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QUANTUM POTENTIAL INTERPRETATION AND QUANTUM COSMOLOGY

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ABSTRACT

We apply Bohm's quantum potential interpretation to quantum cosmology or quantum geometrodynamics. In quantum geometrodynamics based on the Arnowitt-Deser-Misner canonical formalism, the equations of motion for the universe acted on not only by classical potentials, but also by quantum potentials are given. We study in detail the de Sitter minisuperspace model by introducing "extended" Robertson-Walker time. It is shown that the behavior of trajectories of the universe depends, especially in classically forbidden region, on the choice of the operater ordering in the Wheeler-DeWitt equation. We also discuss briefly chaotic inflationary models.

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I. INTRODUCTION

In quantum cosmology, the whole universe is described by a wave function Ψ, which is defined on the superspace (the space of all three metrics and matter field configurations). The wave function Ψ is a solution to the Wheeler-DeWitt(WDW) equation [1], $H\Psi = 0$. Since superspace is of infinite dimensions, the full formalism of quantum cosmology is very difficult to deal with in practice. In addition to the technical difficulties in solving the WDW equation, we also encounter a nontrivial interpretational problem about the wave function of the universe. This problem has been discussed by many authors [2]. It seems that the "Copenhagen" interpretation for the quantum mechanics is inapplicable to quantum cosmology from the following several reasons. Firstly, since the WDW equation is a hyperbolic second-order differential equation, there exists no conserved positive-definite probability density, at least at the very early universe, as in the case of the Klein-Gordon equation. Secondly, the "Copenhagen" formalism assumes a possible division of the world into "observer" and "observed". However, in quantum theory of the universe, there can be no fundamental division into observer and observed. Also, measurement and observer can not be fundamental at early universes, since both can not exist at the early stage of the evolution of the universe. Thirdly, the "collapse" of the wave function in the "Copenhagen" scenarios requires a description involving time evolution. It is not usually easy to introduce "time" in the WDW equation, since the WDW equation is a "timeless" equation.

An alternative interpretation of the quantum theory was proposed by Bohm [3], based on the notion of a particle acted on, not only by classical potential but also by "quantum potential". The quantum potential is determined by a solution to Schrödinger equation. In this quantum potential interpretation of quantum theory, for example, electron is a certain kind of particle following continuous and well-

defined trajectories. In quantum potential interpretation, it was explained that all sorts of quantum processes, such as transition between states, quantum interference in the two-slit experiment are able to take place, in principle, without the need for a observer and also for collapse of the wave function [4]. Therefore, we think that Bohm's quantum potential interpretation is very suitable for quantum theory of the universe. This interpretation was already applied by Vink [5] to the WDW equation for minisuperspace, for a special value ($p = 1$) of the factor-ordering parameter p (see Sec.IV).

In this paper, we apply this quantum potential interpretation to quantum geometrodynamics based on the ADM canonical formalism [6] and give the equations of motion of a universe subject not only to classical potential, but also to quantum potential. We study in detail the de Sitter minisuperspace model by introducing "extended" Robertson-Walker time, which describes well-defined trajectories of the evolution of the universe not only in classically allowed region but also in classically forbidden region. It is shown that this "extended" Robertson-Walker time is identified with the usual one in classically allowed region by an appropriate shift of this time (axis), and the behavior of trajectories of universe (especially in classically forbidden region) depends on the choice of operator ordering in the WDW equation. We also discuss briefly chaotic inflationary minisuperspace model. It is also shown that, in a simple minisuperspace model, probability density is always positive (even in classically forbidden region) if the universe is expanding (see Sec.V). Comparison with other interpretations will not be discussed in this paper. We begin in Sec.II where we briefly review Bohm's quantum potential interpretation. In Sec.III we apply this interpretation to quantum geometrodynamics. In Sec.IV we study in detail the de Sitter minisuperspace model. In Sec.V we discuss chaotic inflationary minisuperspace models. We summarize and conclude in Sec.VI.

II. QUANTUM POTENTIAL INTERPRETATION

We briefly review the main feature of the quantum potential approach. The Schrödinger equation for a single particle is expressed as

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2m}\nabla^2\Psi + V(\mathbf{x})\Psi, \quad (\hbar = 1). \tag{1}$$

We write the wave function in the form $\Psi = R\exp(iS)$, where R and S are real. Then Schrödinger equation reduces to the following two equations:

$$\frac{\partial P}{\partial t} + \nabla\cdot(P\frac{\nabla S}{m}) = 0, \quad \text{with } P = R^2, \tag{2}$$

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + Q = 0, \tag{3}$$

$$Q = \frac{-1}{2m}\frac{\nabla^2 R}{R}. \tag{4}$$

We note that multiplication of the wave function by a constant does not change the quantum potential Q. A causal reinterpretation of the quantum theory is based on the following assumptions [3]. (i) The Ψ-field satisfies Schrödinger equation. (ii) A quantum-mechanical system, such as an electron, consists of a particle with a definite coordinate, which is a well defined continuous function of the time. Its velocity is assumed to be given by $\mathbf{v}(\mathbf{x},t) = \frac{\nabla S}{m}$. (iii) $P = R^2$ is assumed to be the probability distribution of particles in a statistical ensemble of well-defined trajectories. This probability density P satisfies a continuity equation $\frac{\partial P}{\partial t} + \nabla\cdot(P\mathbf{v}) = 0$, which is obtained from eq.(2).

