KEK-TH-566

JUN 4 1998

Fermilab Library
0 1160 0061934 2

KEK-TH-566
KEK Preprint 98-25
April 1998
H

# Nonlinear Model for Trapped Ion interacting with Standing Wave via Gauge-Like Transformations

S. Alam [1,3] *and C. Bentley [2] †

[1]Theory Group, KEK, Tsukuba, Ibaraki 305, Japan

[2]Prairie View Texas A & M University, Texas, USA

[3]University of Peshawar, Peshawar, NWFP, Pakistan

## Abstract

We treat the interaction of a trapped ion with a standing-wave light field relaxing the previous condition of the trapping ion being on the node of the standing wave. In addition our treatment is valid to all orders in the Lamb-Dicke parameter, $\eta$. Only a simple transformation to the "dressed basis" [subject to the constraint of Pauli algebra] is required in order to show that the ion interacting with a standing light-wave may be described by the Jaynes-Cummings like Hamiltonian to first order in the parameter $\eta$. Going beyond the leading order in $\eta$ leads one naturally to a Nonlinear Jaynes-Cummings type Hamiltonian. Nonlinear Jaynes-Cummings dynamics of a trapped ion was recently discussed by Vogel and Filho. However our approach gives/specifies an ansatz how to arrive at the nonlinearity. The ansatz of leaving invariant the Pauli algebra after transformations amounts to a linear transformation in the su(2) algebra describing the two-level system. The outcome is a Jaynes-Cummings like Hamiltonian or a Nonlinear Jaynes-Cummings like Hamiltonian [if nonlinear terms are kept] with the effective coupling dependent on the parameter $\chi$ aimed to figure the trapping location in the standing wave. The main idea, namely leaving the Hamiltonian form invariant while changing the physical parameters, can pave the way to a new understanding of gauge-like transformations in the present context.

*email: sher@theory.kek.jp

†cbentley@diamond.pvsci.edu

**High Energy Accelerator Research Organization (KEK), 1998**

KEK Reports are available from:

Information Resources Division
High Energy Accelerator Research Organization (KEK)
1-1 Oho, Tsukuba-shi
Ibaraki-ken, 305-0801
JAPAN

Phone: 0298-64-5137
Fax: 0298-64-4604
Cable: KEK OHO
E-mail: Library@kekvax.kek.jp
Internet: http://www.kek.jp

## Introduction

Field theories in particle physics of spin one-half fermions [matter fields] interacting with spin-one vector bosons [force fields] subject to the principle of local gauge invariance have been very successful in fundamental processes. The familiar examples are quantum electrodynamics [QED], quantum chromodynamics [QCD] and quantum flavorodynamics [QFD]. QED, the theory of electrically charged spin one-half fermions interacting with massless spin one photons, is remarkably precise in predicting physical quantities up to 15 decimal places. The quantum corrections up to several loop orders in QED precisely and beautifully fit the experimental data. Given the successes of gauge theories in describing the fundamental interactions, it is natural to expect that the physical and mathematical techniques arising from these theories may have direct and/or indirect relevance to the understanding of other areas, such as quantum optics and condensed matter physics. Indeed several attempts to incorporate ideas from gauge theories into condensed matter physics and quantum optics exist.

In the context of quantum optics, the Jaynes-Cummings model is a favorite among theorists and experimentalists. As is well-known when we consider a quantum two-level system interacting with a single mode of quantized field one is led to the familiar Jaynes-Cummings Hamiltonian [JCH], [1] provided one is interested only in the difference of the population of the two levels. The JCH has been extensively used as a model Hamiltonian in fields such as quantum optics, nuclear magnetic resonance, and quantum electronics. It is *well-known* that a two-level system interacting with a single mode of radiation is mathematically equivalent to spin one-half system interacting with the a single mode of radiation. It appears that the authors Yang et al., [2] have overlooked the fact that this similarity between the JC model and that describing a spin-$\frac{1}{2}$ particle in a magnetic field is well-known. To quote them, [2] "We now establish the similarity between JC model and one describing a spin-$\frac{1}{2}$ particle in a magnetic field." A compact method/notation to solve various nonlinear extensions of the JCH will be reported elsewhere [3].

