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Underlying symmetry among the quark and lepton mixing angles

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Abstract

The six quark and lepton mixing angles are generated with the aid of three related forms of symmetry, and three constants that derive from the fine structure constant.

I. Introduction

We demonstrate that the six quark and lepton mixing angles can be generated with the aid of three related forms of symmetry, and three constants that derive from the fine structure constant.

We begin by defining the leptonic mass mixing matrix conventionally, but without the usual phases [1]

$$U = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix}, \tag{1a}$$

where $s_{ij} \equiv \sin(\varphi_{ij})$ and $c_{ij} \equiv \cos(\varphi_{ij})$. These φ_{ij}'s represent the leptonic mixing angles.

Additionally, we note that

$$U = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix}. \tag{1b}$$

We define the quark mass mixing matrix in the same way, but this time we deviate from convention both by omitting its phase, and by swapping its c- and t-quarks [2]. Accordingly, we let

$$V = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix}, \tag{2a}$$

where $s_{ij} \equiv \sin(\theta_{ij})$ and $c_{ij} \equiv \cos(\theta_{ij})$. These θ_{ij}'s represent the quark mixing angles.

Additionally, we note that

$$V = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{td} & V_{ts} & V_{tb} \\ V_{cd} & V_{cs} & V_{cb} \end{bmatrix}, \tag{2b}$$

which is consistent with the earlier-mentioned swap of the c- and t-quarks.

We also define

$$L = \begin{bmatrix} U_{e1}^2 & U_{e2}^2 & U_{e3}^2 \\ U_{\mu 1}^2 & U_{\mu 2}^2 & U_{\mu 3}^2 \\ U_{\tau 1}^2 & U_{\tau 2}^2 & U_{\tau 3}^2 \end{bmatrix} \tag{3a}$$

and

$$Q = \begin{bmatrix} V_{ud}^2 & V_{us}^2 & V_{ub}^2 \\ V_{td}^2 & V_{ts}^2 & V_{tb}^2 \\ V_{cd}^2 & V_{cs}^2 & V_{cb}^2 \end{bmatrix} \tag{3b}$$

in order to impose the *Principal Symmetry Condition*

$$L - L^T = N \times (Q - Q^T) \tag{4}$$

on the quark and lepton mixing angles. Note that N, which represents the number of quark colors, equals 3.

In order to fulfill this condition we now define the following parametric equations.

$$u_{12} = \sqrt{M_L N} \qquad\qquad u_{13} = \sqrt{M_S}$$

$$v_{12} = \sqrt{M_L} \qquad\qquad v_{13} = \sqrt{M_S / N}$$

These in turn generate the quark and lepton mixing angles as follows

$$\varphi_{12} = \sin^{-1}(u_{12}) \qquad\qquad \varphi_{13} = \sin^{-1}(u_{13} \times \sin(\theta'_{23}))$$

$$\theta_{12} = \sin^{-1}(v_{12} \times \sin(\varphi_{23})) \qquad\qquad \theta_{13} = \sin^{-1}(v_{13})$$

where

$$\theta'_{23} = \theta_{23} - 90^\circ \ . \tag{5a}$$

For later use we also define

$$\varphi'_{23} = \varphi_{23} - 90^\circ \ . \tag{5b}$$

II. Generating the quark and lepton mixing angles

Now, by letting

$$M_L = \frac{1}{10} \ , \tag{6a}$$

$$M_S = \frac{1}{30000} \text{, and,} \tag{6b}$$

$$\varphi_{23} = 45^\circ \ , \tag{6c}$$

and by requiring that the principal symmetry condition be fulfilled, we generate the following good approximations of the quark and lepton mixing angles.

$\varphi_{12} = 33.211^\circ$ $\quad\quad$ $\varphi_{13} = 0.014^\circ$ $\quad\quad$ $\varphi_{23} = 45^\circ$

$\theta_{12} = 12.921^\circ$ $\quad\quad$ $\theta_{13} = 0.191^\circ$ $\quad\quad$ $\theta_{23} = 92.367^\circ$

The reader will note that the high value for θ_{23} (about 92.3674418°) is a result of the earlier swap of the c- and t-quarks, and that subtracting 90° from θ_{23} produces this angle expressed in the usual way.