Bohm deduced that the equation of the particle acted on by the classical potential $V(\mathbf{x})$ and the quantum potential $Q(\mathbf{x},t)$ is [3]

$$m\frac{d^2\mathbf{x}}{dt^2} = -\nabla\{V(\mathbf{x}) + Q(\mathbf{x},t)\}. \tag{5}$$

We note that this equation can be obtained from the following equations

$$\dot{\mathbf{x}} = \{\mathbf{x}, H_Q\}_p, \tag{6}$$

$$\dot{\mathbf{p}} = \{\mathbf{p}, H_Q\}_P, \tag{7}$$

where $H_Q = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) + Q(\mathbf{x}, t)$, which is a "Hamiltonian" containing quantum potential, and $\{\ \}_p$ stands for the Poisson bracket. Alternatively, eq.(5) is obtained from the Euler-Lagrange equation for the "Lagrangian" containing quantum potential:$L_Q = \frac{1}{2}m\dot{\mathbf{x}}^2 - (V(\mathbf{x}) + Q(\mathbf{x}, t))$.

III. APPLICATION TO QUANTUM GEOMETRODYNAMICS

We apply the quantum potential interpretation to the quantum geometrodynamics. We recapitulate the ADM canonical formalism. The $(3+1)$ decomposition of the metric reads

$$ds^2 = -(Ndt)^2 + (dx^i + N^i dt)(dx^j + N^j dt)h_{ij}, \tag{8}$$

where N, N^i and h_{ij} are the lapse, shift functions and spatial metrics. The action for the Einstein gravity plus a real scalar field in the unit $16\pi G = 1$, is cast in the form,

$$\begin{aligned} S &= \int d^4x\sqrt{-g}R^{(4)} - \frac{1}{2}\int d^4x\sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)] + \text{surface term} \\ &= \int d^4x N\sqrt{h}(K_{ij}K^{ij} - K^2 + R^{(3)}) + \text{surface term} \\ &+ \int d^4x\sqrt{h}[\frac{1}{2N}\dot{\phi}^2 - \frac{N^i}{N}\partial_i\phi\dot{\phi} - \frac{N}{2}h^{ij}\partial_i\phi\partial_j\phi + \frac{1}{2N}(N^i\partial_i\phi)^2 - \frac{N}{2}V(\phi)], \end{aligned} \tag{9}$$

where $K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - N_{i|j} - N_{j|i})$ is the extrinsic curvature, $K = h_{ij}K^{ij}$ is its trace, a dot stands for the differentiation with respect to time and $R^{(3)}$ denotes the three dimensional scalar curvature. The stroke indicates the covariant derivative defined by the spatial metric h_{ij}. The canonical conjugate momentum π^{ij} to h_{ij} is given by

$$\pi^{ij} = \sqrt{h}(K^{ij} - h^{ij}K). \tag{10}$$

It is straight forward to obtain the action in the phase space,

$$S = \int d^4x[\pi^{ij}\dot{h}_{ij} - NH - N_iH^i] + \text{surface term}, \tag{11}$$

$$\begin{aligned} H &= \frac{1}{\sqrt{h}}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2) - \sqrt{h}R^{(3)} \\ &+ \frac{1}{\sqrt{h}}\frac{\pi_\phi^2}{2} + \frac{1}{2}\sqrt{h}h^{ij}\partial_i\phi\partial_j\phi + \frac{\sqrt{h}}{2}V(\phi), \end{aligned} \tag{12}$$

$$H^i = -2\pi^{ij}_{|j} + \pi_\phi\partial^i\phi, \tag{13}$$

where $\pi = h_{ij}\pi^{ij}$ and $\pi_\phi = \frac{\sqrt{h}}{N}\dot{\phi} - \sqrt{h}\frac{N^i}{N}\partial_i\phi$ is the canonical conjugate momentum to ϕ. The wave function of the universe $\Psi(h_{ij}(x), \phi(x))$ satisfies the equations (assuming that the Universe is closed) [1]

$$[h^{-q}\frac{\delta}{\delta h_{ij}}h^qG_{ijkl}\frac{\delta}{\delta h_{kl}} + \sqrt{h}R^{(3)} + \frac{1}{2\sqrt{h}}\frac{\delta^2}{\delta\phi^2} - \frac{1}{2}\sqrt{h}h^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}\sqrt{h}V(\phi)]\Psi = 0, \tag{14}$$

$$i[2D_j\frac{\delta}{\delta h_{ij}} - h^{ij}\partial_j\phi\frac{\delta}{\delta\phi}]\Psi = 0, \tag{15}$$

where $h = det(h_{ij})$, the parameter q represents the factor-ordering ambiguity, D_j is a covariant derivative in the metric h_{ij}, and $G_{ijkl} = \frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$. Eq.(14) is the Wheeler-DeWit (WDW) equation. We write Ψ in a polar decomposition as $\Psi(h_{ij}, \phi) = A(h_{ij}, \phi)\exp(iS(h_{ij}, \phi))$, in terms of which the WDW equation (14) reads

$$\begin{aligned} G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{kl}} &+ \frac{1}{2}\frac{1}{\sqrt{h}}(\frac{\delta S}{\delta\phi})^2 - \sqrt{h}(R^{(3)} - Q_G) \\ &+ \frac{1}{2}\sqrt{h}h^{ij}\partial_i\phi\partial_j\phi + \frac{1}{2}\sqrt{h}(V(\phi) + Q_M) = 0, \end{aligned} \tag{16}$$

$$Q_G(h_{ij}, \phi) = -h^{-1/2}\frac{1}{A}(G_{ijkl}\frac{\delta^2 A}{\delta h_{ij}\delta h_{kl}} + h^{-q}\frac{\delta h^q G_{ijkl}}{\delta h_{ij}}\frac{\delta A}{\delta h_{kl}}), \tag{17}$$

$$Q_M(h_{ij}, \phi) = -h^{-1}\frac{1}{A}(\frac{\delta^2 A}{\delta\phi^2}), \tag{18}$$

$$\frac{\delta}{\delta h_{ij}}[2h^qG_{ijkl}\frac{\delta S}{\delta h_{kl}}A^2] + \frac{\delta}{\delta\phi}[\frac{h^q}{\sqrt{h}}\frac{\delta S}{\delta\phi}A^2] = 0. \tag{19}$$