An interesting study of the JCH is the periodic spontaneous collapse and revival due to the quantum granularity of the field [4]. In the rotating wave approximation [RWA], the



JCH becomes solvable and it has been widely used in the last several years [5–10]. There are many other interesting works on the JCH.

A question that naturally arises is, why is the JCH so successful and rich in predictions? An answer could be that it is the simplest Hamiltonian one can write for a spin-half system interacting with a single mode of a quantized field and thus it covers a whole spectrum of natural phenomena, since as already mentioned it seems that we can formulate a large class of successful quantum theories based on the interaction of spin one-half fields interacting with spin one fields. Exploiting the general structure of a spin one-half system interacting with a radiation field by using well-known results a compact method/notation has been developed to solve various nonlinear extensions of the JCH [3]. For example in the present note by demanding that the transformed dressed matrices obey the Pauli algebra we are able to demonstrate the correspondence of trapped ion dynamics interacting with a standing-wave light field to the JCH and provide an ansatz and a rationale for arriving at the Nonlinear Jaynes-Cummings like Hamiltonian.

One can also attempt to solve Hamiltonians using the mathematics of supersymmetry [SUSY]. Although SUSY still awaits experimental verification in the context of particle physics, we are free to use SUSY as a mathematical tool to solve various physical problems. In particular SUSY quantum mechanics has proved quite useful in the solution of various problems (see [11] for a review). One example of SUSY quantum mechanics at work is the exact supersymmetry in nonrelativistic hydrogen atom considered by Tangerman and Tjon [11], there are many others. Solutions of the JCH and its extensions using SUSY will be reported elsewhere [12].

The Hamiltonian of the trapped ion interacting with a standing wave, where the trapped ion is on the *node* of a standing wave to first order in the Lamb-Dicke parameter $\eta$ was given by Cirac et al.,. [13]. It was shown in Ref. [13] that the dynamics of the laser cooled two-level trapped ion is described by the JCH. The analogy established in [13] between trapped ion dynamics and cavity QED is an important one and has resulted in many proposals/ideas, for example, quantum non-demolition measurement of final temperature [14,15], experimental study of collapse and revivals [16] and theoretical studies of the same [17,13,18]. One intuitively expects that the dynamics of the ion away from the node can also

be described by JCH. One of the purposes of this note is to extend the result of Ref. [13] by allowing the trapped ion to be anywhere (not just on the node) in the standing-wave light field. Recently Wu and Yang [19] (see Note added) have also attempted to extend the work of Ref. [13]. In their analysis [19] they introduce the usual interaction picture which is not necessary, as is clear from our work. More precisely, the use of interaction picture in our approach is not needed in the context of establishing a correspondence between a trapped two-level ion interacting with a standing-wave light field and the JCH. One simply needs to use the transformation to the dressed basis to establish the correspondence between the trapped ion interacting with the standing-wave light field and JCH. The particular form of the transformation is determined by imposing the condition that the algebra of Pauli matrices be maintained.

We also go beyond the first order in $\eta$ and point out that by keeping second and higher order terms in $\eta$ we are lead to the multi-photon or m-photon JCH [20–23]. Going beyond the leading order in $\eta$ leads one naturally to a Nonlinear Jaynes-Cummings type Hamiltonian. A nonlinear Jaynes-Cummings dynamics of a trapped ion was recently discussed by Vogel and Filho[‡] [24]. However our approach gives/specifies an ansatz how to arrive at the nonlinearity.

Our procedure consists of two new points: the first one consists in replacing $\sin[\eta(a+a^\dagger)]$ by $\cos[\eta(a+a^\dagger)+\chi]$ in the Hamiltonian to describe the general position $\chi$ of the trapped ion and expanding in powers of $\eta$. The second ansatz amounts to a linear transformation in the su(2) algebra describing the two level system which leaves the Pauli commutation rules invariant. The outcome is a JC like Hamiltonian with effective coupling dependent on the parameter $\chi$ aimed to figure the trapping location in the standing wave. The main idea namely, leaving the Hamiltonian form invariant while changing the physical parameters can pave the way to a new understanding of gauge like transformations in the present context.

