FERMILAB
OCT 2 9
KIFR-94-03
KIFR-94-03
Fermilab Library
0 1160 0053180 0

On the Renormalizability of the Wheeler-DeWitt Equation

Tsutomu Horiguchi

Kibi Institute of Fundamental Research, 1384 Tujita, Mabi-cho, Kibi-gun, Okayama-ken, Japan.

(October 1994)

Abstract

We concretely show that, in $3+1$ dimensions, the Wheeler-DeWitt equation to leading order in powers of Planck mass of the wave function is renormalizable, and briefly discuss about the renormalizability of the Wheeler-DeWitt equation for $N+1$ dimensional Lorentzian manifolds ($N > 3$).

NAME | ID No | M.S.
RETURN TO FERMILAB LIBRARY

Typeset using REVTEX

In my previous papers [1] [2], we showed that the dimensional reduction in quantum gravity can occur at the scale which is much smaller than Planck scale under the analysis of the Wheeler-DeWitt (WDW) equation, and showed that "New phase" (the dynamical system described by the three dimensional quantum Einstein gravity) exists at the region beyond Planck scale or at the very early stage of the evolution of the universe. In our new scenario of the evolution of the universe, time axis is born when the size of the created universe from nothing approaches the Planck size. In this paper we concretely show that the WDW equation to leading order in powers of Planck mass of the wave function is renormalizable, and briefly discuss about the renormalizability of the WDW equation for the Lorentzian manifolds with extra dimensions.

It is well known that perturbative approach to quantum gravity which depends on fixed background metric(Minkowski space-time) is not renormalizable [6]. Quantum gravitational effect becomes important at the Planck scale. In the region beyond Planck scale, space-time structure may be very different from Minkowski space-time by quantum effect of space-time. However, usually, perturbative approach to quantum gravity assumes that the space-time structure at the region beyond Planck scale is also Minkowski space-time in the sense of back-ground metric. In ref. [1] [7], it was discussed that dimensional reduction can occur at the region beyond Planck scale by quantum effect of space-time, and, in ref. [1], it was shown that quantum geometrodynamics based on the WDW equation (which does not depend on the background metric) to leading order in powers of the Planck mass of the wave function is renormalizable by nonperturbative effects of gravitation at very small scales.

This picture of space-time structure at the region beyond Planck scale is very different from that of string theories [8]. Superstring theory assumes that, at the region beyond Planck scale, space-time is ten dimensional, and extra dimensional space is compactified to Kähler manifolds with Calabi-Yau metric [9] or orbifolds [10], which are classical spaces with Planck size. It is usually thought that, in superstring theory, our universe was a ten dimensional space-time at the very early stage of the evolution of the universe, and the compactification of extra dimensional space occurs at the early stage in which the size

of our universe is probably Planck size. In string theories which are expected to unify the gravitation and matter fields, nonexistence of ultraviolet divergence comes from the competition of gravitational wave modes and other matter fields modes [8]. In our case (WDW equation), the renormalizability of quantum gravity comes from nonperturbative effects of quantum geometrodynamics, in other wards, from only the quantum effect of space-time.

2. We summarize our results in my previous paper [1]. Wheeler-DeWitt equation [3] [4] is

$$[-\frac{1}{m_p^2}\Delta(x) + m_p^2\sqrt{h(x)}(R^{(3)}(x) - 2\Lambda)]\Psi(h) = 0, \tag{1}$$

$$\Delta(x) \equiv G_{ijkl}(x)\frac{\delta^2}{\delta h_{ij}(x)\delta h_{kl}(x)}, \tag{2}$$

where h_{ij} are spatial metrics, $h = \det h_{ij}$, $R^{(3)}$ denotes the three dimensional scalar curvature, $G_{ijkl} = \frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$, m_p is the Planck mass, $16\pi G = 1/m_p^2$, G is the Newton constant, and Λ is the cosmological constant. The WDW equation stands in need of regularization because the two functional derivatives act at the same point in space. So, for example, $\Delta(x)$ acting on $R^{(3)}(y)$ is proportional to $(\delta(x,y))^2$, which is meaningless. Therefore, we shall construct a regulated operator, Δ_ϕ, that preserves the three dimensional general coordinate transformation. Then, the finite version of the WDW equation will be

$$[-\frac{1}{m_p^2}\Delta_\phi + m_p^2\sqrt{h}(R^{(3)} - 2\Lambda)]\Psi(h) = 0. \tag{3}$$

If we attempt an expansion of the wave function in powers of the Planck mass as follows

$$\Psi(h) \equiv \exp(-S(h)) = \exp(-\sum_{n=1}^{\infty} m_p^{4n} S_n), \tag{4}$$

the WDW equation becomes

$$\frac{1}{m_p^2}\Delta_\phi S - \frac{1}{m_p^2}G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{kl}} + m_p^2\sqrt{h}(R^{(3)} - 2\Lambda) = 0. \tag{5}$$

So that to lowest order, eq.(5) becomes

$$\Delta_\phi S_1 + \sqrt{h}(R^{(3)} - 2\Lambda) = 0. \tag{6}$$

We should not expect our approximation (6) to leading order in our expansion (4) to resemble the WKB approximation (the semi-classical approximation). Eq.(6) gives a very different wave function from that given by the WKB expansion ($S = \sum_{n=1}^{\infty}(\frac{1}{m_p^2})^{n-2}W_n$) which would contain propagators, and so be non-local. We state how to regurate the Wheeler-DeWitt equation. We will replace Δ in the Wheeler-DeWitt equation by a regulated operator Δ_ϕ. This regularization and renormalization procedure introduce an arbitrary function ϕ on which Δ_ϕ depends. This is an inevitable consequence of isolating the finite part of a divergent quantity. Physical quantities should be independent of this arbitrariness, e.g.by absorbing it all into the couplings. To construct Δ_ϕ, we begin by "point-splitting" the functional derivatives. Consider the operator

$$\Delta_\epsilon(x) \equiv \int d^3x' K_{i'j'kl}(x', x; \epsilon)\frac{\delta}{\delta h_{i'j'}(x')}\frac{\delta}{\delta h_{kl}(x)}, \tag{7}$$

where the $K_{i'j'kl}$ is a bi-tensor at both x' and x, satisfying some heat equation with the initial condition

$$\lim_{\epsilon\to 0} K_{i'j'kl}(x', x; \epsilon) = G_{ijkl}(x)\delta(x', x). \tag{8}$$

If Σ denote the space of sums of products of three-dimensional integrals of local functions of h_{ij} with finite coefficients, then Δ acts on a typical element of Σ. We will replace Δ in the WDW equation by a regulated operator, Δ_ϕ, which does have a finite action on elements of Σ and maps Σ to itself. There are many choices of heat-kernel. For simplicity of our analysis, we choose the heat-kernel to satisfy the following heat equation

$$\frac{\partial}{\partial\epsilon}K_{i'j'kl} = D' \cdot D'K_{i'j'kl}, \tag{9}$$

where $D'_{i'}$ is a covariant derivative with respect to $x'_{i'}$ in three dimensional space. We need to know the diagonal matrix elements of $K_{i'j'kl}(x', x; \epsilon)$ for short "times". These may be computed using a standard technique that assumes the existence of expansions of the form [5]

$$K_{i'j'kl}(x', x; \epsilon) = \frac{\Delta^{1/2}(x', x)e^{-\frac{\sigma(x',x)}{2\epsilon}}}{\sqrt{(4\pi\epsilon)^3}}\sum_{n=0}^{\infty}\epsilon^n a^{(n)}_{i'j'kl}(x', x), \tag{10}$$