These two equations (16)(19) come from the real and imaginary part of the WDW equation(14) respectively. In the same way, eq.(15) reads

$$2D_j\frac{\delta A}{\delta h_{ij}} - h^{ij}\partial_j\phi\frac{\delta A}{\delta\phi} = 0, \tag{20}$$

$$2D_j\frac{\delta S}{\delta h_{ij}} - h^{ij}\partial_j\phi\frac{\delta S}{\delta\phi} = 0. \tag{21}$$

We shall call $Q_G$and Q_M the gravitational quantum potential and the matter quantum potential. We can construct a conserved current for the WDW equation as follows

$$J_{ij} = -i[h^q G_{ijkl}(\Psi^*\frac{\delta\Psi}{\delta h_{kl}} - \frac{\delta\Psi^*}{\delta h_{kl}}\Psi)], \tag{22}$$

$$J_\phi = -\frac{i}{2}\frac{h^q}{\sqrt{h}}[\Psi^*\frac{\delta\Psi}{\delta\phi} - \frac{\delta\Psi^*}{\delta\phi}\Psi], \tag{23}$$

$$\frac{\delta J_{ij}}{\delta h_{ij}} + \frac{\delta J_\phi}{\delta\phi} = 0. \tag{24}$$

This current can be regarded as the probability flux in superspace. Using the polar decomposition of Ψ, eq.(22) and eq.(23) become

$$J_{ij} = 2h^q G_{ijkl}\frac{\delta S}{\delta h_{kl}}A^2, \tag{25}$$

$$J_\phi = \frac{h^q}{\sqrt{h}}\frac{\delta S}{\delta\phi}A^2. \tag{26}$$

Inserting (25) and (26) into (24), we see that the continuity equation(24) just agrees with eq.(19), which come from the imaginary part of the the WDW equation. Following quantum potential interpretation, we assume that

$$\frac{\delta S}{\delta h_{kl}} = \pi^{kl} \equiv \sqrt{h}(K^{kl} - h^{kl}K), \quad \frac{\delta S}{\delta\phi} = \pi_\phi \equiv \frac{\sqrt{h}}{N}\dot{\phi} - \sqrt{h}\frac{N^i}{N}\partial_i\phi. \tag{27}$$

Then, eq.(16) and eq.(19) become

$$\begin{aligned} &G_{ijkl}\pi^{ij}\pi^{kl} + \frac{1}{2}\frac{1}{\sqrt{h}}\pi_\phi^2 - \sqrt{h}(R^{(3)} - Q_G) \\ &+ \frac{1}{2}\sqrt{h}h^{ij}\partial_i\phi\partial_j\phi + \frac{1}{2}\sqrt{h}(V(\phi) + Q_M) = 0, \end{aligned} \tag{28}$$

$$\frac{\delta}{\delta h_{ij}}[2h^q A^2 K_{ij}] + \frac{\delta}{\delta\phi}[h^q A^2\frac{1}{\sqrt{h}}\pi_\phi] = 0, \tag{29}$$

where we have used $G_{ijkl}\pi^{kl} = K_{ij}$. Clearly eq.(16) resembles the Hamilton-Jacobi equation except for additional terms Q_G, Q_M. This suggests that we may regard the quantum universe as a universe with momentum $\pi^{ij} = \frac{\delta S}{\delta h_{ij}}, \pi_\phi = \frac{\delta S}{\delta\phi}$ subject not only to the classical potentials, but also to the quantum potentials Q_G, Q_M. If we adopt the gauge choice $N = 1, N_i = 0$, eq.(29) becomes

$$\frac{\delta}{\delta h_{ij}}(P\dot{h}_{ij}) + \frac{\delta}{\delta\phi}(P\dot{\phi}) = 0, \quad \text{with } P \equiv h^q A^2. \tag{30}$$

Eq.(30) is regarded as the continuity equation. $\tilde{J}_{ij} \equiv P\dot{h}_{ij}$, $\tilde{J}_\phi \equiv P\dot{\phi}$ is assumed to be the probability flux of the universe in a statistical ensemble of well-defined trajectories, in the gauge choice $N = 1, N_i = 0$. Generally, from eq.(29), $\tilde{J}_{ij} \equiv 2PK_{ij}$, $\tilde{J}_\phi \equiv P\frac{1}{\sqrt{h}}\pi_\phi$ is regarded as probability flux. The equations of motion of a universe are

$$\dot{h}_{ij} = \frac{2N}{\sqrt{h}}\pi_{ij} - \frac{N}{\sqrt{h}}h_{ij}h_{mn}\pi^{mn} + N_{i|j} + N_{j|i}, \tag{31}$$

