phi_1^{(1)} = \phi_2^{(1)} = \frac{2}{M}\omega_{rot}p_1 k\,T(T+T') \tag{44}$$

for rotation ($\vec{\omega}_{rot} = \omega_{rot}\vec{e}_2$) and vanishing charge q. For curvature one has a different phase shift for the two interferometer geometries:

$$\phi_1^{(1)} = -\frac{1}{2M}R^{0103}\,kp_1T(2T^2+3TT'+(T')^2) - \frac{1}{6M}R^{0303}\,k^2T^2(2T+3T') \quad , \tag{45}$$

and

$$\phi_2^{(1)} = -\frac{1}{2M}R^{0103}\,kp_1T(2T^2+3TT'+(T')^2) - \frac{1}{6M}R^{0303}\,k^2T(4T^2+6TT'+3(T')^2) \tag{46}$$

i.e. $\phi_2^{(1)}$ is more than twice as big as $\phi_1^{(1)}$ for $R^{0303} \neq 0 = R^{0103}$. An experimental method to detect for example $\phi_2^{(1)}$ only has been described by Sterr et al. [15]. The two shifts (43) and (44) are in agreement with the results in ref. [12] and with the usual terms found by a WKB approximation (comp. ref. [10]) if one sets $T = D/v$, $T' = d/v$ where D is the spatial distances between the 1st and 2nd laser beams, d the distance between the 2nd and 3rd beam, and v is the mean velocity of the atom beam. A more detailed discussion which will be given in ref. [17] shows that, while rotation and acceleration phase shifts have already been experimentally verified [6, 8, 18], the influence of spacetime curvature on the shift of the Ramsey fringes should be big enough to be measured in future experiments.

7 Conclusions

We have calculated the shift of the Ramsey fringes caused by a perturbational Hamiltonian $H^{(1)}$ which does not affect the internal degrees of freedom of the matter field to first order in $H^{(1)}$. The result is compared with the phase shift calculated in a WKB approach and agrees with it, if the dependence of $H^{(1)}$ on the position operators, which is assumed to be polynomial, is not too strong, i.e. if the degree of the polynomial does not exceed 3. There are counterexamples for general agreement.

Several applications of the experimental setup are discussed. In connection with frequency standards based on the Ramsey fringes our result allows to decide which classes of unwanted disturbances can be eliminated by laser or atomic beam reversal. In the case the setup is used as an atom interferometer we have discussed the general behaviour of fringe shifts for cooling of the atoms. We have applied our result to rotation, acceleration and spacetime curvature.

Acknowledgements
We thank the group of Prof. Mlynek in Konstanz and Dr. Lämmerzahl for valuable discussions. This work was supported by the European Community DGXII.

References

[1] Ch. J. Bordé, C. Salomon, S. Avrillier, A. Van Leberghe, C. Bréant, D. Bassi and G. Scoles, Phys. Rev. A **30**, 1836 (1984).

[2] Ch. J. Bordé, S. Avrillier, A. Van Leberghe, C. Salomon, D. Bassi and G. Scoles, J. Phys. (Paris) Colloques **48**, C8 (1981).

[3] J. Helmcke, D. Zevgolis and B.Ü. Yen, Appl. Phys. B **28**, 83 (1982).

[4] A. Morinaga, F. Riehle, J. Ishikawa, and J. Helmcke, Appl. Phys. B **48**, 165 (1989).

[5] O. Carnal and J. Mlynek, Phys. Rev. Lett. **66**, 2689 (1991).

[6] F. Riehle, T. Kisters, A. Witte, J. Helmcke and Ch. J. Bordé, Phys. Rev. Lett. **67**, 177 (1991).

[7] D. W. Keith, C. R. Ekstrom, Q. A. Turchette and D. E. Pritchard, Phys. Rev. Lett. **66**, 2693 (1991).

[8] M. Kasevich and S. Chu, Phys. Rev. Lett. **67**, 181 (1991).

[9] Ch. J. Bordé, Phys. Lett. A **140**, 10 (1989).

[10] J. Audretsch,F.-W. Hehl and C. Lämmerzahl in G. Schäfer and J. Ehlers (eds.): *Relativistic gravity research*, Proceedings of the Bad Honnef school on gravitation 1991, Springer Verlag, Berlin 1992.

[11] J. Audretsch and C. Lämmerzahl, Appl. Phys. B **54**, 351 (1992).

[12] Ch. J. Bordé, *Atomic interferometry and laser spectroscopy*, in *Laser spectroscopy 10*, Font-Romeu, France, 1991, edited by M. Ducloy, E. Giacobino and G. Camy, World Scientific, Singapore 1992.

[13] Forthcoming PhD-thesis of K.-P. Marzlin, University of Konstanz.

[14] J. Audretsch and C. Lämmerzahl, Journ. Phys. A **16**, 2457 (1983).

[15] U. Sterr, K. Sengstock, J. H. Müller, D. Bettermann and W. Ertmer, Appl. Phys. B **54**, 341 (1992).

[16] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman and Co., New York 1973.

[17] J. Audretsch and K.-P. Marzlin, *Quantum theory in curved spacetime, noninertial frames and Ramsey interferometry*, manuscript in preparation.

[18] F. Shimizu, K. Shimizu and H. Takuma, Phys. Rev. A **46**, R17 (1992).