III. The primary symmetry's two subordinate symmetries

It is important to realize that the parametric equations embody two additional symmetries. To be specific, if

$$\varphi_{12} = \theta_{12} = \varphi'_{23} = 0 \ , \tag{7a}$$

then any values chosen for

$$M_S, N, \text{ and } \theta'_{23}$$

will produce values for the remaining mixing angles φ_{13} and θ_{13} that are guaranteed to fulfill the principal symmetry condition.

Equally, if

$$\varphi_{13} = \theta_{13} = \theta'_{23} = 0 \quad , \tag{7b}$$

then any values chosen for

$$M_L, N, \text{ and } \varphi'_{23}$$

will produce values—in this case for φ_{12} and θ_{12}—that are guaranteed to fulfill the principal symmetry condition.

These two *Subordinate Symmetries* explain why the parametric equations take the form they do: they are designed to guarantee fulfillment of these two additional symmetries. Hence the form of the parametric equations cannot be regarded as a degree of freedom exploited to fit the mixing data.

IV. The fine structure constant

It is noteworthy that the values for M_L and M_S that fit the mixing data, also fix u_{12}, u_{13}, v_{12}, and v_{13} in such a way as to allow them act in concert to produce this close approximation of the fine structure constant inverse

$$\left(u_{12}^{-2} - v_{13}^{2}\right)^3 + \left(v_{12}^{-2} - u_{13}^{2}\right)^2 = 137.0360000023\ldots \quad . \tag{8}$$

Remarkably, this equation reproduces the fine structure constant inverse to within about two parts per billion of its corresponding 2006 CODATA value of 137.035999679 (94) [3].

Moreover, if one regards the integers $1 / M_L$ and $1 / M_S$ as chosen to fit 137.036, then the quark and lepton mixing angles generated earlier may be said to arise from equations exploiting just one free parameter: φ_{23} ; which is to say the six quark and lepton mixing angles are fully determined by the two fine-structure-related integers ($1 / M_L$ and $1 / M_S$), the three symmetries already explained (the *Principal* and two *Subordinate*), and $\varphi_{23} = 45^\circ$.

However, one can carry this one step further. If Eq. (8) is restated purely in terms of mixing angles

$$\left(\sin(\varphi_{12})^{-2} - \sin(\theta_{13})^{2}\right)^{3} + \left[\left(\frac{\sin(\theta_{12})}{\sin(\varphi_{23})}\right)^{-2} - \left(\frac{\sin(\varphi_{13})}{\sin(\theta'_{23})}\right)^{2}\right]^{2} = 137.0360000023\ldots \quad , \qquad (9)$$

then it is only the equivalence

$$\sin(\varphi'_{23})^{2} = \sin(\varphi_{23})^{2} \qquad (10)$$

which allows it to be rewritten still more symmetrically as

$$\left(\sin(\varphi_{12})^{-2} - \sin(\theta_{13})^{2}\right)^{3} + \left[\left(\frac{\sin(\theta_{12})}{\sin(\varphi'_{23})}\right)^{-2} - \left(\frac{\sin(\varphi_{13})}{\sin(\theta'_{23})}\right)^{2}\right]^{2} = 137.0360000023\ldots \quad . \qquad (11)$$

Accordingly, $\varphi_{23} = 45^\circ$ is not a truly free parameter, but precisely that value needed to make this more symmetrical equation possible. It follows that the earlier parametric equations generate the six quark and lepton mixing angles while exploiting not one free parameter, but essentially no free parameters.

References

[1] B. Kayser, "Neutrino Mass, Mixing, and Flavor Change," in Yao, W. M., *et al.* (Particle Data Group), J. Phys. G 33, 1 (2006).

[2] A. Ceccucci, Z. Ligeti, and Y. Sakai, "The CKM Quark-Mixing Matrix," in Yao, W. M., *et al.* (Particle Data Group), J. Phys. G 33, 1 (2006).

[3] P.J. Mohr, B.N. Taylor, and D.B. Newell (2007), "The 2006 CODATA Recommended Values of the Fundamental Physical Constants" (Web Version 5.0). This database was developed by J. Baker, M. Douma, and S. Kotochigova. Available: http://physics.nist.gov/constants [2007, July 12]. National Institute of Standards and Technology, Gaithersburg, MD 20899.