Our results consist of an extension of Cirac et. al. [13] to ions trapped on other parts of the standing wave (not just the node), and the proposed model opens a way to go beyond the first-order in the Lamb-Dicke parameter $\eta$, allowing naturally the incorporation of nonlinear

[‡]I thank W. Vogel for informing me about his work. We note that Wu and Yang [19] seem to be unaware of Ref. [24].



effects. A nonlinear Jaynes-Cummings dynamics of a trapped ion was recently discussed by Vogel and Filho [24]. It is natural to ask the question why should one incorporate the nonlinearities in the form as suggested in Ref. [24] instead of some other form? Moreover what rules dictate the structure of nonlinear JCH? Our ansatz provides the rationale for the construction of nonlinear JCH, or at least provides some hints in constructing nonlinear JCH.

In addition, and importantly, our algebraic technique is akin to gauge-like transformations, which in the present context, leave the Hamiltonian form invariant while changing the physical parameters.

### JC and m-photon JC Hamiltonians

The master equation of a single two-level ion trapped in a harmonic potential and located at the node of standing-wave light field is, [13],

$$\frac{d\rho}{dt} = -i\left[\nu a^\dagger a + \frac{\Delta}{2}\sigma_z - \frac{\Omega_0}{2}(\sigma_+ + \sigma_-)\sin[\eta(a+a^\dagger)], \rho\right] + \frac{\Gamma}{2}(2\sigma_-\tilde{\rho}\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-), \tag{1}$$

where $\Delta$ is the detuning of the two-level transition from the laser frequency, $\nu$ is the trap frequency, $\Omega_0$ is the laser Rabi frequency, and $\eta = \pi a_0/\lambda$ is the Lamb-Dicke parameter with $a_0$ the amplitude of the ground state of the trap and $\lambda$ the optical wavelength. $a^\dagger$ and $a$ are the creation and destruction operators for phonons, respectively. The $\sigma_i$, $i = x, y, z$ are the usual Pauli matrices [4] with $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, the raising and lowering operators for the two-level system. We note the familiar relations governing $\sigma$'s, i.e., $\{\sigma_i, \sigma_j\} = \delta_{ij}$, $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, $i,\ j = x,\ y,\ z$. $\tilde{\rho}$, does not concern us here and is defined in Ref. [13]. As mentioned in Ref. [13], it accounts for the momentum transfer associated with the spontaneous emission of a photon.

The Hamiltonian for a single two-level ion trapped in a harmonic potential and located at the node of a standing-wave light field is easily read from Eq. 1. We have

$$\mathcal{H} = \hbar\nu a^\dagger a + \hbar\frac{\Delta}{2}\sigma_z - \hbar\frac{\Omega_0}{2}(\sigma_+ + \sigma_-)\sin[\eta(a+a^\dagger)]. \tag{2}$$

We note that the population occupation operators are defined in the usual manner: $\sigma_{22} - \sigma_{11} = \sigma_z$ and $\sigma_{22} + \sigma_{11} = 1$ or alternatively $\sigma_{22} = \frac{1}{2}(1+\sigma_z)$ and $\sigma_{11} = \frac{1}{2}(1-\sigma_z)$.

