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# DIMENSIONAL REDUCTION IN A QUANTUM UNIVERSE

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## Abstract

We try to solve the Wheeler-DeWitt equation for the wave function of the universe as an expansion in powers of the Planck mass. Expectation values computed with the leading order approximation are reduced to the expectation value in three dimensional Euclidean Einstein gravity theory. This may mean that our universe was a three dimensional Euclidean Universe effectively at the very early stage of the evolution of the universe, in which the size of the universe is much smaller than Planck size.

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1. The wave function of the universe is defined on the superspace (the space of all three metrics and matter-field configurations). The wave function $\Psi$ obeys the Wheeler-DeWitt (WDW) equation [1], $H\Psi = 0$. It is usually thought that, at the early stage of the evolution of the universe, the quantum state of the whole universe is well described by the WDW equation [4] [5] [6]. However, the region which is much smaller than Planck scale has not been analyzed. We will investigate the region.

Generally speaking, there are mainly two problems in the WDW equation. One of the problems is concerned with the interpretation of the wave function of the universe [7]. The other problem is the technical difficulty to solve the WDW equation on the infinite dimensional space (superspace). In this paper, we try to solve the WDW equation by using heat-kernel regularization [8]. We attempt an expansion of the wave function of the universe in powers of Planck mass and discuss the physical meaning of the wave function of the universe to leading order.

In Ref. [2], G. 't Hooft stated that nature is much more crazy at the Planck scale than even String theorists could have imaged, and stated that, at Planck scale, our world is not 3+1 dimensional, i.e. dimensional reduction occurs at this scale under the requirement that the physical phenomena associated with gravitational collapse should be duly reconciled with the postulates of quantum mechanics. We shall show that the dimensional reduction in quantum gravity can occur at the scale which is much smaller than Planck scale under the analysis of the WDW equation.

2. Wheeler-DeWitt equation [1] [3] is

$$[-\frac{1}{m_p^2}\Delta(x) + m_p^2\sqrt{h(x)}(R^{(3)}(x) - 2\Lambda)]\Psi(h) = 0, \tag{1}$$

$$\Delta(x) \equiv G_{ijkl}(x)\frac{\delta^2}{\delta h_{ij}(x)\delta h_{kl}(x)}, \tag{2}$$

where $h_{ij}$ are spatial metrics, $h = \det h_{ij}$, $R^{(3)}$ denotes the three dimensional scalar curvature, $G_{ijkl} = \frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$, $m_p$ is the Planck mass, $G = 1/m_p^2$, $G$ is the Newton constant, and $\Lambda$ is the cosmological constant. The WDW equation stands in need of regularization because the two functional derivatives acts at the same point in space.

So, for example, $\Delta(x)$ acting on $R^{(3)}(y)$ is proportional to $(\delta(x,y))^2$, which is meaningless. Therefore, in the next section, we shall construct a regulated operator, $\Delta_\phi$, that preserves the three dimensional general coordinate transformation. Then, the finite version of the WDW equation will be

$$[-\frac{1}{m_p^2}\Delta_\phi + m_p^2\sqrt{h}(R^{(3)} - 2\Lambda)]\Psi(h) = 0. \tag{3}$$

If we attempt an expansion of the wave function of the universe in powers of the Planck mass as follows

$$\Psi(h) \equiv \exp(-S(h)) \tag{4}$$

$$= \exp(-m_p^{4n}\sum_{n=1}^{\infty} S_n), \tag{5}$$

the WDW equation becomes

$$\frac{1}{m_p^2}\Delta_\phi S - \frac{1}{m_p^2}G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{kl}} + m_p^2\sqrt{h}(R^{(3)} - 2\Lambda) = 0. \tag{6}$$

So that to lowest order, eq.(6) becomes

$$\Delta_\phi S_1 + \sqrt{h}(R^{(3)} - 2\Lambda) = 0. \tag{7}$$

On the other hand, in the WKB approximation (the semi-classical approximation), if we expand $S = \sum_{n=1}^{\infty}(\frac{1}{m_p^2})^{n-2}W_n$, we obtain the Hamilton-Jacobi equation up to leading order

$$-G_{ijkl}\frac{\delta W_1}{\delta h_{ij}}\frac{\delta W_1}{\delta h_{kl}} + \sqrt{h}(R^{(3)} - 2\Lambda) = 0. \tag{8}$$

We should not expect our approximation (7) to leading order in our expansion (5) to resemble the semi-classical approximation (8). Eq.(7) gives a very different wave function of the universe from that (8) given by the WKB expansion which would contain propagators, and so be non-local.