where the bi-scalar $\sigma(x',x)$, which we shall call the geodetic integral is equal to one half the square of the distance along the geodesic between x and x', and $\Delta(x',x)$ is a bi-scalar, $\Delta(x',x) \equiv h^{-1/2}(x')D(x',x)h^{-1/2}(x)$, $D \equiv -\det(D_{i'j})$, $D_{i'j} \equiv -\sigma_{;i'j}$. $;i'j$ denotes the covariant differentiations with respect to $x^{i'}$ and x^j [5]. $a^{(n)}_{i'j'kl}(x',x)$ is a bi-tensor at both x' and x. The prefactor in (10) solves the ordinary heat equation. The initial conditions (8) are satisfied by taking

$$\lim_{x'\to x} a^{(0)}_{i'j'kl}(x',x) = \sqrt{h}G_{ijkl}(x). \tag{11}$$

Inserting (10) into eq.(9) and making use of $\frac{1}{2}\sigma_{;i'}\sigma^{;i'} = \sigma$ and $\sigma^{;j'}\sigma_{;j'i} = \sigma_{;i}$ [5], one finds [5]

$$\sigma^{;m'}(x',x)a^{(0)}_{i'j'kl;m'}(x',x) = 0, \tag{12}$$

$$\begin{aligned}\sigma^{;m'}(x',x)a^{(n+1)}_{i'j'kl;m'}(x',x) &+ (n+1)a^{(n+1)}_{i'j'kl}(x',x) \\ &= \Delta^{-1/2}(x',x)(\Delta^{1/2}(x',x)a^{(n)}_{i'j'kl}(x',x))_{;m'}{}^{m'} \quad \text{for} \quad n = 0,1,2,3,\cdots. \end{aligned} \tag{13}$$

Using (10),(11) and the recursion relations (12), (13), after the tedious calculation, we have

$$\begin{aligned}\Delta_\epsilon \int d^3x\sqrt{h}R^{(3)} &= \alpha_1\epsilon^{-5/2}\sqrt{h} + \alpha_2\epsilon^{-3/2}\sqrt{h}R^{(3)} + \epsilon^{-1/2}(\alpha_3\sqrt{h}(R^{(3)})^2 + \alpha_4\sqrt{h}R^{(3)}_{ij}R^{(3)ij} \\ &+ \alpha_5\sqrt{h}R^{(3)i}_{;i}) + O(\epsilon^{1/2}),\end{aligned} \tag{14}$$

$$\Delta_\epsilon \int d^3y\sqrt{h} = \beta_1\epsilon^{-3/2}\sqrt{h} + \beta_2\epsilon^{-1/2}\sqrt{h}R^{(3)} + O(\epsilon^{1/2}), \tag{15}$$

where α_i and β_i are numerical constants, $\alpha_1 = 3/(2(4\pi)^{-3/2})$, $\alpha_2 = -11/(24(4\pi)^{-3/2})$, and $\beta_1 = -21/(8(4\pi)^{-3/2})$. The proper-time ϵ has the dimensions of $[m_p]^{-2}$, whilst Δ_ϵ has dimensions of $[m_p]^6$. These results (14),(15) are also suggested by dimensional analysis. We can extract a finite result for $\Delta_\epsilon\sigma$ when $\epsilon = 0$ using analytic continuation in an analogous fashion to zeta function regularization, where $\sigma\epsilon\Sigma$. We take the action of Δ_ϕ on an arbitrary element, $\sigma\epsilon\Sigma$ to be [13]

$$\Delta_\phi\sigma \equiv \lim_{s\to 0} s\int_0^\infty dp p^{s-1}\phi(p)\Delta_{p^2}\sigma, \tag{16}$$

where the integral on the right-hand side is defined by analytic continuation in s. It is assumed that $\phi(p)$ is a differentiable function equal to one at the origin ($\phi(0) = 1$) and has the property $\lim_{p\to\infty} \phi(p)f(p) = 0$, where $f(p)$ is also a differentiable function. For the more detailed discussion about this regularization scheme, see ref. [13]. We take Δ at proper-time p^2 rather than p because of the square roots appearing in (14) and (15). Then (14) and (15) become

$$\Delta_\phi \int d^3x\sqrt{h}R^{(3)} = \frac{\alpha_1}{5!}\phi^{(5)}(0)\sqrt{h} + \frac{\alpha_2}{3!}\phi^{(3)}(0)\sqrt{h}R^{(3)} + \phi^{(1)}(0)(\alpha_3\sqrt{h}(R^{(3)})^2 + \alpha_4\sqrt{h}R^{(3)}_{ij}R^{(3)ij} + \alpha_5\sqrt{h}R^{(3)i}_{;i}), \tag{17}$$

$$\Delta_\phi \int d^3x\sqrt{h} = \frac{\beta_1}{3!}\phi^{(3)}(0)\sqrt{h} + \beta_2\phi^{(1)}(0)\sqrt{h}R^{(3)}, \tag{18}$$

where $\phi^{(n)}(0) \equiv \frac{d^n\phi(p)}{dp^n}|_{p=0}$, for $n = 1, 2, 3, \cdots$. We have done part of the job of renormalization. If the quantum theory of gravitation based on the WDW equation is not anomalous, the regularized Hamiltonian constraint ($\tilde{H} \approx 0$) and momentum constraints ($H^i \approx 0$) should constitute closed algebras such as in the classical theory of gravitation based on Arnowitt-Deser-Misner canonical formalism [11] [12]. If we compute the following commutator , we get

$$[\int d^3x\xi(x)\tilde{H}(x), \int d^3y\eta(x)\tilde{H}(y)] = i\int d^3x(\xi(x)\eta(x)_{;i} - \xi(x)_{;i}\eta(x))H^i(x) + \Delta\Gamma, \tag{19}$$

$$\Delta\Gamma = c\phi^{(1)}(0)\int d^3x\sqrt{h}(\xi(x)\eta(x)_{;i} - \xi(x)_{;i}\eta(x))R^{(3)i}(x), \tag{20}$$

where c is a numerical constant. An additional term (second term of right hand side) arises from the cross commutator of Δ_ϕ and $\sqrt{h}R^{(3)}$ as we will concretely show later. This term is also suggested by dimensional analysis. The closure of the algebra (19) requires that $\phi^{(1)}(0) = 0$. Under the consideration of (17) and (18), we find that the solution to eq.(6) is

$$\Psi[h] = \exp(-m_p^4 S_1) = \exp(-\frac{1}{G_\phi}\int d^3y\sqrt{h}(R^{(3)} - 2\Lambda_\phi)), \tag{21}$$

where

$$G_\phi = -\frac{\phi^{(3)}(0)\alpha_2}{3!m_p^4}, \quad \Lambda_\phi = \frac{3!\alpha_1\phi^{(5)}(0)}{2\cdot 5!\beta_1\phi^{(3)}(0)} + \frac{\alpha_2}{\beta_1}\Lambda. \tag{22}$$

In our formulation, the renormalizability of the theory requires that all physical quantities must be independent of the arbitrary functions $\phi^{(n)}(0)$ except for $\phi^{(1)}(0)$, so that, to leading order, the wave function should be independent of $\phi^{(n)}(0)$ except for $\phi^{(1)}(0)$. We may define a β-function for m_p^4 as $\beta \equiv \mu\frac{\partial}{\partial\mu}(m_p^4)$ where $\mu \equiv (\phi^{(3)}(0))^{1/3}$ which is a parameter of the dimension of $[m_p]$. If we require that the wavefunction to leading order is independent of μ, we get $\beta = 3m_p^4(\mu)$. This means that $m_p^4(\mu) \propto \mu^3$ ($G(\mu) \propto \mu^{-3/2}$), where $16\pi G(\mu) = 1/m_p^2(\mu)$. Thus gravitational constant $G(\mu)$ decreases as μ increases.