$$\begin{aligned} \dot{\pi}^{ij} = &- N\sqrt{h}\{R^{ij} - \frac{1}{2}h^{ij}(R^{(3)} - Q_G) + \frac{\delta Q_G}{\delta h_{ij}}\} \\ &+ \frac{N}{2}\frac{h^{ij}}{\sqrt{h}}(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2) - \frac{2N}{\sqrt{h}}(\pi^{im}\pi^j_m - \frac{1}{2}h_{mn}\pi^{mn}\pi^{ij}) \\ &- N^i_{|m}\pi^{mj} - N^j_{|m}\pi^{mi} + (\pi^{ij}N^m)_{|m} + \sqrt{h}(N^{|ij} - h^{ij}N^{|m}_{|m}) \\ &+ \frac{N}{2}[\frac{1}{2}\frac{h^{ij}}{\sqrt{h}}\pi_\phi^2 - \frac{1}{2}h^{ij}h^{mn}\partial_m\phi\partial_n\phi - \sqrt{h}\partial^i\phi\partial^j\phi \\ &- \frac{1}{2}\sqrt{h}h^{ij}(V(\phi) + Q_M) - \sqrt{h}\frac{\delta Q_M}{\delta h_{ij}}], \end{aligned} \tag{32}$$

$$\dot{\phi} = N\frac{\pi_\phi}{\sqrt{h}} + N^i\partial_i\phi, \tag{33}$$

$$\dot{\pi}_\phi = \partial_j(N\sqrt{h}h^{ij}\partial_i\phi) - N\frac{\sqrt{h}}{2}(\frac{\delta V(\phi)}{\delta\phi} + \frac{\delta Q_M}{\delta\phi}) - N\sqrt{h}\frac{\delta Q_G}{\delta\phi} + \partial_i(N^i\pi_\phi). \tag{34}$$

These equations can be obtained by the variation of the following "action" containing quantum potentials:

$$S_Q = \int d^4x[\pi^{ij}\dot{h_{ij}} + \pi_\phi\dot{\phi} - NH_Q(\pi^{ij}, h_{ij}) - N_iH^i(\pi^{ij}, h_{ij})], \tag{35}$$

$$H_Q = \frac{1}{\sqrt{h}}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2) - \sqrt{h}(R^{(3)} - Q_G) + \frac{1}{\sqrt{h}}\frac{\pi_\phi^2}{2} + \sqrt{h}\frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + \frac{\sqrt{h}}{2}(V(\phi) + Q_M), \tag{36}$$

$$H^i = -2\pi^{ij}_{;j} + \pi_\phi\partial^i\phi. \tag{37}$$

To solve eq.(31) $\sim$ eq.(34) is very difficult, since we must solve the WDW equation (14) first and obtain the quantum potentials Q_G, Q_M. Hereafter, our attention will be concentrated on minisuperspace models in which the infinite number of degrees of freedom of the gravitational and matter fields are restricted to a finite number.

IV. THE DE SITTER MINISUPERSPACE MODEL

We shall consider the de Sitter minisuperspace model [7][9]. It is described by the closed Robertson-Walker metric

$$ds^2 = \sigma^2[-N^2dt^2 + a(t)^2d\Omega_3^2]. \tag{38}$$

where $\sigma^2 = \frac{2G}{3\pi}$, $a(t)$ is the scale factor and $d\Omega_3^2$ is the metric on the unit three-sphere. The Einstein action with (rescaled) cosmological constant for this metric is

$$S = \frac{1}{2}\int dtN[-\frac{a\dot{a}^2}{N^2} + a - \Lambda a^3]. \tag{39}$$

The WDW equation in this model is given by [9]

$$\{\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - a^2[1 - a^2\Lambda]\}\Psi(a) = 0, \tag{40}$$

where the parameter p represents the factor-ordering ambiguity. We write Ψ in a polar decomposition as $\Psi(a) = A(a)\exp\{iS(a)\}$, in terms of which eq.(40) can be rewritten as

$$-(\frac{\partial S}{\partial a})^2 - a^2 + a^4(\Lambda + Q_G(a)) = 0, \tag{41}$$

$$\partial_a(a^pA^2\partial_aS) = 0. \tag{42}$$

These two equations come from the real and imaginary part of eq.(40), respectively, and the gravitational quantum potential is

$$Q_G(a) = a^{-4}\frac{1}{A}(\frac{\partial^2 A}{\partial a^2} + \frac{p}{a}\frac{\partial A}{\partial a}). \tag{43}$$

Following quantum potential interpretation, we identify ∂_aS with the canonical momentum p_a to a:

$$-a\dot{a} \equiv p_a = \frac{\partial S}{\partial a}. \tag{44}$$

Then, eq.(41) and eq.(42) become

$$\dot{a}^2 = -1 + a^2(\Lambda + Q_G(a)), \tag{45}$$

$$\partial_a(P\dot{a}) = 0, \quad \text{with } P \equiv a^{p+1}A^2, \tag{46}$$

where a dot stands for the differentiation with respect to "extended" Robertson-Walker time, which describes causally deterministic trajectory of the universe not only in classically allowed region ($\Lambda a^2 > 1$), but also in classically forbidden region ($\Lambda a^2 < 1$). In the region of minisuperspace where $|Q_G| \ll \Lambda$, eq.(45) approaches to the classical equation of motion for the de Sitter space. We note that the probability density $\tilde{J}_a \equiv P\dot{a}$ is conserved if $a(t)$ is a solution to eq.(44). The continuity equation (46) implies conservation of probability density. The choice $p = 1$ corresponds to

the minisuperspace model (with a scalar field) dealt with in ref.[5]. We will keep the value of p unspecified. With the choice $p = -1$, eq.(40) can be solved exactly [8]. The solution corresponding to the Vilenkin boundary condition is (disregarding the normalization constant)