In eq.(7), $\Delta_\phi$ will act on integrals of the invariant functions under three dimensional general coordinate transformation to produce three dimensional functions with finite coefficients. So we can look for a solution for $S_1$ that is itself the integrals of a local function. In the next section we will show that

$$m_p^4 S_1 = \frac{1}{G_\phi} \int d^3x \sqrt{h}(R^{(3)} - 2\Lambda_\phi), \tag{9}$$

where $G_\phi$, $\Lambda_\phi$ are the "renormalized" gravitational constant and the "renormalized" cosmological constant, as we will see in the next section. To leading order the wave function of the universe is

$$\Psi(h) = \exp(-\frac{1}{G_\phi} \int d^3x \sqrt{h}(R^{(3)} - 2\Lambda_\phi)). \tag{10}$$

Now that we have an approximation for $\Psi$ we will be able to compute expectation values of some operators $\Omega[h_{ij}]$ such as the Wilson loop [9] in which case $\Omega$ can be taken independent of $-i\frac{\delta}{\delta h_{ij}}$;

$$< \Omega > = \frac{\int Dh_{ij} \Psi^*(h_{ij}) \Omega[h_{ij}] \Psi(h_{ij})}{\int Dh_{ij} \Psi^*(h_{ij}) \Psi(h_{ij})} \tag{11}$$

$$\simeq \frac{\int Dh_{ij} \Omega[h_{ij}] \exp(-\frac{1}{\tilde{G}_\phi} \int d^3x \sqrt{h}(R^{(3)} - 2\Lambda_\phi))}{\int Dh_{ij} \exp(-\frac{1}{\tilde{G}_\phi} \int d^3x \sqrt{h}(R^{(3)} - 2\Lambda_\phi))}. \tag{12}$$

where $\tilde{G}_\phi = \frac{1}{2} G_\phi$. But this is the functional integral for the vacuum expectation value of $\Omega$ in the three dimensional Euclidean Einstein action with couplings $\tilde{G}_\phi$, $\Lambda_\phi$. What does this expression(12) mean ? This may mean that time axis vanish at the scale which is much smaller than Planck scale, and four dimensional Einstein Gravity is described by three dimensional Einstein Gravity effectively at this scale, or our universe is described by three dimensional Euclidean universe when the size of our universe was much smaller than Planck size, as we will discuss later. This picture may resolve the problem about the origin of the beginning of time (axis).

3. We will now address the crucial question of how to regulate the Wheeler-DeWitt equation. We will replace $\Delta$ in the Wheeler-DeWitt equation by a regulated operator $\Delta_\phi$. This regularization and renormalization procedure introduces an arbitrary function $\phi$ on which $\Delta_\phi$ depends. This is an inevitable consequence of isolating the finite part of a divergent quantity. Physical quantities should be independent of this arbitrariness, e.g.by absorbing it all into the coupling. To construct $\Delta_\phi$, we begin by "point-splitting" the functional derivatives. Consider the operator

$$\Delta_\epsilon(x) \equiv \int d^3x' K_{i'j'kl}(x', x; \epsilon) \frac{\delta}{\delta h_{i'j'}(x')} \frac{\delta}{\delta h_{kl}(x)}, \tag{13}$$

where the $K_{i'j'kl}$ is a bi-tensor at both $x'$ and $x$, satisfying some heat equation with the initial condition

$$\lim_{\epsilon \to 0} K_{i'j'kl}(x', x; \epsilon) = G_{ijkl}(x)\delta(x', x). \tag{14}$$

If $\Sigma$ denote the space of sums of products of three-dimensional integrals of local functions of $h_{ij}$ with finite coefficients, then $\Delta$ acts on a typical element of $\Sigma$. We will replace $\Delta$ in the WDW equation by a regulated operator, $\Delta_\phi$, which does have a finite action on elements of $\Sigma$ and maps $\Sigma$ to itself.