We shall briefly discuss the WDW equation to next leading order in powers of Planck mass of the wave function. The WDW equation to next leading order is given by

$$\Delta_\phi S_2 - G_{ijkl}\frac{\delta S_1}{\delta h_{ij}}\frac{\delta S_1}{\delta h_{kl}} = 0. \tag{23}$$

From (21) and eq.(23), the wave function to next leading order would be of the form

$$\Psi[h] = \exp\{-m_p^4 S_1 - m_p^8 S_2\}, \tag{24}$$

$$m_p^4 S_1 = \frac{1}{G_\phi}\int d^3y\sqrt{h}(R^{(3)} - 2\Lambda_\phi), \tag{25}$$

$$m_p^8 S_2 = (\frac{m_p}{\mu})^4\frac{m_p^4}{\mu}\int d^3y(\bar{\alpha}_\phi\sqrt{h} + \bar{\beta}_\phi\sqrt{h}\frac{R^{(3)}}{\mu^2} + \frac{1}{\mu^4}(\bar{\gamma}_\phi\sqrt{h}(R^{(3)})^2 + \bar{\delta}_\phi\sqrt{h}R^{(3)}_{ij}R^{(3)ij})), \tag{26}$$

where we have introduced a mass parameter μ. $\bar{\alpha}_\phi, \bar{\beta}_\phi, \bar{\gamma}_\phi$ and $\bar{\delta}_\phi$ are dimensionless constants expressed by $\bar{\phi}^{(n)}(0) = \phi^{(n)}(0)\mu^{-n}$. From the above expression (26), we see that the next leading contribution will be suppressed if we take $\mu \gg m_p$. This may mean that if we take a "renormalization" scale μ to be much greater than m_p, the wave function may be given by

$$\Psi[h] \simeq \exp(-\frac{1}{G_\phi}\int d^3y\sqrt{h}(R^{(3)} - 2\Lambda_\phi))$$
$$\text{for} \quad \mu \gg m_p(\nu \ll l_p) \text{ and } R^{(3)} \sim \mu^2(\nu^{-2}),\ R^{(3)}_{ij}R^{(3)ij} \sim \mu^4(\nu^{-4}). \tag{27}$$

where $\nu \equiv \mu^{-1}$, $l_p = 1/m_p$, l_p is the Planck length. The exponent of the wave function (27) is the three dimensional Einstein gravity action with the "renormalized" Newton's constant G_ϕ and the "renormalized" cosmological constant Λ_ϕ. Three dimensional Einstein gravity theory does not contain gravitational wave modes, and the theory is the dynamical system of the finite degrees of freedom which come from different topologies of the manifolds. As we have an approximation for Ψ we will be able to compute expectation values of some operators $\Omega[h_{ij}]$ such as the Wilson loop [14] in which case Ω can be taken independent of $-i\frac{\delta}{\delta h_{ij}}$;

$$< \Omega > = \frac{\int Dh_{ij}\Psi^*(h_{ij})\Omega[h_{ij}]\Psi(h_{ij})}{\int Dh_{ij}\Psi^*(h_{ij})\Psi(h_{ij})} \tag{28}$$

$$\simeq \frac{\int Dh_{ij}\Omega[h_{ij}]\exp(-\frac{1}{\tilde{G}_\phi}\int d^3x\sqrt{h}(R^{(3)} - 2\Lambda_\phi))}{\int Dh_{ij}\exp(-\frac{1}{\tilde{G}_\phi}\int d^3x\sqrt{h}(R^{(3)} - 2\Lambda_\phi))}, \tag{29}$$

where $\tilde{G}_\phi = \frac{1}{2}G_\phi$. But this is the functional integral for the vacuum expectation value of Ω in the three dimensional Euclidean Einstein action with $\tilde{G}_\phi$, Λ_ϕ. This expression (29) may mean that time axis vanishes at the scale which is much smaller than Planck scale, and four dimensional Einstein Gravity is described by three dimensional Einstein Gravity effectively at this scale, or our universe was a three dimensional Euclidean universe effectively at the very early stage of the evolution of the universe, in which the size of the universe is much smaller than Planck size.

3. Now we shall concretely show that $\Delta\Gamma$ (20) arises from the cross commutator Δ_ϕ and $\sqrt{h}R^{(3)}$, within our choice of heat equation (9). Classically, Hamiltonian constraint ($H \approx 0$) and momentum constraints ($H^i \approx 0$) constitute the following closed algebras [11] [12]

$$\{ \int d^3x\xi(x)H(x), \int d^3y\eta(y)H(y)\}_p$$
$$= \int d^3x(\xi(x)\partial_i\eta(x) - (\partial_i\xi(x))\eta(x))H^i(x), \tag{30}$$

$$\{ \int d^3x\xi^i(x)H_i(x), \int d^3y\eta^j(y)H_j(y)\}_p$$
$$= \int d^3x(\xi^i(x)\partial_i\eta^j(x) - (\partial_i\xi^j(x))\eta^i(x))H_j(x), \tag{31}$$

$$\{H(x), \int d^3y \xi^i(y) H_i(y)\}_p = -\partial_i(\xi^i(x) H(x)), \tag{32}$$

where

$$H(x) \equiv \frac{1}{m_p^2} G_{ijkl} \pi^{ij} \pi^{kl} + m_p^2 \sqrt{h}(R^{(3)} - 2\Lambda), \tag{33}$$

$$H_i(x) \equiv -2h_{ij} D_k \pi^{jk}, \tag{34}$$

π^{ij} are the canonical conjugate momenta to h_{ij}. $\{\ \}_p$ represents the Poisson brackets. Such closed algebras ((30), (31) and (32)) should be also realized for the quantum theory of gravitation described by the reguralized WDW equation if our theory is not anomalous. The reguralized Hamiltonian constraint ($\tilde{H} \approx 0$) contains heat kernel $K_{i'j'kl}$. Left-hand side of eq. (19) is given by

$$\begin{aligned} &\lim_{\epsilon \to 0} \int d^3x d^3y \xi(x) \eta(y) [\tilde{H}(x;\epsilon), \tilde{H}(y;\epsilon)] \\ &= \lim_{\epsilon \to 0} \int d^3x d^3y \xi(x) \eta(y) \{[\tilde{T}(x,\epsilon), \tilde{T}(x;\epsilon)] + [\tilde{T}(x;\epsilon), \tilde{V}(y)] + [\tilde{V}(x), \tilde{T}(y;\epsilon)] \\ &+ [\tilde{V}(x), \tilde{V}(y)]\}, \end{aligned} \tag{35}$$

where

$$\tilde{H}(x,\epsilon) \equiv \tilde{T}(x;\epsilon) + \tilde{V}(x), \tag{36}$$

$$\tilde{T}(x;\epsilon) \equiv \frac{1}{m_p^2} \int d^3x' K_{i'j'kl}(x', x; \epsilon) \pi^{kl}(x) \pi^{i'j'}(x'), \tag{37}$$

$$\tilde{V}(x) \equiv m_p^2 \sqrt{h(x)}(R^{(3)}(x) - 2\Lambda). \tag{38}$$

We shall calculate the commutators $[\tilde{T}, \tilde{T}]$, $[\tilde{T}, \tilde{V}]$, $[\tilde{V}, \tilde{T}]$, $[\tilde{V}, \tilde{V}]$. Firstly, we calculate $[\tilde{T}, \tilde{T}]$.