It is straightforward to extend Eq. 2 to the case in which the ion is not necessarily on the node by letting $-\sin[\eta(a+a^\dagger)] \rightarrow -\sin[\pi/2]\sin[\eta(a+a^\dagger)] \rightarrow \cos[\eta(a+a^\dagger)+\pi/2] \rightarrow \cos[\eta(a+a^\dagger)+\chi]$. We thus obtain the Hamiltonian for a trapped ion in a harmonic potential and located anywhere on a standing-wave light field,

$$\mathcal{H} = \hbar\nu a^\dagger a + \hbar\frac{\Delta}{2}\sigma_z + \hbar\frac{\Omega_0}{2}(\sigma_+ + \sigma_-)\cos[\eta(a+a^\dagger)+\chi]. \tag{3}$$

We can write $\cos[\eta(a+a^\dagger)+\chi] = \cos[\eta(a+a^\dagger)]\cos\chi - \sin[\eta(a+a^\dagger)]\sin\chi$ and expand Eq. 4 in powers of the Dicke parameter $\eta$ to obtain,

$$\begin{aligned}\mathcal{H} = {} & \hbar\nu a^\dagger a + \hbar\frac{\Delta}{2}\sigma_z \\ & + \hbar\frac{\Omega_0}{2}(\cos\chi)(\sigma_+ + \sigma_-) - \eta\hbar\frac{\Omega_0}{2}(\sin\chi)(\sigma_+ + \sigma_-)(a+a^\dagger) \\ & - \frac{\eta^2}{2!}\hbar\frac{\Omega_0}{2}(\cos\chi)(\sigma_+ + \sigma_-)(a+a^\dagger)^2 \\ & + \frac{\eta^3}{3!}\hbar\frac{\Omega_0}{2}(\sin\chi)(\sigma_+ + \sigma_-)(a+a^\dagger)^3 + O(\eta^4).\end{aligned} \tag{4}$$

For the moment ignoring the term of order equal to or greater than $\eta^2$ and using the definition/relation $\sigma_+ + \sigma_- = \sigma_x$, we may rewrite Eq. 4 as

$$\mathcal{H} = \hbar\nu a^\dagger a + \hbar\frac{\Delta}{2}\sigma_z + \hbar\frac{\Omega_0}{2}(\cos\chi)\sigma_x - \eta\hbar\frac{\Omega_0}{2}(\sin\chi)(\sigma_+ a + \sigma_- a^\dagger), \tag{5}$$

where in the last line we have gone to the usual RWA. We have gone to the RWA since we wish to compare our Hamiltonian to the JCH, which is usually written using RWA. In order to cast Eq. 5 into the JC form, we can combine the second and third terms in Eq. 5 into a single term by defining the transformation

$$\begin{aligned}\Sigma_z &= \frac{\Delta}{\Omega}\sigma_z + \frac{\Omega_0\cos\chi}{\Omega}\sigma_x, \\ \Omega &= \sqrt{\Delta^2 + \Omega_0^2\cos^2\chi}\ .\end{aligned} \tag{6}$$

We note that $\Sigma_z^2 = 1$. Using 6 in Eq. 5 we can write the latter as

$$\mathcal{H} = \hbar\nu a^\dagger a + \hbar\frac{\Omega}{2}\Sigma_z - \eta\hbar\frac{\Omega_0}{2}\sin\chi(\sigma_+ a + \sigma_- a^\dagger), \tag{7}$$



In order to have a consistent Pauli algebra between the $\Sigma$ matrices, we must supplement $\Sigma_z$ in 6 with $\Sigma_x$ and $\Sigma_y$, where

$$\begin{aligned}\Sigma_x &= \frac{\Delta}{\Omega}\sigma_x - \frac{\Omega_0\cos\chi}{\Omega}\sigma_z, \\ \Sigma_y &= \sigma_y, \\ \Sigma_\pm &= \frac{1}{2}(\Sigma_x \pm i\Sigma_y),\end{aligned} \tag{8}$$

which are/[can be] found/determined with some simple algebra. For example, a simple way to determine $\Sigma_x$ is to assume that it is some arbitrary linear combination of $\sigma_x$ and $\sigma_z$ and then determine the coefficients appearing in the linear combination by using the fact that $\{\Sigma_z, \Sigma_x\} = 0$. It is easily checked that $\Sigma_x^2 = 1$, $\Sigma_y^2 = 1$, $\{\Sigma_x, \Sigma_y\} = 0$, $\{\Sigma_y, \Sigma_z\} = 0$ $\{\Sigma_z, \Sigma_x\} = 0$, $[\Sigma_x, \Sigma_y] = 2i\Sigma_z$, $[\Sigma_y, \Sigma_z] = 2i\Sigma_x$ and $[\Sigma_z, \Sigma_x] = 2i\Sigma_y$. These relations can be summarized or put into the usual compact notation namely $\{\Sigma_i, \Sigma_j\} = \delta_{ij}$, $[\Sigma_i, \Sigma_j] = 2i\epsilon_{ijk}\Sigma_k$, $i,\ j = x,\ y,\ z$. We thus have a consistent Pauli algebra.