The coefficients of integrals resulting from the action of $\Delta_\epsilon$ on $\Sigma$ will contain powers of $\epsilon$ some of which diverge as $\epsilon \to 0$. These powers of $\epsilon$ may be determined from dimensional analysis. We will choose $K_{i'j'kl}$ to preserve three dimensional general coordinate transformation invariance, so, given the local character of $\Delta_\epsilon$, we have, for example,

$$\Delta_\epsilon \int d^3x \sqrt{h} R^{(3)} = \alpha_1 \epsilon^{-5/2} \sqrt{h} + \alpha_2 \epsilon^{-3/2} \sqrt{h} R^{(3)} + \epsilon^{-1/2} (\alpha_3 \sqrt{h} (R^{(3)})^2 + \alpha_4 \sqrt{h} R^{(3)}_{ij} R^{(3)ij} + \alpha_5 \sqrt{h} R^{(3)i}_{;i}) + O(\epsilon^{1/2}), \tag{15}$$

$$\Delta_\epsilon \int d^3y \sqrt{h} = \beta_1 \epsilon^{-3/2} \sqrt{h} + \beta_2 \epsilon^{-1/2} \sqrt{h} R^{(3)} + O(\epsilon^{1/2}), \tag{16}$$

where the proper-time $\epsilon$ has the dimensions of $[m_p]^{-2}$, whilst $\Delta_\epsilon$ has dimensions of $[m_p]^6$, and $\alpha_i$ and $\beta_i$ are numerical constants, which should be determined from the heat equation. We note that $R^{(3)}_{ijkl}$ does not arise in eq.(15) since, in three dimensions, $R^{(3)}_{ijkl}$ can be expressed in terms of $R^{(3)}$ and $R^{(3)}_{ij}$. We can extract a finite result for $\Delta_\epsilon \sigma$ when $\epsilon = 0$ using analytic continuation in an analogous fashion to zeta function regularization, where $\sigma \epsilon \Sigma$. We take the action of $\Delta_\phi$ on an arbitrary element, $\sigma \epsilon \Sigma$ to be [8]

$$\Delta_\phi \sigma \equiv \lim_{s \to 0} s \int_0^\infty dp p^{s-1} \phi(p) \Delta_{p^2} \sigma, \tag{17}$$

where the integral on the right-hand side is defined by analytic continuation in s [1]. We take $\Delta$ at proper-time $p^2$ rather than $p$ because of the square roots appearing in (15),(16). For example,

$$\Delta_\phi \int d^3x\sqrt{h}R^{(3)} = \frac{\alpha_1}{5!}\phi^{(5)}(0)\sqrt{h} + \frac{\alpha_2}{2!}\phi^{(3)}(0)\sqrt{h}R^{(3)} + \phi^{(1)}(0)(\alpha_3\sqrt{h}(R^{(3)})^2 + \alpha_4\sqrt{h}R^{(3)}_{ij}R^{(3)ij} + \alpha_5\sqrt{h}R^{(3)i}_{;i}), \tag{18}$$

$$\Delta_\phi \int d^3x\sqrt{h} = \frac{\beta_1}{3!}\phi^{(3)}(0)\sqrt{h} + \beta_2\phi^{(1)}(0)\sqrt{h}R^{(3)}, \tag{19}$$

where $\phi^{(n)}(0) \equiv \frac{d^n\phi(p)}{dp^n}|_{p=0}$, for $n = 1, 2, 3, \cdots$. We have done part of the job of renormalization. However, (10) can not become the solution to the equation (7) generally under the consideration of (18) and (19), if we do not take the requirement $\phi^{(1)}(0) = 0$. Therefore, for the moment, we require the "strange" condition $\phi^{(1)}(0) = 0$. This condition will be discussed later. The rest, ensuring that Physical quantities are independent of $\phi^{(n)}(0)$ except for $\phi^{(1)}(0)$, remains to be done. We now turn to the choice of heat-kernel. There are many choices of heat-kernel. Here we choose the following simple form