$$\begin{aligned} &\lim_{\epsilon \to 0} [\ \tilde{T}(x;\epsilon), \tilde{T}(y;\epsilon)] \\ &= \lim_{\epsilon \to 0} [\frac{1}{m_p^2} \int d^3x' K_{i'j'kl}(x', x; \epsilon) \pi^{kl}(x) \pi^{i'j'}(x'), \frac{1}{m_p^2} \int d^3y' K_{m'n'pq}(y', y; \epsilon) \pi^{pq}(y) \pi^{m'n'}(y')] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\frac{1}{m_p^4}\int d^3x'd^3y'\{K_{i'j'kl}(x',x;\epsilon)\pi^{kl}(x)[\pi^{i'j'}(x'),K_{m'n'pq}(y',y;\epsilon)]\pi^{pq}(y)\pi^{m'n'}(y') \\
&+ K_{i'j'kl}(x',x;\epsilon)[\pi^{kl}(x),K_{m'n'pq}(y',y;\epsilon)]\pi^{i'j'}(x')\pi^{pq}(y)\pi^{m'n'}(y) \\
&+ K_{m'n'pq}(y',y;\epsilon)[K_{i'j'kl}(x',x;\epsilon),\pi^{pq}(y)]\pi^{m'n'}(y)\pi^{kl}(x)\pi^{i'j'}(x') \\
&+ K_{m'n'pq}(y',y;\epsilon)\pi^{pq}(y)[K_{i'j'kl}(x',x;\epsilon),\pi^{m'n'}(y')]\pi^{kl}(x)\pi^{i'j'}(x')\} \\
&= \frac{1}{m_p^4}\int d^3x'\{\lim_{\epsilon\to 0}K_{i'j'kl}(x',x;\epsilon)\pi^{kl}(x)[\pi^{i'j'}(x'),G_{m'n'pq}(y)]\pi^{pq}(y)\pi^{m'n'}(y) \\
&+ \lim_{\epsilon\to 0}K_{i'j'kl}(x',x;\epsilon)[\pi^{kl}(x),G_{m'n'pq}(y)]\pi^{i'j'}(x')\pi^{pq}(y)\pi^{m'n'}(x')\} \\
&+ \frac{1}{m_p^4}\int d^3y'\{\lim_{\epsilon\to 0}K_{m'n'pq}(y',y;\epsilon)[G_{i'j'kl}(x),\pi^{pq}(y)]\pi^{m'n'}(y')\pi^{kl}(x)\pi^{i'j'}(x) \\
&+ \lim_{\epsilon\to 0}K_{m'n'pq}(y',y;\epsilon)\pi^{pq}(y)[G_{i'j'kl}(x),\pi^{m'n'}(y')]\pi^{kl}(x)\pi^{i'j'}(x)\} \\
&= 0, \qquad (39)
\end{aligned}
$$

where we have used (8) and $[\pi^{i'j'}(x'),G_{m'n'pq}(y)] \equiv A^{i'j'}_{m'n'pq}(x')\delta(y,x')$ etc, where $A^{i'j'}_{m'n'pq}(x') = \frac{i}{4\sqrt{h}}\{h^{rs}\delta^{(i'j')}_{(rs)}(h_{m'p}h_{n'q}+h_{m'q}h_{n'p}-h_{m'n'}h_{pq}) - (\delta^{(i'j')}_{(m'p)}h_{n'q}+h_{m'p}\delta^{(i'j')}_{(n'q')}+\delta^{(i'j')}_{(m'q')}h_{n'p}+h_{m'q}\delta^{(i'j')}_{(n'p)}-\delta^{(i'j')}_{(m'n')}h_{pq}-h_{m'n'}\delta^{(i'j')}_{(pq)})\}$, $\delta^{(i'j')}_{(rs)} \equiv \frac{1}{2}(\delta^{i'}_r\delta^{j'}_s+\delta^{i'}_s\delta^{j'}_r)$, etc. Clearly, we have

$$
[\tilde{V}(x),\tilde{V}(y)] = 0. \qquad (40)
$$

Then, (35) becomes

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int d^3xd^3y\xi(x)\eta(y)\{[\tilde{T}(x;\epsilon),\tilde{V}(y)]+[\tilde{V}(x),\tilde{T}(y;\epsilon)]\} \\
&= \lim_{\epsilon\to 0}\int d^3xd^3yd^3x'\xi(x)\eta(y)[\frac{1}{m_p^2}K_{i'j'kl}(x',x;\epsilon)\pi^{kl}(x)\pi^{i'j'}(x'),m_p^2\sqrt{h(y)}(R^{(3)}(y)-2\Lambda)] \\
&- \text{(same, but with } \xi\leftrightarrow\eta) \\
&= \lim_{\epsilon\to 0}\int d^3xd^3yd^3x'\xi(x)\eta(y)K_{i'j'kl}(x',x;\epsilon) \\
&\times \{[\pi^{kl}(x),\sqrt{h(y)}(R^{(3)}(y)-2\Lambda)]\pi^{i'j'}(x')+[\pi^{i'j'}(x'),\sqrt{h(y)}(R^{(3)}(y)-2\Lambda)]\pi^{kl}(x) \\
&+ [\pi^{kl}(x),[\pi^{i'j'}(x'),\sqrt{h(y)}(R^{(3)}(y)-2\Lambda)]]\} \\
&- \text{(same, but with } \xi\leftrightarrow\eta) \\
&= i\int d^3x(\xi(x)\partial_i\eta(x)-(\partial_i\xi(x))\eta(x))H^i(x) \\
&- \lim_{\epsilon\to 0}\int d^3xd^3x'\xi(x)K_{i'j'kl}(x',x;\epsilon)\frac{\delta}{\delta h_{kl}(x)}\frac{\delta}{\delta h_{i'j'}(x')}\int d^3y\eta(y)\sqrt{h(y)}R^{(3)}(y)
\end{aligned}
$$

$$+ \quad \lim_{\epsilon\to 0}\int d^3x d^3x' \eta(x) K_{i'j'kl}(x',x;\epsilon)\frac{\delta}{\delta h_{kl}(x)}\frac{\delta}{\delta h_{i'j'}(x')}\int d^3y \xi(y)\sqrt{h(y)}R^{(3)}(y). \tag{41}$$

We define last two terms in eq.(41) as

$$\Delta\Gamma \equiv -\lim_{\epsilon\to 0}\int d^3x d^3x' \xi(x) K_{i'j'kl}(x',x;\epsilon)\frac{\delta}{\delta h_{kl}(x)}\frac{\delta}{\delta h_{i'j'}(x')}\int d^3y \eta(y)\sqrt{h(y)}R^{(3)}(y)$$
$$- \text{(same, but with } \xi \leftrightarrow \eta). \tag{42}$$

