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ДВЕ РАБОТЫ ПО ГЕОМЕТРИЧЕСКОМУ КВАНТОВАНИЮ

G.I.Kolerov

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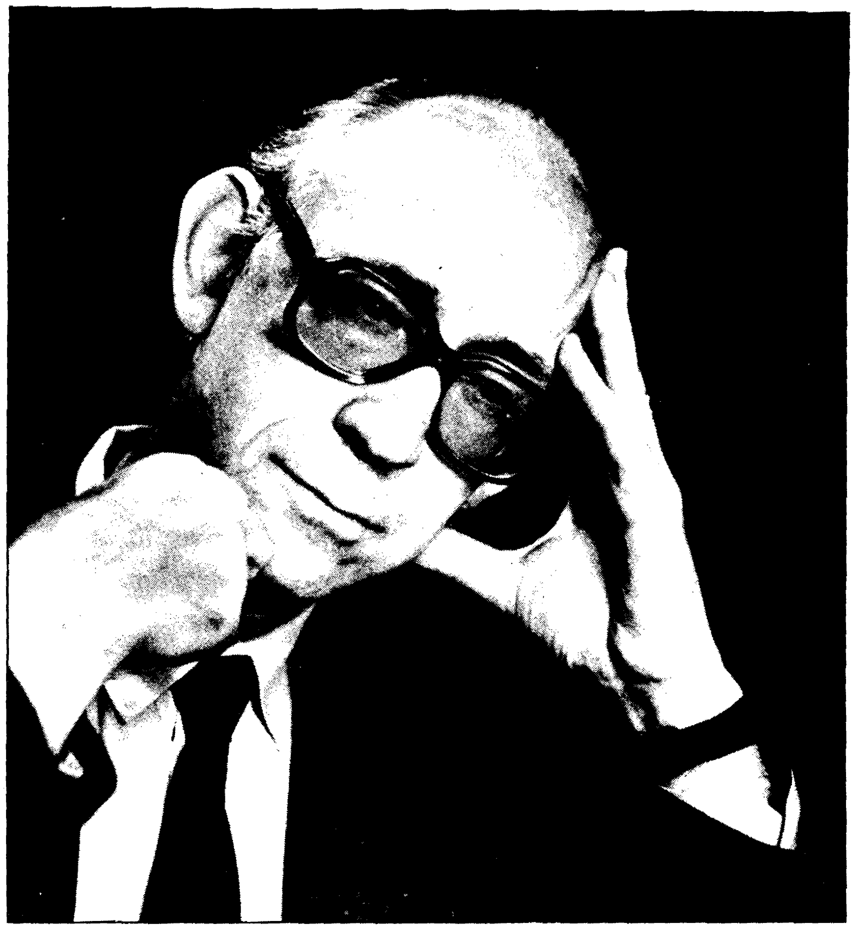

03.08.1936 – 15.07.1990

Пять лет минуло со дня кончины Генриха Ивановича Колерова – замечательного ученого, доброго, чуткого человека. При жизни Генрих Иванович не уставал восхищаться блестящими работами таких корифеев математики, как Давид Гильберт и Феликс Клейн, и их не менее блестящему умению находить важное физическое приложение полученным математическим результатам. В публикуемых двух работах Генриха Ивановича, относящихся к последнему периоду его научной деятельности, он, как нам кажется, стремился следовать именно этим путем. Здесь мы находим гармоничное приложение абстрактного, на первый взгляд, математического аппарата (теория Морса и проективная геометрия) к исследованию принципиальных проблем теоретической физики, в данном случае к выяснению геометрической основы в общей схеме квантования и к вычислению континуальных (или функциональных) интегралов.

Five years passed since the death of Genrikh Ivanovich Kolerov, an outstanding scientist and a good friend. Genrikh Ivanovich was filled with admiration for brilliant studies by great mathematicians David Hilbert and Felix Klein and their high skill to find important physical applications of mathematical results. In his last two papers Genrikh Ivanovich tried, as we think, to keep this way. The papers contain harmonic application of the abstract, at first sight, mathematical apparatus (Morse theory and projective geometry) to study fundamental problems of theoretical physics, in particular, the geometrical basis in the general scheme of quantization and calculation of continual (functional) integrals.