$$K_{i'j'kl}(x', x; \epsilon) \equiv \exp(\epsilon D' \cdot D')_{i'j'kl}\delta(x', x), \tag{20}$$

where $D'_{i'}$ is a covariant derivative with respect to $x'_{i'}$ in three dimensional space. Eq.(20) satisfy the heat equation

$$\frac{\partial}{\partial\epsilon}K_{i'j'kl} = D' \cdot D'K_{i'j'kl}, \tag{21}$$

subject to the initial conditions

$$\lim_{\epsilon\to 0} K_{i'j'kl}(x', x; \epsilon) = G_{ijkl}(x)\delta(x', x). \tag{22}$$

We need to know the diagonal matrix elements of $K_{i'j'kl}(x', x; \epsilon)$ for short "times". These may be computed using a standard technique that assumes the existence of expansions of the form [10] [11]

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[1] For the more detailed discussion about this regularization scheme, see ref. [8]

$$K_{i'j'kl}(x',x;\epsilon) = \frac{\Delta^{1/2}(x',x)e^{-\frac{\sigma(x',x)}{\epsilon}}}{\sqrt{4\pi\epsilon^3}} \sum_{n=0}^{\infty} \epsilon^n a^{(n)}_{i'j'kl}(x',x), \tag{23}$$

where the bi-scalar $\sigma(x',x)$, which we shall call the geodetic integral is equal to one half the square of the distance along the geodesic between $x$ and $x'$, and $\Delta(x',x)$ is a bi-scalar, $\Delta(x',x) \equiv h^{-1/2}(x')D(x',x)h^{-1/2}(x)$, $D \equiv -\det(D_{i'j})$, $D_{i'j} \equiv -\sigma_{;i'j}$. $;i'j$ denotes the covariant differentiations with respect to $x^{i'}$ and $x^j$ [10]. $a^{(n)}_{i'j'kl}(x',x)$ is a bi-tensor at both $x'$ and $x$. The prefactor in (23) solves the ordinary heat equation. The initial conditions(14) are satisfied by taking

$$\lim_{x'\to x} a^{(0)}_{i'j'kl}(x',x) = \sqrt{h}G_{ijkl}(x). \tag{24}$$

Inserting (23) into eq.(21) and making use of $\frac{1}{2}\sigma_{;i'}\sigma^{;i'} = \sigma$ and $\sigma^{;j'}\sigma_{;j'i} = \sigma_{;i}$ [10], one finds [10]

$$\sigma^{;m'}(x',x)a^{(0)}_{i'j'kl;m'}(x',x) = 0, \tag{25}$$

$$\begin{aligned}\sigma^{;m'} a^{(n+1)}_{i'j'kl;m'}(x',x) &+ (n+1)a^{(n+1)}_{i'j'kl}(x',x) \\ &= \Delta^{-1/2}(x',x)(\Delta^{1/2}(x',x)a^{(n)}_{i'j'kl}(x',x))^{m'}_{;m'} \quad \text{for} \quad n = 0,1,2,3,\cdots.\end{aligned} \tag{26}$$

Using (23),(24) and the recursion relations (25),(26), we have

$$\lim_{x'\to x} K_{i'j'kl}(x',x;\epsilon) = \frac{1}{(4\pi\epsilon)^{3/2}}\sqrt{h}G_{i'j'kl}(x) + O(\epsilon^{-1/2}), \tag{27}$$

$$\lim_{x'\to x} K_{i'j'kl;p'}(x',x;\epsilon) = O(\epsilon^{-1/2}), \tag{28}$$

$$\begin{aligned}\lim_{x'\to x} K_{i'j'kl;p'q'}(x',x;\epsilon) &= \epsilon^{-5/2}[-\frac{1}{2(4\pi)^{3/2}}h_{p'q'}(x)\sqrt{h}G_{i'j'kl}(x)] \\ &+ \epsilon^{-3/2}\frac{1}{(4\pi)^{3/2}}[(-\frac{1}{6}R^{(3)}_{p'q'}(x) + \frac{1}{12}R^{(3)}(x)h_{p'q'}(x))\sqrt{h}G_{i'j'kl}(x) \\ &- \frac{1}{2}\sqrt{h}(G_{m'j'kl}(x)R^{(3)m'}_{i'p'q'}(x) + G_{i'm'kl}(x)R^{(3)m'}_{j'p'q'}(x))] + O(\epsilon^{-1/2}).\end{aligned} \tag{29}$$