If we compute functional derivatives in (42) [5], we have

$$\Delta\Gamma \equiv \Delta\Gamma_1 + \Delta\Gamma_2 + \Delta\Gamma_3 + \Delta\Gamma_4, \tag{43}$$

where

$$\Delta\Gamma_1 \equiv \frac{1}{4}\lim_{\epsilon\to 0}\int d^3x d^3x' \xi(x) K_{i'j'kl}(x',x;\epsilon)\sqrt{h(x')}\eta(x')(h^{i'm'}(x')h^{j'n'}(x')$$
$$+ h^{i'n'}(x')h^{j'm'}(x') - h^{i'j'}(x')h^{m'n'}(x'))h^{p'q'}(x')$$
$$\times [(\delta^{(kl)}_{(p'q')}(x',x))_{;m'n'} + (\delta^{(kl)}_{(m'n')}(x',x))_{;p'q'} - (\delta^{(kl)}_{(m'p')}(x',x))_{;n'q'} - (\delta^{(kl)}_{(n'p')}(x',x))_{;m'q'}]$$
$$- \text{(same, but with } \xi \leftrightarrow \eta), \tag{44}$$

$$\Delta\Gamma_2 \equiv -\frac{1}{2}\lim_{\epsilon\to 0}\int d^3x d^3x' \xi(x) K_{i'j'kl}(x',x;\epsilon)$$
$$\times \sqrt{h(x')}\eta(x')[h^{i'm'}(x')R^{(3)j'n'}(x') + h^{i'n'}(x')R^{(3)j'm'}(x') + h^{j'm'}(x')R^{(3)i'n'}(x')$$
$$+ h^{j'n'}(x')R^{(3)i'm'}(x') - h^{i'j'}(x')R^{(3)m'n'}(x') - h^{m'n'}(x')R^{(3)i'j'}(x')$$
$$- \frac{1}{2}(h^{i'm'}(x')h^{j'n'}(x') + h^{i'n'}(x')h^{j'm'}(x') - h^{i'j'}(x')h^{m'n'}(x'))R^{(3)}(x')]\delta^{(kl)}_{(m'n')}(x',x)$$
$$- \text{(same, but with } \xi \leftrightarrow \eta), \tag{45}$$

$$\Delta\Gamma_3 \equiv -\frac{1}{2}\lim_{\epsilon\to 0}\int d^3x d^3x' \xi(x) K_{i'j'kl}(x',x;\epsilon)\sqrt{h(x')}[h^{m'n'}(x')(h^{i'j'}(x')\eta_{;p'}{}^{p'}(x') - \eta^{i'j'}(x'))$$
$$- (h^{i'm'}(x')h^{j'n'}(x') + h^{i'n'}(x')h^{j'm'}(x'))\eta_{;p'}{}^{p'}(x')$$
$$- 2h^{i'j'}(x')\eta^{;m'n'}(x') + h^{i'm'}(x')\eta^{;j'n'}(x') + h^{i'n'}(x')\eta^{;j'm'}(x') + h^{j'm'}(x')\eta^{;i'n'}(x')$$
$$+ h^{j'n'}(x')\eta^{;i'm'}(x')]\delta^{(kl)}_{(m'n')}(x',x)$$
$$- \text{(same, but with } \xi \leftrightarrow \eta), \tag{46}$$

$$\Delta\Gamma_4 \equiv -\frac{1}{4}\lim_{\epsilon\to 0}\int d^3x d^3x' \xi(x) K_{i'j'kl}(x',x;\epsilon)\sqrt{h(x')}(h^{i'm'}(x')h^{j'n'}(x') + h^{i'n'}(x')h^{j'm'}(x')$$
$$- 2h^{i'j'}(x')h^{m'n'}(x'))\eta^{;p'}(x')$$
$$\times [(\delta^{(kl)}_{(p'm')}(x',x))_{;n'} + (\delta^{(kl)}_{(p'n')}(x',x))_{;m'} - (\delta^{(kl)}_{(m'n')}(x',x))_{;p'}]$$
$$- (\text{same, but with } \xi \leftrightarrow \eta), \tag{47}$$

where $\delta^{(k,l)}_{(m'n')}(x',x) \equiv \frac{1}{2}(\delta^k_{m'}\delta^l_{n'} + \delta^k_{n'}\delta^l_{m'})\delta(x',x)$, etc. If we integrate over x' in (44), (45), (46) and (47), we get

$$\Delta\Gamma_1 = \frac{1}{4}\lim_{\substack{x'\to x\\ \epsilon\to 0}}\int d^3x\sqrt{h(x)}\xi(x)(h^{i'm'}(x')h^{j'n'}(x') + h^{i'n'}(x')h^{j'm'}(x') - h^{i'j'}(x')h^{m'n'}(x'))h^{p'q'}$$
$$\times \{(\eta_{;n'm'}(x')K_{i'j'kl}(x',x;\epsilon) + \eta_{;n'}(x')K_{i'j'kl;m'}(x',x,;\epsilon) + \eta_{;m'}(x')K_{i'j'kl;n'}(x',x;\epsilon))\delta^{(kl)}_{(p'q')}$$
$$+ (\eta_{;q'p'}(x')K_{i'j'kl}(x',x;\epsilon) + \eta_{;q'}(x')K_{i'j'kl;p'}(x',x,;\epsilon) + \eta_{;p'}(x')K_{i'j'kl;q'}(x',x;\epsilon))\delta^{(kl)}_{(m'n')}$$
$$- (\eta_{;q'n'}(x')K_{i'j'kl}(x',x;\epsilon) + \eta_{;q'}(x')K_{i'j'kl;n'}(x',x;\epsilon) + \eta_{;n'}(x')K_{i'j'kl;q'}(x',x;\epsilon))\delta^{(kl)}_{(m'p')}$$
$$- (\eta_{;q'm'}(x')K_{i'j'kl}(x',x;\epsilon) + \eta_{;q'}(x')K_{i'j'kl;m'}(x',x;\epsilon) + \eta_{;m'}(x')K_{i'j'kl;q'}(x',x;\epsilon))\delta^{(kl)}_{(n'p')}\}$$
$$- (\text{same, but with } \xi \leftrightarrow \eta), \tag{48}$$

$$\Delta\Gamma_2 = -\frac{1}{2}\lim_{\substack{x'\to x\\ \epsilon\to 0}}\int d^3x\xi(x)\eta(x')K_{i'j'kl}(x',x;\epsilon)$$
$$\times \sqrt{h(x)}[h^{i'm'}(x')R^{(3)j'n'}(x') + h^{i'n'}(x')R^{(3)j'm'}(x') + h^{j'm'}(x')R^{(3)i'n'}(x')$$
$$+ h^{j'n'}(x')R^{(3)i'm'}(x') - h^{i'j'}(x')R^{(3)m'n'}(x') - h^{m'n'}(x')R^{(3)i'j'}(x')$$
$$- \frac{1}{2}(h^{i'm'}(x')h^{j'n'}(x') + h^{i'n'}(x')h^{j'm'}(x') - h^{i'j'}(x')h^{m'n'}(x'))R^{(3)}(x')]\delta^{(kl)}_{(m'n')}$$
$$- (\text{same, but with } \xi \leftrightarrow \eta)$$
$$= 0, \tag{49}$$

$$\Delta\Gamma_3 = -\frac{1}{2}\lim_{\substack{x'\to x\\ \epsilon\to 0}}\int d^3x\sqrt{h(x)}\xi(x)K_{i'j'kl}(x',x;\epsilon)[h^{m'n'}(x')(h^{i'j'}(x')\eta^{\ p'}_{;p'}(x') - \eta^{;i'j'}(x'))$$
$$- (h^{i'm'}(x')h^{j'n'}(x') + h^{i'n'}(x')h^{j'm'}(x'))\eta^{\ p'}_{;p'}(x')$$
$$- 2h^{i'j'}(x')\eta^{;m'n'}(x') + h^{i'm'}(x')\eta^{;j'n'}(x') + h^{i'n'}(x')\eta^{;j'm'}(x') + h^{j'm'}(x')\eta^{;i'n'}(x')$$
$$+ h^{j'n'}(x')\eta^{;i'm'}(x')]\delta^{(kl)}_{(m'n')}$$
$$- (\text{same, but with } \xi \leftrightarrow \eta), \tag{50}$$