$$\Psi_V = \frac{\mathrm{Ai}(z) + i\mathrm{Bi}(z)}{\mathrm{Ai}(z_0) + i\mathrm{Bi}(z_0)}, \tag{47}$$

where

$$z = (2\Lambda)^{-\frac{2}{3}}(1 - a^2\Lambda), \tag{48}$$

$z_0 = z(a = 0)$, and $\mathrm{Ai}(z)$, $\mathrm{Bi}(z)$ are Airy functions. In this case, gravitational quantum potential (43) becomes

$$\begin{aligned} Q_G(a) = &(2V)^{\frac{2}{3}} a^{-2}\{(\mathrm{Ai}'^2 + z\mathrm{Ai}^2 + \mathrm{Bi}'^2 + z\mathrm{Bi}^2)(\mathrm{Ai}^2 + \mathrm{Bi}^2)^{-1} \\ &- (\mathrm{Ai}\mathrm{Ai}' + \mathrm{Bi}\mathrm{Bi}')^2(\mathrm{Ai}^2 + \mathrm{Bi}^2)^{-2}\}, \end{aligned} \tag{49}$$

where dash stands for the differentiation with respect to z. "Total potential" $W(a) \equiv 1 - a^2(\Lambda + Q_G(a))$ is shown in Fig.1. Fig.1 shows that the universe can start at $a \approx 0$. The asymptotic forms of the Airy functions at large value of $z (z \to +\infty)$ are [10]:

$$Ai(z) \sim \frac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \exp(-\zeta), \tag{50}$$

$$Bi(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \exp(\zeta), \tag{51}$$

$$Ai(-z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \sin(\zeta + \pi/4), \tag{52}$$

$$Bi(-z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \cos(\zeta + \pi/4), \tag{53}$$

where $\zeta = \frac{2}{3} z^{\frac{3}{2}}$. We assume that $0 < \Lambda \ll 1$ in Planck unit. Using these asymptotic forms, gravitational quantum potential (49) for small scales ($a \to 0$) and large scales ($a \to \infty$) is given by

$$Q_G(a) \approx a^{-2} - \Lambda, \quad \text{for } a \ll \frac{1}{\sqrt{\Lambda}}, \tag{54}$$

$$Q_G(a) \approx 0, \quad \text{for } a \gg \frac{1}{\sqrt{\Lambda}}. \tag{55}$$

Then, eq.(45) becomes

$$\dot{a}^2 \approx 0, \quad \text{for } a \ll \frac{1}{\sqrt{\Lambda}}, \tag{56}$$

$$\dot{a}^2 \approx -1 + a^2\Lambda, \quad \text{for } a \gg \frac{1}{\sqrt{\Lambda}}. \tag{57}$$

For the exact gravitational quantum potential (49), we have solved eq.(45) numerically (see Fig.2). Fig.2 and eq.(57) shows explicitly that "extended" Robertson-Walker time is identified with the usual one at large scales ($\Lambda a^2 \gg 1$) in classically allowed region ($\Lambda a^2 > 1$) by an appropriate shift of this time (axis).

Another proposed boundary condition is the Hartle-Hawking wave function, which in the choice $p = -1$ is given by [9]

$$\Psi_H = \frac{Ai(z)}{Ai(z_0)}. \tag{58}$$

In this case, gravitational quantum potential (43) becomes

$$Q_G(a) = a^{-2}(1 - a^2\Lambda). \tag{59}$$

Then, for all scales, eq.(45) becomes

$$\dot{a}^2 = 0. \tag{60}$$

Naively, eq.(60) means that universe is at rest. This result is an unsurprising fact because the wave function Ψ_H is real, so that we obtain $p_a = \frac{\partial S}{\partial a} = 0$ (see (44)). This situation resembles that of the problems of a stationary state of an atom with zero angular momentum, or a "free" particle contained between two impenetrable and

perfectly reflecting walls, which have been discussed by Bohm [3]. Quantum potential interpretation to the Hartle-Hawking wave function will not be further discussed in this paper.

Next, we shall discuss the quantum potential interpretation to the wave function for any value of the factor-ordering parameter p. The wave function is calculated by using the WKB approximation. The WKB approximation is an expansion in power of $\Lambda \ll 1$. We look for a solution to eq.(40) in the form ($\Psi = \exp(iS)$) of an expansion

$$S = S_0 + S_1 + \cdots, \tag{61}$$

where $S_n \propto \Lambda^{n-1}$. We obtain the following expression for the wave function up to S_1 in the classically forbidden region ($\Lambda a^2 < 1$) [8]:

$$\Psi(a) = Ca^{-(p+1)/2}(1-\Lambda a^2)^{-1/4}\exp\{-\frac{1}{3\Lambda}[1-(1-\Lambda a^2)^{3/2}]\}, \tag{62}$$

where C is a constant. The wave function in the classically allowed region ($\Lambda a^2 > 1$) can be found by means of analytic continuation [8]:

$$\Psi(a) = Ce^{i\pi/4}a^{-(p+1)/2}(\Lambda a^2-1)^{-1/4}\exp\{-\frac{1}{3\Lambda}[1+i(\Lambda a^2-1)^{3/2}]\}. \tag{63}$$