Using Eq. 8 we may rewrite Eq. 7 completely in terms of $\Sigma$'s, viz,

$$\mathcal{H} = \hbar\nu a^\dagger a + \hbar\frac{\Omega}{2}\Sigma_z - \eta\hbar\frac{\Omega_0}{2}\frac{\Delta}{\Omega}\sin\chi(\Sigma_+ a + \Sigma_- a^\dagger). \tag{9}$$

Comparing the Hamiltonian of Eq. 9 with the JCH [13],

$$\mathcal{H} = \hbar\omega_f a^\dagger a + \frac{1}{2}\omega_0\sigma_z + \hbar g(\sigma_+ a + \sigma_- a^\dagger), \tag{10}$$

we see a direct correspondence. In particular by making the identifications $\nu \leftrightarrow \omega_f$, $\Omega \leftrightarrow \omega_0$, and $g \leftrightarrow \eta\frac{\Omega_0}{2}\frac{\Delta}{\Omega}\sin\chi$, we immediately see that there is a direct correspondence between the two-level trapped ion interacting with a standing-wave light field and the familiar JCH. The effective coupling $\eta\frac{\Omega_0}{2}\frac{\Delta}{\Omega}\sin\chi$ is dependent on the Rabi frequency and can be adjusted in experiments. It is clear that Eq. 9 reduces in the special case of the ion at the node (i.e., setting $\chi = \pi/2$) to the previous result [13].

Going back to Eq. 4, we see that the second-order term in $\eta$ corresponds in the RWA approximation to the two-photon term in the m-photon JCH [20,21,23]. The higher-order terms in $\eta$ similarly correspond to the multi-photon terms in the m-photon JCH. We have thus arrived at a nonlinear Jaynes Cummings Hamiltonian. One type of nonlinear Jaynes-Cummings dynamics of a trapped ion was recently discussed by Vogel and Filho [24]. However our approach gives/specifies for the first time an ansatz how to arrive at the nonlinearity.

One can see that the fluorescence spectrum without the inclusion of the decay rate will consist of three peaks, as noted in Ref. [13]. In our case the three peaks will be centered at $\nu = 0, \ \pm\Omega$ with $\Omega$ given in 6. We note that our $\Omega_0$ corresponds to $\Omega$ in Ref. [13]. If the decay rate is included, $\Omega$ will be changed to something like $\Omega = \sqrt{\Delta^2 + \Omega_0^2 \cos^2\chi + c\,\Gamma^2}$, where $c$ is a constant. It is claimed that $c = 5$ in [19]. However, we have not checked this.

## Conclusions

In conclusion:

- We have treated the interaction of a trapped ion with a standing-wave light field, relaxing the previous condition that the trapped ion remain on the node of the standing-wave [13]. It has been shown here that the two-level trapped ion dynamics both off and on the node are described by JCH .
- Our analysis relies on a transformation to the dressed basis, the transformation in turn is determined by demanding that the Pauli algebra hold between the transformed matrices.
- We have also shown/indicated that by going to higher orders, i.e., beyond first order in Lamb-Dicke parameter $\eta$, we are lead to the m-photon JCH. Going beyond the first order in $\eta$ coupled to the constrain of maintaining the Pauli algebra between the transformed matrices naturally leads us to the nonlinear JCH. This for the first time, to our knowledge, provides an ansatz and a rationale for obtaining a nonlinear JCH.