$$
\begin{aligned}
\Delta\Gamma_4 = \frac{1}{4}\lim_{\substack{x'\to x\\ \epsilon\to 0}} \int d^3x\sqrt{h(x)}\xi(x)(h^{i'm'}(x')h^{j'n'}(x') + h^{i'n'}(x')h^{j'm'}(x') - 2h^{i'j'}(x')h^{m'n'}(x'))\\
\times \{(\eta^{ip'}{}_{n'}(x')K_{i'j'kl}(x',x;\epsilon) + \eta^{ip'}(x)K_{i'j'kl;n'}(x',x;\epsilon))\delta^{(kl)}_{(p'm')}\\
+ (\eta^{ip'}{}_{m'}(x')K_{i'j'kl}(x',x;\epsilon) + \eta^{ip'}(x')K_{i'j'kl;m'}(x',x;\epsilon))\delta^{(kl)}_{(p'n')}\\
- (\eta^{ip'}{}_{p'}(x')K_{i'j'kl}(x',x;\epsilon) + \eta^{ip'}(x)K_{i'j'kl;p'}(x',x;\epsilon))\delta^{(kl)}_{(m'n')}\}\\
- (\text{same, but with } \xi \leftrightarrow \eta).
\end{aligned} \tag{51}
$$

For the further calculation of $\Delta\Gamma_1, \Delta\Gamma_3$ and $\Delta\Gamma_4$, we need to compute $\lim_{x'\to x} K_{i'j'kl}(x',x;\epsilon)$ and $\lim_{x'\to x} K_{i'j'kl;m'}(x',x;\epsilon)$. To this end, firstly, we compute $\lim_{x'\to x} a^{(0)}_{i'j'kl;m'}(x',x)$ etc. If $D_{m'}$ operates on eq.(12), we have

$$
\sigma^{ip'}{}_{m'}(x',x)a^{(0)}_{i'j'kl;p'}(x',x) + \sigma^{ip'}(x',x)a^{(0)}_{i'j'kl;p'm'}(x',x) = 0. \tag{52}
$$

Using (52), $\lim_{x'\to x}\sigma^{ip'}{}_{m'}(x',x) = \delta^{p'}_{m'}$ and $\lim_{x'\to x}\sigma^{ip'}(x',x) = 0$ [5], we have

$$
\lim_{x'\to x} a^{(0)}_{i'j'kl;m'}(x',x) = 0. \tag{53}
$$

If $D_{n'}D_{m'}$ operates on eq.(12), we have

$$
\begin{aligned}
&\sigma^{iq'}{}_{m'n'}(x',x)a^{(0)}_{i'j'kl;q'}(x',x) + \sigma^{iq'}{}_{m'}(x',x)a^{(0)}_{i'j'kl;q'n'}(x',x)\\
&\quad + \quad \sigma^{iq'}{}_{n'}(x',x)a^{(0)}_{i'j'kl;q'm'}(x',x) + \sigma^{iq'}(x',x)a^{(0)}_{i'j'kl;q'm'n'}(x',x) = 0.
\end{aligned} \tag{54}
$$

Using (54), $\lim_{x'\to x}\sigma^{iq'}{}_{m'n'}(x',x) = 0$, $\lim_{x'\to x}\sigma^{iq'}_{m'}(x',x) = \delta^{q'}_{m'}$, $\lim_{x'\to x}\sigma^{iq'}(x',x) = 0$ [5] and $a^{(0)}_{i'j'kl;n'm'}(x,x) = a^{(0)}_{i'j'kl;m'n'}(x,x) - R^{(3)p'}{}_{i'n'm'}(x)a^{(0)}_{p'j'kl}(x,x) - R^{(3)p'}{}_{j'n'm'}(x)a^{(0)}_{i'p'kl}(x,x)$, we have

$$
\lim_{x'\to x} a^{(0)}_{i'j'kl;m'n'}(x',x) = -\frac{1}{2}\sqrt{h}(R^{(3)p'}{}_{i'm'n'}G_{p'j'kl} + R^{(3)p'}{}_{j'm'n'}G_{i'p'kl}). \tag{55}
$$

In the same way, if $D_{p'}D_{n'}D_{m'}$ operates on eq.(13), we have

$$
\begin{aligned}
\lim_{x'\to x} a^{(0)}_{i'j'kl;m'n'p'}(x',x) = &-\frac{1}{3}\sqrt{h}(R^{(3)q'}{}_{i'm'n';p'} - R^{(3)q'}{}_{i'p'm';n'})G_{q'j'kl}\\
&- \frac{1}{3}\sqrt{h}(R^{(3)q'}{}_{j'm'n';p'} - R^{(3)q'}{}_{j'p'm';n'})G_{i'q'kl}.
\end{aligned} \tag{56}
$$

where we have used (53). For $n = 0$, eq.(13) becomes

$$\sigma^{;p'}(x',x)a^{(1)}_{i'j'kl;p'}(x',x) + a^{(1)}_{i'j'kl}(x',x)$$
$$= \Delta^{-1/2}(x',x)(\Delta^{1/2}(x',x)a^{(0)}_{i'j'kl}(x',x))^{;p'}_{\ p'}. \tag{57}$$

Using (57), $\lim_{x'\to x}\Delta^{-1/2}(x',x) = 1$, $\lim_{x'\to x}\Delta^{1/2;p'}(x',x) = 0$ and $\lim_{x'\to x}\Delta^{1/2;p'}_{\ \ p'}(x',x) = -\frac{1}{6}R^{(3)}$ [5], we have

$$\lim_{x'\to x} a^{(1)}_{i'j'kl}(x',x) = -\frac{1}{6}\sqrt{h}R^{(3)}G_{i'j'kl}. \tag{58}$$

If $D_{m'}$ operates on (13), we have

$$\sigma^{;q'}{}_{m'}(x',x)a^{(1)}_{i'j'kl;q'}(x',x) + \sigma^{;q'}(x',x)a^{(1)}_{i'j'kl;q'm'}(x',x) + a^{(1)}_{i'j'kl;m'}(x',x)$$
$$+ \Delta^{-1/2}_{;m'}(x',x)(\Delta^{1/2}(x',x)a^{(0)}_{i'j'kl}(x',x))^{;q'}{}_{q'} + \Delta^{-1/2}(x',x)(\Delta^{1/2}(x',x)a^{(0)}_{i'j'kl}(x',x))^{;q'}{}_{q'm'}. \tag{59}$$

From eq.(59), we have

$$\lim_{x'\to x} a^{(1)}_{i'j'kl;m'}(x',x) = -\frac{1}{12}R^{(3)}_{;m'}\sqrt{h}G_{i'j'kl} + \frac{1}{6}\sqrt{h}(R^{(3)p'}{}_{i'm'}{}^{q'}{}_{;q'}G_{p'j'kl} + R^{(3)p'}{}_{j'm'}{}^{q'}{}_{;q'}G_{i'p'kl}) \tag{60}$$

where we have used (53), $\lim_{\epsilon\to 0}\Delta^{1/2}(x',x) = 1$, $\lim_{x'\to x}\Delta^{1/2;q'}{}_{q'm'}(x',x) = -\frac{1}{12}(R^{(3)q'}{}_{q';m'} + R^{(3)}{}_{q'm'}{}^{;q'} + R^{(3)}{}_{m'}{}^{q'}{}_{;q'})$ [5] and $(R^{(3)}_{m'q'} - \frac{1}{2}h_{m'q'}R^{(3)})^{;q'} = 0$. From (10),(11), (58) and (60), we have