Now we shall calculate $\Delta_\epsilon \int d^3y\sqrt{h}$ and $\Delta_\epsilon \int d^3y\sqrt{h}R^{(3)}$ from the heat kernel. If $\Delta_\epsilon(x)$ (13) operates on $\int d^3y\sqrt{h}$, we have

$$\Delta_\epsilon(x)\int d^3y\sqrt{h} = -\frac{1}{4}\sqrt{h}K_{ijkl}(x,x)(h^{ik}h^{jl}+h^{il}h^{jk}-h^{ij}h^{kl}). \tag{30}$$

Using (30) and (27), we get

$$\Delta_\epsilon(x)\int d^3y\sqrt{h} = a_1\epsilon^{-3/2}\sqrt{h}+O(\epsilon^{-1/2}), \tag{31}$$

where

$$a_1 = -\frac{21}{8(4\pi)^{3/2}}. \tag{32}$$

If $\Delta_\epsilon(x)$ (13) operates on $\int d^3y\sqrt{h}R^{(3)}$, we have

$$\begin{aligned}\Delta_\epsilon(x)\int d^3y\sqrt{h}R^{(3)} &= \frac{1}{2}\sqrt{h}K_{ijkl}(x,x;\epsilon)[h^{ik}R^{(3)jl}+h^{il}R^{(3)jk}+h^{jk}R^{(3)il}+h^{il}R^{(3)ik}\\ &- h^{ij}R^{(3)kl}-h^{kl}R^{(3)ij}-\frac{1}{2}(h^{ik}h^{jl}+h^{il}h^{jk}-h^{ij}h^{kl})R^{(3)}]\\ &- \frac{1}{4}\sqrt{h}(h^{i'm'}h^{j'n'}+h^{i'n'}h^{j'm'}-h^{i'j'}h^{m'n'})h^{p'q'}\\ &\times [K_{i'j'kl;n'm'}\delta^{kl}_{p'q'}+K_{i'j'kl;q'p'}\delta^{kl}_{m'n'}\\ &- K_{i'j'kl;q'n'}\delta^{kl}_{m'p'}-K_{i'j'kl;q'm'}\delta^{kl}_{n'p'}]|_{x'=x},\end{aligned} \tag{33}$$

where $\delta^{kl}_{ij} \equiv \frac{1}{2}(\delta^k_i\delta^l_j+\delta^l_i\delta^k_j)$. Using (27),(28),(29) and (33), we get

$$\Delta_\epsilon(x)\int d^3y\sqrt{h}R^{(3)} = b_1\epsilon^{-5/2}\sqrt{h}+b_2\epsilon^{-3/2}\sqrt{h}R^{(3)}+O(\epsilon^{-1/2}), \tag{34}$$

where

$$b_1 = \frac{3}{2(4\pi)^{3/2}}, \quad b_2 = -\frac{11}{24(4\pi)^{3/2}}. \tag{35}$$

Our "renormalization" procedure (17) tells us that [8]

$$(\epsilon^{-n/2})_R|_{\epsilon=0} = \frac{1}{n!}\phi^{(n)}(0), \quad \text{for} \quad n=1,2,\cdots, \tag{36}$$

$$(\epsilon^{n/2})_R|_{\epsilon=0} = 0, \quad \text{for} \quad n=0,1,2,\cdots \tag{37}$$