This wave function(63) corresponds to a positive frequency out-going mode. In classically allowed regions the positive frequency mode corresponds to the expanding universe [8][9][11]. From (62), (63) and (43), quantum potential is given by

$$Q_G(a) = -\frac{(p+1)(p-3)}{4}a^{-6} + \frac{5}{4}\Lambda^2 a^{-2}(1-\Lambda a^2)^{-2} + a^{-2}(1-\Lambda a^2), \quad \text{for } \Lambda a^2 < 1, \tag{64}$$

$$Q_G(a) = -\frac{(p+1)(p-3)}{4}a^{-6} + \frac{5}{4}\Lambda^2 a^{-2}(1-\Lambda a^2)^{-2}, \quad \text{for } \Lambda a^2 > 1. \tag{65}$$

Then, "total potential" $W(a) \equiv 1 - a^2(\Lambda + Q_G(a))$ is given by

$$W(a) = \frac{(p+1)(p-3)}{4}a^{-4} - \frac{5}{4}\Lambda^2(1-\Lambda a^2)^{-2}, \quad \text{for } \Lambda a^2 < 1, \tag{66}$$

$$W(a) = 1 - a^2\Lambda + \frac{(p+1)(p-3)}{4}a^{-4} - \frac{5}{4}\Lambda^2(1-\Lambda a^2)^{-2}, \quad \text{for } \Lambda a^2 > 1. \tag{67}$$

The behavior of $W(a)$ with respect to a depends on the value of p (see Fig.3 and Fig.4). For $-1 \le p \le 3$, $W(a)$ is always negative. So, the universe can start at $a \approx 0$. For $\Lambda a^2 \ll 1$ and $-1 < p < 3$, $W(a) \approx \frac{(p+1)(p-3)}{4}a^{-4}$. Then, from eq.(45), the time evolution of scale factor is obtained as

$$a(t) = [\frac{3}{2}\{(p+1)(3-p)\}^{1/2}t + t_0]^{1/3}, \quad \text{for } \Lambda a^2 \ll 1. \tag{68}$$

where t_0 is an initial time. For $\Lambda a^2 < 1$ and $p = -1$ or $p = 3$, $W(a) = -\frac{5}{4}\Lambda^2(1-\Lambda a^2)^{-2}$. Then, in the same way, we have

$$a(t) = \frac{1}{\sqrt{\Lambda}}\tanh\{\frac{\sqrt{5}}{2}\Lambda^{3/2}(t+t_0)\}, \quad \text{for } \Lambda a^2 < 1. \tag{69}$$

For $p < -1$ or $p > 3$, $W(a) < 0$ for $a > c$ and $W(a) > 0$ for $a < c$, where

$$c = (\frac{(p+1)(p-3) - \sqrt{5(p+1)(p-3)}}{\{(p+1)(p-3)-5\}\Lambda})^{1/2}, \quad \text{for } p \neq -2 \text{ and } p \neq 4, \tag{70}$$

$$c = \frac{1}{\sqrt{2\Lambda}}, \quad \text{for } p = -2 \text{ or } p = 4. \tag{71}$$

So, the universe can start at $a = c$ ($c \sim \frac{1}{\sqrt{\Lambda}}$, since $0 < \Lambda \ll 1$). Thus, in quantum potential interpretation, the behavior of causally determined trajectories of the universe depends on the choice of factor-ordering parameter p, especially in classically forbidden region. For $-1 \le p \le 3$, the universe can start at $a \approx 0$. On the other hand, for $p < -1$ or $p > 3$, the universe can not start at $a \approx 0$.

V. CHAOTIC INFLATIONARY MINISUPERSPACE MODELS

Finally, we briefly discuss chaotic inflationary minisuperspace models. For closed Robertson-Walker metric, the action is

$$S = \int dt\frac{1}{2}[-\frac{a\dot{a}^2}{N} + \frac{a^3\dot{\phi}^2}{N} + Na - Na^3V(\phi)], \tag{72}$$

where the scalar field ϕ is assumed to be homogeneous and isotropic, and the matter potential $V(\phi)$ is the chaotic inflationary type ($V(\phi) = m^2\phi^2$ or $\lambda\phi^4$) [12]. The canonical conjugate momenta to a and ϕ are

$$p_a = -a\dot{a}, \quad p_\phi = a^3\dot{\phi}. \tag{73}$$

where we set $N = 1$. The corresponding WDW equation is [9]

$$\{\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial \phi^2} - a^2[1 - a^2V(\phi)]\}\Psi(a,\phi) = 0. \tag{74}$$

Then the conserved current $j = (j_a,\ j_\phi)$ is given by [8][9]

$$j_a = \frac{i}{2}a^p(\Psi^*\partial_a\Psi - \Psi\partial_a\Psi^*), \tag{75}$$

$$j_\phi = -\frac{i}{2}a^{p-2}(\Psi^*\partial_\phi\Psi - \Psi\partial_\phi\Psi^*), \tag{76}$$

$$\partial_a j_a + \partial_\phi j_\phi = 0. \tag{77}$$

We write the wave function in the form $\Psi = A(a,\phi)\exp(iS(a,\phi))$. Then the WDW equation (74) can be rewritten as

$$-(\partial_a S)^2 + \frac{1}{a^2}(\partial_\phi S)^2 - a^2 + a^4(V(\phi) + Q_G + Q_M) = 0, \tag{78}$$

$$Q_G(a,\phi) = a^{-4}\frac{1}{A}(\partial_a^2 A + \frac{p}{a}\partial_a A), \tag{79}$$

$$Q_M(a,\phi) = -a^{-6}\frac{1}{A}\partial_\phi^2 A, \tag{80}$$

$$-\partial_a(a^pA^2\partial_a S) + \partial_\phi(a^{p-2}A^2\partial_\phi S) = 0. \tag{81}$$