QUANTIZATION AND PROJECTIVE GEOMETRY.*

G. I. Kolerov

One of the principal ideas of the geometric quantization consists in the introduction of the linear connection in a certain bundle space [1, 2], the Hilbert space of states H being constructed of section of the given bundle space. In this case the space of sections F can be determined as the space in which sections are nullified by the covariant differentiation along any vector $\hat{\xi}$. In our paper we tried to demonstrate that the geometry of the quantization can be developed from the classic geometry, particularly, from the projective one.

Let us construct the bundle space $\mathcal{L}$ over the symplectic space with the fibre C (the complex plane)

$$\begin{array}{ccc} C & \rightarrow & \mathcal{L} \\ & & \downarrow \pi \\ & & \mathcal{M} \end{array}$$

by introducing the linear connection

$$\nabla_x f_\alpha = \frac{1}{i\hbar} \mathcal{Q}(x) f_\alpha \quad , \tag{1}$$

where $\mathcal{Q}(x)$ is a vector on $\mathcal{M}$: $\mathcal{Q}(x) = \varphi_i(x)\, dx^i$; and $f_\alpha(x)$ is the section of the bundle space. If in the space $\mathcal{M}$ for a some coordinate neighborhood U_i with the coordinates $\{x^i, p_i\}$ we define the trajectory along the curve $c(\tau)$ with the infinitesimal generator $\hat{X}$, then

$$X = \sum \left\{ \frac{dx^\alpha}{d\tau} \frac{d}{dx^\alpha} - \frac{dp_\alpha}{d\tau} \frac{d}{dp_\alpha} \right\} \quad . \tag{2}$$

*Poster report at VIII International Conference on Mathematical Physics, Marseille, France, July 1986.

In this case for every function φ the following equalities take place:

$$\begin{aligned} \nabla_{\varphi\hat{x}} &= \varphi \cdot \nabla_{\hat{x}} \\ \nabla_{\hat{x}}(\varphi_f) &= (\hat{x}\cdot\varphi)f + \varphi\cdot\nabla_{\hat{x}}\cdot f \quad . \end{aligned} \tag{3}$$

From (3) and (1) we obtain

$$\nabla_{\frac{d}{dx^\alpha}} \cdot \hat{e}_\beta = \Gamma^m_{\alpha\beta}\hat{e}_m \quad . \tag{4}$$

Where $\Gamma^m_{\alpha\beta} = (1/i\hbar)\varphi_\alpha\delta^m_\beta$, $f_\alpha = \mathrm{Re}f_\alpha + i\,\mathrm{Im}f_\alpha = e_m f^m_\alpha$. Using (2) and (4) we obtain the expression for the covariant derivative along the curve $c(\tau)$

$$\frac{\mathcal{D}f_\alpha}{d\tau} = \frac{df}{d\tau} + \Gamma^\beta_{\alpha m}\frac{dx^m}{d\tau} \quad , \tag{5}$$

or in the differential form

$$\mathcal{D}f_\alpha = df_\alpha + \frac{1}{i\hbar}\mathcal{Q}(x)f_\alpha \quad . \tag{6}$$

The expression for sections which are nullified by the covariant derivative has the following form

$$\mathcal{D}f_\alpha(x) = 0 \quad . \tag{7}$$

In a particular case, if $\mathcal{Q}(x) = p_i\, dx^i$, then

$$f_\alpha(x) \simeq e^{\frac{i}{\hbar}p_i\,\delta x^i} \quad . \tag{8}$$

Thus, we obtain the expression for a plane wave.

Let $\{U_\alpha\}$ be the covering of $\mathcal{M}$ and $g_{\alpha\beta}$ are the transition function

$$f_\alpha(x) = g_{\alpha\beta}(x)f_\beta(x)\,; \quad x \in U_\alpha \cap U_\beta \quad . \tag{9}$$