Then, eq.(31) and eq.(34) become

$$\Delta_\phi(x)\int d^3y\sqrt{h} = \frac{a_1}{3!}\phi^{(3)}(0)\sqrt{h}, \tag{38}$$

$$\Delta_\phi(x)\int d^3y\sqrt{h}R^{(3)} = \frac{b_1}{5!}\phi^{(5)}(0)\sqrt{h} + \frac{b_2}{3!}\phi^{(3)}(0)\sqrt{h}R^{(3)}, \tag{39}$$

where we have taken $\phi^{(1)}(0) = 0$. Substituting (9) to eq.(7), and using (38) and (39), we find that to leading order the wave function of the universe is

$$\Psi[h] = \exp(-\frac{1}{G_\phi}\int d^3y\sqrt{h}(R^{(3)} - 2\Lambda_\phi)), \tag{40}$$

where

$$G_\phi = -\frac{\phi^{(3)}(0)b_2}{3!m_p^4}, \tag{41}$$

$$\Lambda_\phi = \frac{3!b_1\phi^{(5)}(0)}{2\cdot 5!a_1\phi^{(3)}(0)} + \frac{b_2}{a_1}\Lambda. \tag{42}$$

In our formulation, the "renormalizability" of the theory requires that all physical quantities must be independent of the arbitrary function $\phi^{(n)}(0)$ except for $\phi^{(1)}(0)$, so that, to leading order, the wave function is independent of $\phi^{(n)}(0)$ except for $\phi^{(1)}(0)$. We may define a $\beta$-function for $m_p^4$ as $\beta = \mu\frac{\partial}{\partial\mu}(m_p^4)$ where $\mu \equiv (\phi^{(3)}(0))^{1/3}$ which is a parameter of the dimension of $[m_p]$. If we require that the wavefunction to leading order is independent of $\mu$, we get $\beta = 3/m_p^4(\mu)$. This means that $m_p^4(\mu) \propto \mu^3(G(\mu) \propto \mu^{-3/2})$, where $G(\mu) = 1/m_p^2(\mu)$, $G(\mu)$ is the Newton constant.

4. Next we shall discuss the physical meaning of our expansion (5) of the wave function. Firstly we shall briefly discuss the WDW equation to next leading order in powers of Planck mass. The WDW equation to next leading order is given by

$$\Delta_\phi S_2 - G_{ijkl}\frac{\delta S_1}{\delta h_{ij}}\frac{\delta S_1}{\delta h_{kl}} = 0. \tag{43}$$

Substituting (9) into (43), we have

$$\Delta_\phi S_2[h] = -\frac{3}{8}c_\phi^2\sqrt{h} - \frac{1}{4}c_\phi d_\phi\sqrt{h}R^{(3)} + \frac{1}{2}d_\phi^2(-\frac{3}{4}\sqrt{h}(R^{(3)})^2 + 2\sqrt{h}R_{ij}^{(3)}R^{(3)ij}), \tag{44}$$

where $c_\phi = -\frac{2\Lambda_\phi}{m_p^4 G_\phi}$, $d_\phi = \frac{1}{m_p^4 G_\phi}$. This equation (44) would be solved by the ansatz

$$S_2 = \int d^3y\{e'\sqrt{h} + f'\sqrt{h}R^{(3)} + g'\sqrt{h}(R^{(3)})^2 + h'\sqrt{h}R_{ij}^{(3)}R^{(3)ij}\}, \tag{45}$$

where $e', f', g', h'$ are some constants. The term $\int dy^3\sqrt{h}R^{(3)i}_{;i}$ is not included in $S_2$ since it vanishes identically. Let us introduce a mass parameter $\mu$. In terms of $\mu$, we may write $e' \equiv \bar{e}'\mu^{-5}, f' \equiv \bar{f}'\mu^{-7}, g' \equiv \bar{g}'\mu^{-9}, h' \equiv \bar{h}'\mu^{-9}$. Here, all the coefficients $\bar{e}', \bar{f}', \bar{g}', \bar{h}'$ are dimensionless constants. Thus, up to the next leading order, the wave function of the universe would be found to be of the form

$$\Psi[h] = \exp\{-m_p^4 S_1 - (\frac{m_p}{\mu})^4 m_p^4 S_2\} \tag{46}$$

$$= \exp\{-m_p^4 S_1 - (\frac{\nu}{l_p})^4 m_p^4 S_2\}, \tag{47}$$

$$m_p^4 S_1 = \frac{1}{G_\phi}\int d^3y\sqrt{h}(R^{(3)} - 2\Lambda_\phi), \tag{48}$$