Eq.(78) and eq.(81) come from the real and imaginary part of eq.(74), respectively. In quantum potential interpretation, $\partial_a S$ and $\partial_\phi S$ are identified with momenta p_a and p_ϕ:

$$-a\dot{a} \equiv p_a = \partial_a S, \quad a^3\dot{\phi} \equiv p_\phi = \partial_\phi S. \tag{82}$$

Then eq.(78) and eq.(81) become

$$-\dot{a}^2 + a^2\dot{\phi}^2 - 1 + a^2(V(\phi) + Q_G + Q_M) = 0, \tag{83}$$

$$\partial_a(P\dot{a}) + \partial_\phi(P\dot{\phi}) = 0, \quad \text{with } P = a^{p+1}A^2. \tag{84}$$

Eq.(84) is the continuity equation, which implies conservation of probability flux $\tilde{J}_a \equiv P\dot{a}$, $\tilde{J}_\phi \equiv P\dot{\phi}$. Also, the continuity equation (77) just agrees with eq.(81), which comes from the imaginary part of the WDW equation (74), because, using polar decomposition of Ψ, eq.(75) and eq.(76) become $j_a = -a^pA^2\partial_a S$, $j_\phi = a^{p-2}A^2\partial_\phi S$. $\tilde{J}_a$ can be interpreted as the probability density for ϕ at a given value of a. We note that the probability density $\rho_\phi(a) = \tilde{J}_a = P\dot{a}$ is always positive (even in classically forbidden region) if the universe is expanding ($\dot{a} > 0$). Equations of motion of the universe acted on by classical potentials and quantum potentials are

$$\frac{\ddot{a}}{a} + 2\dot{\phi}^2 - (V(\phi) + Q_G + Q_M) - \frac{a}{2}(\partial_a Q_G + \partial_a Q_M) = 0, \tag{85}$$

$$\ddot{\phi} + 3(\frac{\dot{a}}{a})\dot{\phi} + \frac{1}{2}(\partial_\phi V(\phi) + \partial_\phi Q_G + \partial_\phi Q_M) = 0. \tag{86}$$

These equations can be obtained from the following equations

$$\dot{a} = \{a, H_Q\}_p, \quad \dot{p}_a = \{p_a, H_Q\}_p, \tag{87}$$

$$\dot{\phi} = \{\phi, H_Q\}_p, \quad \dot{p}_\phi = \{p_\phi, H_Q\}_p, \tag{88}$$

where

$$H_Q = -\frac{1}{2a}[p_a^2 - \frac{1}{a^2}p_\phi^2 + a^2 - a^4(V(\phi) + Q_G + Q_M)]. \tag{89}$$

Alternatively, eq.(85) and eq.(86) are obtained from the Euler-Lagrange equations for the "Lagrangian" containing quantum potentials:

$$L_Q = \frac{1}{2}[-\frac{a\dot{a}^2}{N} + \frac{a^3\dot{\phi}^2}{N} + Na - Na^3(V(\phi) + Q_G + Q_M)]. \tag{90}$$

To solve equation (74), we make an assumption that V is small compared to Planck scale ($0 < V \ll 1$) and is a slowly varying function of ϕ, i.e.

$$|\frac{dV}{d\phi}| \ll \max\{V, \frac{1}{a^2}\}. \tag{91}$$

Then the derivative with respect to ϕ in eq.(74) can be neglected. If we choose Vilenkin boundary conditions [8][9], the wave function with the choice $p = -1$ is, (up to a numerical coefficient), given by

$$\Psi = \frac{Ai(z) + iBi(z)}{Ai(z_0) + iBi(z_0)}, \tag{92}$$

where $z = (2V)^{-2/3}(1 - a^2V)$ and $z_0 = z(a = 0)$. Using the asymptotic forms of Airy functions (50)~(53), we have

$$\Psi = (1 - a^2V)^{-1/4}\exp\{-\frac{1}{3V}[1 - (1 - a^2V)^{3/2}]\} \quad \text{for } a^2V < 1, \tag{93}$$

$$\Psi = e^{i\pi/4}(a^2V - 1)^{-1/4}\exp\{-\frac{1}{3V}[1 + i(a^2V - 1)^{3/2}]\} \quad \text{for } a^2V > 1. \tag{94}$$

In the quantum potential interpretation, the WDW wave function defines a family of trajectories distributed with probability flux $\tilde{J}_a = P\dot{a}$, $\tilde{J}_\phi = P\dot{\phi}$. The probability distribution for ϕ at a given value of a in classically allowed region can be found by using eq.(94) and $\rho_a(\phi) \equiv \tilde{J}_a$ with an appropriate normalization constant as follows

$$\rho_a(\phi) \sim \exp(-\frac{2}{3V(\phi)})/\int d\phi \exp(-\frac{2}{3V(\phi)}). \tag{95}$$

The factor $\exp(-\frac{2}{3V(\phi)})$ means that the probability density for small values of $V(\phi)$ is exponentially suppressed. This is favorable for the chaotic inflation scenario.