Let the point x in the neighborhood U_α be specified by the homogeneous coordinates $\{x^i, x_0, i = 1\ldots 4\}$. By introducing the absolute [3] we obtain

$$\varphi(x,x) = a_{mn}x^m x^n = 0 \quad . \tag{10}$$

Then the distance between two points x and y is expressed by the logarithm of the unharmonic relation of the group constituent by these two points and the points (j,i) where the line which connects them crosses the absolute. In other words

$$S_\alpha(x,y) = \frac{k}{2}\ln\frac{\varphi(x,y) + \sqrt{\varphi^2(x,y) - \varphi(x,x)\varphi(y,y)}}{\varphi(x,y) - \sqrt{\varphi^2(x,y) - \varphi(x,x)\varphi(y,y)}} \quad . \tag{11}$$

It may be written in the other form

$$S_\alpha(x,y) = \frac{k}{2}\ln\frac{\varphi(x,y)+\sqrt{\varphi^2(x,y)-\varphi(x,x)\varphi(y,y)}}{\sqrt{\varphi(x,x)\varphi(y,y)}} \quad . \tag{12}$$

The expression $\varphi^2(x,y)-\varphi(x,x)\varphi(y,y)$ represents the quadratic function

$$\phi(p,p) = -\sum A_{iklm}p_{ik}p_{lm} \tag{13}$$

of the variables $p_{ik} = x_i y_k - x_k y_i$ which are called the Plücker coordinates. The coefficients of this function are the following

$$A_{iklm} = a_{ik}a_{lm} - a_{im}a_{lk} \quad . \tag{14}$$

The unharmonic relation of four points (x,y,j,i) which is under the logarithm in the expression (11) can be written in the form

$$\mathcal{D}(x,y;j,i) = \frac{y-j}{y-i} : \frac{x-j}{x-i} \quad . \tag{15}$$

By virtue of the unharmonic relation properties the following formula holds

$$S_\alpha(x,z) = S_\alpha(x,y) + S_\alpha(y,z) \quad . \tag{16}$$

Let us take expression (4) as the absolute [4]

$$\varphi(x,x) = \vec{x}^2 - c^2x_4^2 + R^2x_0^2 = 0 \quad . \tag{17}$$

If in formula (12) we take the limit $R \to \infty$ so that $k/R \to$ const., then for $S_\alpha(x,y)$ in heterogeneous coordinates $\bar{x}^i = x^i/x_0$ the following can be obtained:

$$S_\alpha(x,y) = \text{const.}\sqrt{\overrightarrow{(\bar{x}-\bar{y})}^2 - c^2(\bar{x}_4-\bar{y}_4)^2} \quad . \tag{18}$$

In other words, we obtain the expression for an interval in the case of the pseudo-Euclidean geometry. Such an extreme limit corresponds to the situation when the points (j,i) pass through the straight line (x,y) with the absolute (17) and are merged into the unique point on this line which is situated at infinity. In this case the absolute degenerates into cyclic points.

Assuming that const. $= mc$ we obtain the expression for a free particle which can be written as the following

$$S_\alpha(x,x) = mc\sqrt{x^2} = p_i x^i \quad , \tag{19}$$

where $p_i = mcx^i/\sqrt{x^2}$. Taken $y^i = x^i + dx^i$, we determine from (12)

$$g_{\alpha\beta}(x) = \mathcal{D}_{\alpha\beta}(x, x + dx; j, i) = \exp\left\{\frac{\delta S}{k}\right\} \quad . \tag{20}$$

From (19) at $1/R \ll 1$ we obtain

$$g_{\alpha\beta}(x) \sim \exp\left\{\frac{i}{\hbar}p_i\delta x^i\right\} \quad . \tag{21}$$

Thus, formula (21) coincides with expression (8) obtained from geometric quantization.

If we calculate the variation δS at $y^i = x^i + dx^i$ with an accuracy of the third order in dx^i by making of (11) we arrive at the Fubini-Study metrics which determines the natural metrics in the complex projective space between the couple of straight lines crossing the origin of coordinates. The differential from corresponding to this metric is

$$\omega = k\partial\partial^2 \ln|\varphi(x, x)| \tag{22}$$

Now let us add, to the available images of our geometry, the infinitely remote elements

$$\vec{x}^2 - c^2x_4^2 = 0\,; \quad x_0 = 0 \tag{23}$$

end consider the transformation group retaining these elements. If the points x and y are unharmonically placed with relation to (23), that is

$$\mathcal{D}(x, y; j, i) = -1 \tag{24}$$

then

$$S = \frac{k}{2}\{\ln(-1) + 2\pi i n\} \quad . \tag{25}$$

Assuming that in (25) $k = ih$ (where h is the Planck constant) we obtain

$$S = \pi h(n + \frac{1}{2}) \tag{26}$$

that coincides with the condition of quantization in case of the quasiclassic approximation. Here from (24) and (13) it follows that the Plücker coordinates have the form

$$p_{ik} = c\delta_{ik}\;, \tag{27}$$

where c is the constant.