$$\lim_{x'\to x} K_{i'j'kl}(x',x;\epsilon) = \lim_{x'\to x}\frac{\Delta^{1/2}(x',x)}{(4\pi\epsilon)^{3/2}}\exp(-\frac{\sigma(x',x)}{2\epsilon})\sum_{n=0}^{\infty}\epsilon^n a^{(n)}_{i'j'kl}(x',x)$$
$$= \frac{1}{(4\pi\epsilon)^{3/2}}\{\lim_{x'\to x} a^{(0)}_{i'j'kl}(x',x) + \epsilon\lim_{x'\to x} a^{(1)}_{i'j'kl}(x',x) + O(\epsilon^2)\}$$
$$= \frac{1}{(4\pi\epsilon)^{3/2}}\sqrt{h}G_{i'j'kl}(x)\{1 - \epsilon\frac{1}{6}R^{(3)}(x)\} + O(\epsilon^{1/2}), \tag{61}$$

$$\lim_{x'\to x} K_{i'j'kl;m'}(x',x;\epsilon)$$
$$= \lim_{x'\to x}[\frac{\Delta^{1/2}(x',x)}{(4\pi\epsilon)^{3/2}}\exp(-\frac{\sigma(x',x)}{2\epsilon})\sum_{n=0}^{\infty}\epsilon^n a^{(n)}_{i'j'kl}(x',x)]_{;m'}$$
$$= \frac{1}{(4\pi\epsilon)^{3/2}}\{\lim_{\epsilon\to 0} a^{(0)}_{i'j'kl;m'}(x',x) + \epsilon\lim_{x'\to x} a^{(1)}_{i'j'kl;m'}(x',x) + O(\epsilon^2)\}$$
$$= \frac{1}{(4\pi)^{3/2}\epsilon^{1/2}}\{-\frac{1}{12}R^{(3)}_{;m'}(x)\sqrt{h(x)}G_{i'j'kl}(x)$$
$$+ \frac{1}{6}\sqrt{h(x)}(R^{(3)p'}{}_{i'm'}{}^{q'}{}_{;q'}(x)G_{p'j'kl}(x) + R^{(3)p'}{}_{j'm'}{}^{q'}{}_{;q'}(x)G_{i'p'kl}(x))\} + O(\epsilon^{1/2}). \tag{62}$$

Substituting (61) and (62) into (48), (50), (51), we have

$$\begin{aligned}\Delta\Gamma_1 &= \lim_{\epsilon\to 0}\frac{1}{(4\pi)^{3/2}\epsilon^{1/2}}[\frac{1}{48}-\frac{1}{96}-\frac{1}{96}+\frac{21}{48}-\frac{21}{96}-\frac{21}{96}-\frac{7}{48}+\frac{17}{96}-\frac{1}{32}-\frac{7}{48}+\frac{17}{96}-\frac{1}{32}]\\ &\times\int d^3x\sqrt{h}(\xi(x)\eta_{;m}(x)-\xi_{;m}(x)\eta(x))R^{(3);m}(x)+O(\epsilon^{1/2}),\\ &=0,\end{aligned}\tag{63}$$

$$\Delta\Gamma_3=\lim_{\epsilon\to 0}\frac{1}{4(4\pi)^{3/2}\epsilon^{1/2}}\int d^3x\sqrt{h}(\xi(x)\eta_{;m}(x)-\xi_{;m}(x)\eta(x))R^{(3);m}(x)+O(\epsilon^{1/2}),\tag{64}$$

$$\begin{aligned}\Delta\Gamma_4 &= \lim_{\epsilon\to 0}\frac{1}{(4\pi)^{3/2}\epsilon^{1/2}}[\frac{1}{6}-\frac{3}{16}+\frac{1}{6}-\frac{3}{16}-\frac{1}{2}+\frac{1}{4}]\\ &\times\int d^3x\sqrt{h}(\xi(x)\eta_{;m}(x)-\xi_{;m}(x)\eta(x))R^{(3);m}(x)+O(\epsilon^{1/2})\\ &=-\frac{7}{24(4\pi)^{3/2}\epsilon^{1/2}}\int d^3x\sqrt{h}(\xi(x)\eta_{;m}(x)-\xi_{;m}(x)\eta(x))R^{(3);m}(x)+O(\epsilon^{1/2}).\end{aligned}\tag{65}$$

Thus, from (43), (49), (63), (64) and (65), we have

$$\begin{aligned}\Delta\Gamma &= \Delta\Gamma_1+\Delta\Gamma_2+\Delta\Gamma_3+\Delta\Gamma_4\\ &=-\lim_{\epsilon\to 0}\frac{1}{24(4\pi)^{3/2}\epsilon^{1/2}}\int d^3x\sqrt{h}(\xi(x)\eta_{;m}(x)-\xi_{;m}(x)\eta(x))R^{(3);m}(x)+O(\epsilon^{1/2})\end{aligned}\tag{66}$$

From (16), our renormalization procedure tells us that

$$\lim_{\epsilon\to 0}\epsilon^{-n/2}\to\frac{1}{n!}\phi^{(n)}(0)\quad\text{for } n=0,1,2,\cdots,\tag{67}$$

$$\lim_{\epsilon\to 0}\epsilon^{n/2}\to 0\quad\text{for } n=1,2,\cdots,\tag{68}$$

where $\phi(0)=1$. Thus, from (66) and (67), finally, we get

$$\begin{aligned}&[\int d^3x\xi(x)\tilde{H}(x),\int d^3y\eta(y)\tilde{H}(y)]\\ &=i\int d^3x(\xi(x)\partial_i\eta(x)-(\partial_i\xi(x))\eta(x))H^i(x)+\Delta\Gamma,\end{aligned}\tag{69}$$

where

$$\Delta\Gamma=-\frac{1}{24(4\pi)^{3/2}}\phi^{(1)}(0)\int d^3x\sqrt{h}(\xi(x)\eta_{;m}(x)-\xi_{;m}(x)\eta(x))R^{(3);m}(x).\tag{70}$$

This additional term ($\Delta\Gamma$) is the second term of right-hand side of (19). The closure of the algebra (69) guarantees that (21) is the solution to eq. (6). From the quantum mechanical commutator $\lim_{\epsilon\to 0}[\tilde{H}(x;\epsilon), i\int d^3y\xi^i(y)H_i(y)]$ which corresponds to the left-hand side of (32) in the classical theory of gravitation, an additional term does not arise.

4. Next we shall briefly discuss about the renormalizability of the WDW equation in powers of the "Planck mass" of the wave function for $N+1$ dimensional Lorentzian manifold. The metric is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\tilde{N}^2dt^2 + h_{ij}(N^idt + dx^i)(N^jdt + dx^j)$ where $\tilde{N}$ and N_i are the lapse and shift functions. In this case, the regularized WDW equation is given by

$$[-\frac{1}{M^2_{p(N)}}\Delta_{\epsilon(N)}(x) + M^2_{p(N)}\sqrt{h}(R^{(N)} - 2\Lambda)]\Psi = 0, \tag{71}$$

$$\Delta_{\epsilon(N)}(x) \equiv \int d^Nx' K_{i'j'kl}(x', x;\epsilon)\frac{\delta}{\delta h_{i'j'}(x')}\frac{\delta}{\delta h_{kl}(x)}, \tag{72}$$

$$\lim_{\epsilon\to 0} K_{i'j'kl}(x', x;\epsilon) = G_{ijkl}(x)\delta^{(N)}(x', x), \tag{73}$$

$$G_{ijkl} = \frac{1}{2\sqrt{h}}(h_{ik}h_{jl} + h_{il}h_{jk} - \frac{2}{N-1}h_{ij}h_{kl}), \tag{74}$$

where $M^2_{p(N)} = \frac{1}{16\pi G_N}$, G_N is the gravitational constant for $N+1$ dimensional space-time, $R^{(N)}$ denotes the N dimensional scalar curvature. $\Delta_{\epsilon(N)}$ has the dimension of $[m_p]^{2N}$, $M^2_{p(N)}$ has the dimension of $[m_p]^{N-1}$. If we attempt an expansion of the wave function of the WDW equation as follows

$$\Psi = \exp(-S) = \exp(-\sum_{k=1}^{\infty} M^{4k}_{p(N)}S_k). \tag{75}$$

To leading order, eq.(71) becomes

$$\Delta_{\phi(N)}S_1 + \sqrt{h}(R^{(N)} - 2\Lambda) = 0. \tag{76}$$

We will investigate whether eq.(76) has the following solution or not

$$M^4_{p(N)}S_1 = \frac{1}{G^{(N-1)}_\phi}\int d^Nx\sqrt{h}(R^{(N)} - 2\Lambda^{(N-1)}_\phi). \tag{77}$$