$$m_p^4 S_2 = \frac{1}{G_\phi}\int d^3y(\alpha_\phi\sqrt{h}R^{(3)} + \beta_\phi(\sqrt{h}R^{(3)})^2 + \gamma_\phi\sqrt{h}R^{(3)}_{ij}R^{(3)ij}), \tag{49}$$

where $l_p = 1/m_p$, $\nu = 1/\mu$, $l_p$ is the Planck length. $\alpha_\phi, \beta_\phi, \gamma_\phi$ are constants expressed by $\phi^{(n)}(0)$. From the above expression (46), (47), we see that the next leading contribution will be suppressed if we take $\mu \gg m_p$ ($\nu \ll l_p$). This may mean that if we take a "renormalization" scale $\mu$ to be much greater than $m_p$, or if we take a "renormalization" scale $\nu$ to be much smaller than $l_p$, the wave function of the universe may be given by

$$\Psi[h] \simeq \exp(-\frac{1}{G_\phi}\int d^3y\sqrt{h}(R^{(3)} - 2\Lambda_\phi)) \quad \text{for} \quad \mu \gg m_p \ (\nu \ll l_p). \tag{50}$$

The exponent of the wave function (50) is the three dimensional Einstein gravity action with the "renormalized" Newton's constant $G_\phi$ and the "renormalized" cosmological constant $\Lambda_\phi$. Three dimensional Einstein gravity theory does not contain gravitational wave modes, and the theory is the dynamical system of the finite degrees of freedom which come from different topologies of the manifolds. Thus the expression (12) may mean that "New Phase" (the dynamical system described effectively by the three dimensional quantum Einstein Gravity theory) exists at the scale which is much smaller than Planck scale.

To see physical meaning of our expansion (5) from the viewpoint of minisuperspace models, we shall discuss the de Sitter minisuperspace model. The WDW equation in this model is given by [4] [5]

$$\{\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - a^2[1 - \Lambda a^2]\}\Psi(a) = 0, \tag{51}$$

where we have set $m_p = 1$, and $a$ is the scale factor, the parameter $p$ represents the factor-ordering ambiguity. With the choice $p = -1$, this equation can be solved exactly . If we choose Hartle-Hawking boundary condition [4], the solution is given by [5]

$$\Psi[a] = c_0 A_i(z), \tag{52}$$

where $z = (4\Lambda^2)^{-1/3}(1 - \Lambda a^2)$, $c_0$ is a real constant, and $A_i(z)$ is Airy function [12]. If we write

$$\Psi[a] = e^{-S(a)}, \tag{53}$$

the WDW equation becomes

$$-\frac{d^2S}{da^2} + (\frac{dS}{da})^2 + \frac{1}{a}\frac{dS}{da} - a^2(1 - \Lambda a^2) = 0. \tag{54}$$

We shall consider the small scales ($a \to 0$). In this limit the asymptotic behavior of Airy function is [12]

$$A_i(z) \sim \frac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\zeta}, \tag{55}$$

where $\zeta = \frac{2}{3}z^{3/2}$. Using (52), (53) and (55), we have

$$\frac{d^2S}{da^2} = 1, \quad \frac{dS}{da} = a, \quad \text{for} \quad \Lambda a^2 \ll 1 \quad \text{and} \quad 0 < \Lambda \ll 1. \tag{56}$$

Then, we get

$$\frac{d^2S}{da^2}, \quad \frac{1}{a}\frac{dS}{da} \gg (\frac{dS}{da})^2 \quad \text{for} \quad 0 < a \ll 1. \tag{57}$$

This relation (57) in our simple minisuperspace model suggests that the WDW equation (1) is approximated as eq.(7) when the radius of the universe is much smaller than Planck length. Therefore, "New Phase"( the system described by three dimensional quantum Einstein gravity theory) would exist at the very early stage of the evolution of the universe.