VI. SUMMARY AND DISCUSSION

We have applied quantum potential interpretation to quantum geometrodynamics based on the ADM canonical formalism, and the equations of motion of the universe acted on not only by classical potentials, but also by quantum potentials were given in Sec.III. In Sec.IV we have studied in detail the de Sitter minisuperspace model from the viewpoint of the quantum potential interpretation, by introducing the "extended" Robertson-Walker time which exists even in classically forbidden region. It was shown that "extended" Robertson-Walker time is identified with usual one at large scales ($\Lambda a^2 \gg 1$) in classically allowed region ($\Lambda a^2 > 1$) by an appropriate shift of this time (axis), and the behavior of the trajectories of the universe described by the Vilenkin wave function depends on the choice of operator-ordering parameter p in the WDW equation: for $-1 \leq p \leq 3$, the universe can start at $a \approx 0$, for $p < -1$ or $p > 3$, the universe can not start at $a \approx 0$, but can start $a = c$, ($c \sim \frac{1}{\sqrt{\Lambda}}$, for $0 < \Lambda \ll 1$) (see Sec.IV). We believe that the correct value of the operator-ordering parameter exists in the range $-1 \leq p \leq 3$ from the viewpoint of "creation of the universe from nothing"[8][9]. It was also shown that, in quantum potential interpretation, the probability density $\rho_a(\phi) = \tilde{J}_a = P\dot{a}$ is always positive (even in classically forbidden region) if the universe is expanding ($\dot{a} > 0$). In quantum potential interpretation, the universe described by Hartle-Hawking wave function is, naively speaking, at rest. But we know that our universe is now expanding. A further investigation of the quantum potential interpretation to the real wave function of the universe will be needed if we want to describe our universe by the real wave function. In Sec.V we have briefly studied quantum potential approach to chaotic inflationary minisuperspace models. Further investigation of quantum potential approach to inflationary universe models will be reported elsewhere.

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FIGURES

FIG. 1. "Total potential" $W(a) \equiv 1 - a^2(\Lambda + Q_G(a))$ for $Q_G(a)$ given by eq.(49) (solid curve), $Q_G(a) = 0$ (dashed curve), with $\Lambda = 0.5$ in Planck unit.

FIG. 2. Solutions $a(t)$ to eq.(45) for $Q_G(a)$ given by eq.(49)(label A), $Q_G(a) = 0$ (label B), with $\Lambda = 0.5$ in Planck unit. The equation (45) for $Q_G(a)$ given by eq.(49) was integrated numerically, by starting from $t = 0$ at $a \approx 0$ (in this example, we have actually used $t = 0$ at $a = 0.01$). The curve labeled by B represents the de Sitter space $a(t) = \frac{1}{\sqrt{\Lambda}}\cosh(\sqrt{\Lambda}t)$ for $t \geq 0$.

FIG. 3. For $-1 < p < 3$, the behaviour of "total potential" $W(a) \equiv 1 - a^2(\Lambda + Q_G(a))$ for $Q_G(a)$ given by eq.(64) and eq.(65) (solid curve), $Q_G(a) = 0$ (dashed curve).

FIG. 4. For $p < -1$ or $p > 3$, the behaviour of "total potential" $W(a) \equiv 1 - a^2(\Lambda + Q_G(a))$ for $Q_G(a)$ given by eq.(64) and eq.(65) (solid curve), $Q_G(a) = 0$ (dashed curve).

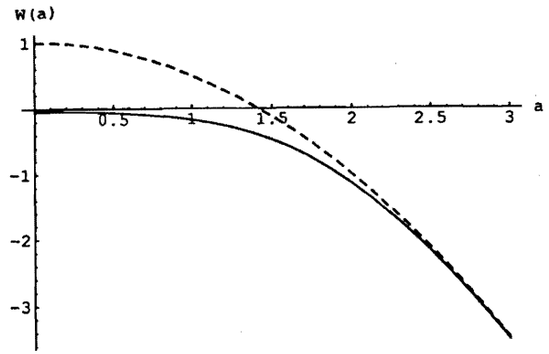

Fig.1

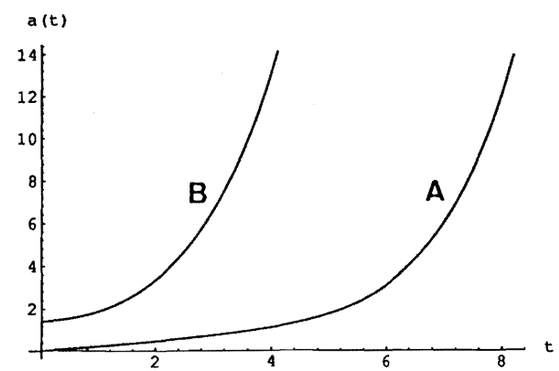

Fig.2

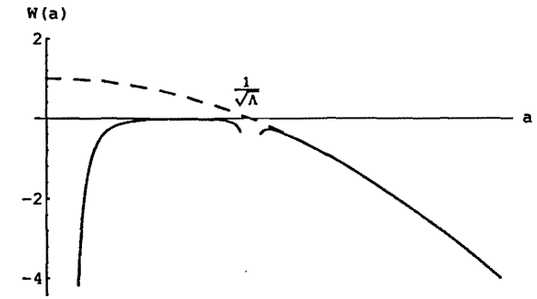

Fig.3

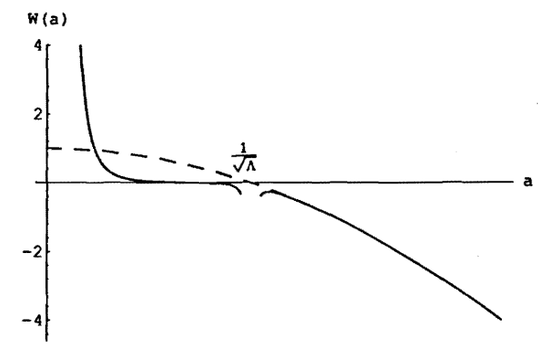

Fig.4