We may choose the heat kernel and the heat equation as

$$K^{(N)}_{i'j'kl}(x',x;\epsilon) = \frac{\Delta^{1/2}(x',x)}{\sqrt{(4\pi\epsilon)^N}}\exp(-\frac{\sigma(x',x)}{2\epsilon})\sum_{n=0}^{\infty}\epsilon^n a^{(n)}_{i'j'kl}(x',x), \tag{78}$$

$$\frac{\partial}{\partial\epsilon}K^{(N)}_{i'j'kl} = D'\cdot D'K^{(N)}_{i'j'kl}, \tag{79}$$

where $D'_{i'}$ is a covariant derivative with respect to $x'_{i'}$ in N dimensional space. From (78) and (79), we have

$$\begin{aligned}&\Delta_{\epsilon(N)}\int d^N x\sqrt{h}R^{(N)}\\ &= a_1^{(N)}\epsilon^{-\frac{N+2}{2}} + a_2^{(N)}\epsilon^{-\frac{N}{2}}\sqrt{h}R^{(N)} + a_3^{(N)}\epsilon^{-\frac{N-2}{2}}\sqrt{h}((R^{(N)})^2+\cdots)+\cdots,\end{aligned} \tag{80}$$

$$\begin{aligned}&\Delta_{\epsilon(N)}\int d^N x\sqrt{h}\\ &= b_1^{(N)}\epsilon^{-\frac{N}{2}} + b_2^{(N)}\epsilon^{-\frac{N-2}{2}}\sqrt{h}R^{(N)} + b_3^{(N)}\epsilon^{-\frac{N-4}{2}}\sqrt{h}((R^{(N)})^2+\cdots)+\cdots.\end{aligned} \tag{81}$$

$$\begin{aligned}&[\int d^N x\xi(x)H(x), \int d^N y\eta(y)H(y)]\\ &= i\int d^N x(\xi(x)\eta_{;i}(x) - \xi_{;i}(x)\eta(x))H^i(x) + \Delta\Gamma,\end{aligned} \tag{82}$$

$$\Delta\Gamma = c^{(N)}\phi^{(N-2)}(0)\int d^N x\sqrt{h}(\xi(x)\eta_{;i}(x) - \xi_{;i}(x)\eta(x))R^{(N);i}(x), \tag{83}$$

where ϵ has the dimension of $[m_p]^{-2}$, $\phi^{(N-2)}(0)$ has the dimensions of $[m_p]^{N-2}$, $a_i^{(N)}$, $b_i^{(N)}$ and $c^{(N)}$ are numerical constants which are determined from the heat equation. The closure of the algebra (82) requires that $\phi^{(N-2)}(0) = 0$. However, in order that (77) becomes the solution to eq.(76), following conditions will be needed

$$\phi^{(N-2)}(0) = \phi^{(N-4)}(0) = \phi^{(N-6)}(0) = \cdots = \phi(0) = 0 \quad \text{for} \quad N = \text{even integer}, \tag{84}$$

$$\phi^{(N-2)}(0) = \phi^{(N-4)}(0) = \phi^{(N-6)}(0) = \cdots = \phi^{(1)}(0) = 0 \quad \text{for} \quad N = \text{odd integer}. \tag{85}$$

Thus, in the theory with extra dimensions, (77) can not become the solution to eq.(76) generally. It seems to be difficult to renormalise the WDW equation for Riemanian manifolds with extra dimensions, within our approach.

5. In summary, we have concretely shown that an additional term ($\Delta\Gamma$) arises from the commutator $[\tilde{H}, \tilde{H}]$ where $\tilde{H} \approx 0$ is the regularized Hamiltonian constraint. The closure ($\Delta\Gamma = 0$) of the algebras for the reguralized Hamiltonian constraint and momentum constraints guarantees that (21) is the solution to eq.(6) in 3+1 dimensional theory, but does not guarantee that (77) is the solution to eq.(76) in N+1 dimensional theory ($N > 3$). This means that (77) can not become the solution to eq.(76) generally for $N > 3$. This result about the renormalizability of the WDW equation for $N+1$ dimensions may partially answer the question of "why we live in $3+1$ dimensions".

In order to renormalize up to next leading order in 3+1 dimensional theory, we may need to choose the heat equation with additional parameters(m, γ);

$$\frac{\partial}{\partial\epsilon}K_{i'j'kl} = (D' \cdot D' - m^2 + \gamma R^{(3)})K_{i'j'kl}. \tag{86}$$

In this case, An additional term ($\Delta\Gamma$) in the commutator (19) becomes

$$\Delta\Gamma = -\frac{1}{(4\pi)^{3/2}}(\frac{1}{24} + \gamma)\phi^{(1)}(0)\int d^3x\sqrt{h}(\xi(x)\eta_{;m}(x) - \xi_{;m}(x)\eta(x))R^{(3);m}(x). \tag{87}$$

Next leading order contribution will be discussed, in more detail, in another paper.

REFERENCES

[1] T. Horiguchi, "Dimensional Reduction in a Quantum Universe", Preprint KIFR-94-01, KIFR-94-01 (Revised Version).

[2] T. Horiguchi, "Dimensional Reduction in a Quantum Universe from the Viewpoint of a Minisuperspace Model", Preprint KIFR-94-02.

[3] B. S. DeWitt, Phys. Rev. **160** (1967) 1113.

[4] E. Alvarez, Rev. Mod. Phys. **61** (1989) 561.

[5] B. S. DeWitt, Dynamical theory of Groups and Fields (Gordon and Breach, Inc., New York, 1965).

[6] G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincaré **20**, 69-94 (1974).

[7] G. 't Hooft, Preprint THU-93/26.

[8] M. Green, J. H. Schwartz and E. Witten, Superstring theory, Vol. 1 and Vol.2 (Canbrige University Press 1987).

[9] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B 258 (1985) 46; G. Horowitz, in Unified String theories (edited by M. Green and D. Gross), World Scientific, Singapore, 1986.

[10] L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. B 261, (1985) 678; **B 274** (1986) 286; C. Vafa, Nucl. Phys.**B 273** (1986) 592.

[11] R. Arnowitt, S. Deser and C. W. Misner, in Gravitation, ed. L. Witten (Wiley, 1962).

[12] S. A. Hojman, K. Kuchař and C. Teitelboim, Ann, Phys. (N. Y) **96** (1976) 88.

[13] P. Mansfield, Nucl. Phys. B **418** (1994) 113.

[14] K. G. Wilson, Phys. Rev. D **10** (1974) 2445.