A particularly interesting aspect of our work is the introduction of the ansatz which results from the observation that one must leave the Pauli algebra invariant after the transformations. This amounts to the introduction of a linear transformation in the su(2) algebra describing the two level system. The outcome is a Jaynes-Cummings like Hamiltonian with effective coupling dependent on the parameter $\chi$ aimed to figure the trapping location in the standing wave. The main idea, namely leaving the Hamiltonian form invariant while changing the physical parameters, can pave the way to a new understanding of gauge like transformations in the present context.



One could use the well-established field of ion trapping as a testing ground for strongly coupled QED because for a trapped ion the coupling parameters can be varied by the laser field strength. The ability to control/vary the coupling is an attractive feature of the trapped ion system.

Note added: We recently became aware of the work by Wu and Yang, [19]which also deals with the extension of Ref. [13]. However, during the course of their analysis, they go to the interaction picture and in the end drop the time-dependences. In contrast, our approach is based on maintaining the algebra of the Pauli matrices after the transformation to the "dressed basis". Hence the use of interaction picture in our approach is not needed in the context of establishing a correspondence between a trapped two-level ion interacting with a standing-wave light field and the JCH.

## ACKNOWLEDGMENTS

The work of S. Alam was supported in part by a COE Fellowship from the Japanese Ministry of Education and culture and in part by the JSPS.

## REFERENCES


[1] E. T. Jaynes and F. W. Cummings, Proc. IEEE 51 (1963) 89.

[2] Xiaoxue Yang et al., Phys. Rev. A55 (1997) 4545.

[3] S. Alam, *Compact Method/Notation for Nonlinear extensions of Jaynes-Cummings Model*, in preparation.

[4] P. Meystre, M. Sargent III, Elements of Quantum Optics. (Springer-Verlag, Berlin Heidelberg 1990).

[5] J.H. Eberly, N.B. Narozhny, and J.J. Sanchez-Mondragon, Phys. Rev. Lett. 44 (1980) 1323.

[6] R. R. Schlicher, Opt. Comm. 70 (1989) 97.

[7] Hong-bin Huang and Hong-yi Fan, Phys. Lett. A 166 (1992) 308.

[8] Ho Trung Dung and A.S. Shumovsky, Phys. Lett. A 169 (1992) 379.

[9] C. A. Miller, J. Hilsenbeck, and H. Risken, Phys. Rev. A46 (1992) 4323.

[10] S. V. Prants and L. S. Yacoupova, J. Mod. Opt. 39 (1992) 961.

[11] A. Lahari et. al., Int. J. Mod. Phys. A5 (1990) 1383; R. D. Tangerman and J. A. Tjon, Phys. Rev. A48 (1993)1089.

[12] S. Alam, *Supersymmetry based Methods for Jaynes-Cummings Model and its extensions*, in preparation.

[13] J. I. Cirac et al., Phys. Rev. Lett. 70 (1993) 762.

[14] J. I. Cirac et al., Phys. Rev. A 49 (1994) 1202.

[15] C. D'Helon and G. J. Milburn, Phys. Rev. A 52 (1995) 4755.

[16] D. M. Meekhof et al., Phys. Rev. Lett. 76 (1996) 1796; C. Monroe et al., Phys. Rev. Lett. 75 (1995) 4011.

[17] J. I. Cirac et al., Phys. Rev. A 46 (1992) 2668.

[18] I. Marzoli et al., Phys. Rev. A 49 (1994) 2771.

[19] Ying Wu and Xiaoxue Yang, Phys. Rev. Lett. 78 (1997) 3086.

[20] B. Buck and C. V. Sukumar, Phys. Lett. 81 (1981) 132.

[21] C. V. Sukumar and B. Buck, J. Phys. A: Math. Gen 17 (1984) 885.

[22] V. Bartzis and N. Nayak, J. Opt. Soc. Am. B8 (1991) 1779.

[23] S. Alam, Progress of Theoretical Physics Vol. 98, No. 2, 1997.





[24] W. Vogel and R. L. de Matos Filho, Phys. Rev. A 52 (1995) 4214.