Since the plane equation in the projective space has the following form

$$x^i y_i = (x, y) = 0 \tag{28}$$

we can compare the points $\{x\}$ and $\{y\}$ of our projective space with the vectors $\bar{x}$ and $\bar{y}$. The condition $(\bar{x}, \bar{y}) = 0$ may be considered as the condition of the appurtenance of the point (x^i) to the polar plane of the point (y^i) relatively to the quadric (23). Moreover, the vector $\bar{y}$ has the normalization condition $\bar{y}^2 = \rho^2$.

Assuming that the vector $\bar{y}$ has the meaning of the momentum vector p with the normalization condition $p = (mc)^2$, we may obtain from (24), (13), (27), the following expression

$$[p, x] = ihI \tag{29}$$

at the appropriate choice of constant c in formula (27). Here the brackets mean the oblique scalar product. For each polar point p equation (28) describes a circle. In fact, by introducing the coordinates

$$\begin{aligned} x_1 &= \tfrac{\rho}{2}(z + \bar{z}) \quad x_3 = \tfrac{\rho}{2}(1 - z\bar{z}) \\ x_2 &= \tfrac{\rho}{2}(z - \bar{z}) \quad x_4 = \tfrac{\rho}{2}(1 + z\bar{z}) \end{aligned} \tag{30}$$

eq. (28) may be rewritten as the equation of the circle in the complex plane (z):

$$az\bar{z} + bz + \bar{b}\bar{z} - d = 0 \quad . \tag{31}$$

For simplicity let us be restricted to two measurements and write the equation of the circle in the coordinates (x_3, x_4)

$$-x_4^2 + x_3^2 = \rho^2 \quad .$$

If the vector p is normalized as $-p^2 = (mc)^2$ and ρ is equal to $(mc)^2$, then from the unharmonic condition (24) and by neglecting the indices we may obtain

$$p \cdot x \sim \lambda \quad \text{where } \lambda = \frac{p_4}{p_3} = \frac{x_3}{x_4} \quad \text{at } \lambda = \hbar \quad .$$

This can be considered as the analogue of the uncertainty relation.

Now consider the conformal mapping of the plane onto the plane by using the analytic function $w = f(z)$. If its linear elements are related by $ds' = e^p \, ds$, then in consequence of the mapping being conformal, it follows that $p = \ln|f'(z)|$ whence $\varphi(z) = \ln f'(z) = p + iq$ where q is the function conjugate to p.

Taking the Schwartz derivative of the function $f(z)$.

$$\{w, z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{32}$$

it may be shown that

$$\mathrm{Im}\{w, z\}\, dz^2 = (\nabla_i \tilde{p}_j - p_i \tilde{p}_j)\, du^i\, du^j \tag{33}$$

where ∇_i is the covariant derivative; $\{u^i\}$ are the curved coordinates of the plane.

Conformal mapping between the planes is called circular if it maps the circle onto the circle or onto the straight line. In this case the lines of the constant curvature pass through the same points and from this condition it follows that

$$(\nabla_i \tilde{p}_j - p_i \tilde{p}_j) = 0 \quad . \tag{34}$$

Therefore $\{w, z\} = 0$ whence $w = (az + b)/(cz + d)$. In the general case [5]

$$\{w, z\} = \mathcal{R}(z) \quad , \tag{35}$$

where $\mathcal{R}(z)$ is a regular function over the whole plane except the finite number of points. Equation (35) can be written in the form

$$g''(z) + \frac{1}{2}\mathcal{R}(z)g(z) = 0 \quad . \tag{36}$$

Then $w = g_1(z)/g_2(z)$, where $g_1(z)$ and $g_2(z) \neq 0$ are linearly independent solution of eq. (36).