5. Finally, we briefly comment on renormalizability of the WDW equation. Our strange condition $\phi^{(1)}(0) = 0$ is obtained from the closure of algebras for the regularized Hamiltonian constraint $\tilde{H}(x)$ and momentum constraints $\tilde{H}^i(x)$ because, in our heat kernel regularization, the regularized Hamiltonian constraint $\tilde{H}(x)$ contains heat-kernel $K_{i'j'kl}$;

$$\tilde{H}(x) = \frac{1}{m_p^2}\int d^3x' K_{i'j'kl}(x', x; \epsilon)\pi^{i'j'}(x')\pi^{kl}(x) + m_p^2\sqrt{h}(R^{(3)}(x) - 2\Lambda), \tag{58}$$

where $\pi^{ij}$ is the canonical conjugate to the spatial metrics $h_{ij}$. Additional term $\Delta\Gamma$ arise from the cross commutator of $\Delta_\phi$ and $\sqrt{h}R^{(3)}$ as follows,

$$[\int d^3x\xi(x)\tilde{H}(x), \int d^3y\eta(x)\tilde{H}(y)] \tag{59}$$

$$= i\int d^3x(\xi(x)\eta(x)_{;i} - \xi(x)_{;i}\eta(x))H^i + \Delta\Gamma, \tag{60}$$

$$\Delta\Gamma = \alpha\phi^{(1)}(0)\int d^3x(\xi(x)\eta(x)_{;i} - \xi(x)_{;i}\eta(x))R^{(3);i}. \tag{61}$$

where $\alpha$ is a constant, $\phi^{(1)}(0)$ has the dimension of $[m_p]$, $H(x)$ and $\Delta\Gamma$ have the dimensions of $[m_p]^4$ and $[m_p]^2$. $\Delta\Gamma$ is antisymmetric under the exchange of $\xi$ and $\eta$. The closure of the algebra requires that $\phi^{(1)}(0) = 0$. Therefore the WDW equation may be renormalizable.

6. In summary, we have tried to solve the WDW equation for the wave function of the universe as an expansion in powers of the Planck mass, by using the heat-kernel regularization. We have shown that the wave function of the universe to leading order is expressed by three-dimensional Euclidean Einstein action, and expectation values computed with the leading order approximation are reduced to the expectation value in three dimensional Euclidean Einstein gravity theory. This may mean that "New phase"( the dynamical system described effectively by the three dimensional quantum Einstein gravity theory) exists at the scale which is much smaller than Planck scale, or our universe was a three dimensional Euclidean universe effectively at the very early stage of the evolution of the universe, in which the size of the universe is much smaller than Planck size. The renormalizability of the WDW equation and the next order contribution in our expansion (5) will be discussed, in more detail, in another paper.

## REFERENCES


[1] B. S. DeWitt, Phys. Rev. **160** (1967) 1113.

[2] G. 't Hooft, Preprint THU-93/26.

[3] E. Alvarez, Rev. Mod. Phys. **61** (1989) 561.

[4] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28** (1983) 2960; S. W. Hawking, Nucl. Phys. B **239** (1984) 257.

[5] A. Vilenkin, Phys. Rev. D **33** (1986) 3560, Phys. Rev. D **37** (1988) 888.

[6] T. Horiguchi, Mod. Phys. Lett. A **8** (1993) 777, Phys. Rev.D **48** (1993) 5764.

[7] Conceptual Problems of Quantum Gravity, ed. A. Ashtekar, J. Stachel, (The center for Einstein Studies, Boston University, 1991);T. Horiguchi in Proceeding of the Third Workshop on General Relativity and Gravitation ( January 17-20, 1994) ed. K. Maeda et al; T. Horiguchi, Mod. Phys. Lett. A **9** (1994) 1429.

[8] P. Mansfield, Nucl. Phys. B **418** (1994) 113.

[9] K. G. Wilson, Phys. Rev. D **10** (1974) 2445.

[10] B. S. DeWitt, Dynamical theory of Groups and Fields (Gordon and Breach, Inc., New York, 1965).

[11] B. S. DeWitt, Phys. Reports **19** (1975) 295; H. P. McKean, I. M. Singer, J. Diff. Geom. **5**(1971) 233; P. B. Gilkey, J. Diff. Geom. **10** (1975) 601, Proc. Symp. Pure Math. **27** (1975) 265.

[12] M. Abramowitz and I. A. Stegun, Hand book of mathematical functions ( Dover, New York, 1972).