Consider g_1 and g_2 as the current coordinates of the vector radius $\vec{R}$ in some subsidiary plane. Using the new variables $w = g_1 g_2^{-1}$, $dz = g_1\, dg_2 - g_2\, dg_1$ we obtain

$$dz =< \vec{R}, d\vec{R} > \quad , \tag{37}$$

where $< \vec{R}, d\vec{R} >$ is the oblique product. From (37) it follows that

$$\left\langle \vec{R}, \frac{d\vec{R}}{dz} \right\rangle = 1\, ; \quad \left\langle \vec{R}, \frac{d^2\vec{R}}{dz^2} \right\rangle = 0 \tag{38}$$

and, from (38), consequently, follows that the vectors $d^2\vec{R}/dz^2$, and $\vec{R}$ are collinear. In this case

$$\frac{d^2\vec{R}}{dz^2} + k\vec{R} = 0\, , \tag{39}$$

where

$$k = -\frac{1}{2[f'(z)]^2}\{w, z\} \quad . \tag{40}$$

Expression (36),(39) and (40) determine the variation of the curved lines at the conformal mappings.

Using the Kobayashi metric based on the conception of a great set $\mathcal{O}(U, \mathcal{M})$ of holomorphic mappings of the unit circle U on the manifold $\mathcal{M}$ and fixing the points p and $q \in \mathcal{M}$ we define the chain of mappings on $\mathcal{M}$ from p in q consisting of a certain number m of holomorphic mappings $f^i \in \mathcal{O}$, the number m of pairs of points $\xi'_j, \xi''_j \in U(j = 1\dots m)$ so that $f^1(\xi'_1) = p$, $f^m(\xi''_m) = q$, $f^j(\xi''_j) = f^{j+1}(\xi'_{j+1})$. For $(j = 1\dots m-1)$ we may define the distance between the points p and q

$$\gamma_m(p, q) = \inf \sum_{j=1}^{m} \rho(\xi'_j, \xi''_j) \tag{41}$$

where ρ is the Lobachevski distance in a unit circle. The bottom facet is taken from all chains $\{f^j, \xi'_j, \xi''_j\}_{j=1}^m$ on $\mathcal{M}$ from p to q with any numbers of links.

By shaping the form (22) in the fibre bundle space $T(\mathcal{M}) = \bigcup_p T_p(\mathcal{M})$ and using (20, 33, 40) we can turn to the limit $m \to \infty$ in formula (41) and thus obtain the expression for the Feynman path integral.

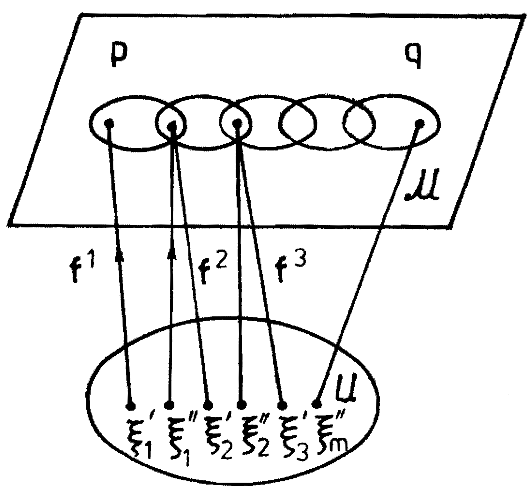

Fig.1

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LINEAR CONNECTION AND CONTINUAL INTEGRAL (APPLICATION OF THE MORSE THEORY) *

G. I. Kolerov

The method of continual or functional integrals is of great importance in theoretical physics due to its universality. By using this method, one can represent a quantum quantity as a sum of contributions of classical virtual trajectories. Simple dependence on the Planck constant $\hbar$ demonstrates that the main contribution to a quantum quantity as $\hbar \to 0$ is ensured by a real classical trajectory.

However, practical calculation of continual integrals encounter difficulties. On the one hand, the traditional approach developed by Winner assumes the continual integral as an integral over measure in a functional space, and therefore, is unadequate. On the other hand, the continual integral as a limit of finite-dimensional approximations is sensitive to their choice and gives no unique values. In this case the ambiguity caused by a different set of approximations has the same meaning as that of quantization. There are some other attempts to use different methods to study the continual integrals.

In this connection, the fundamental Morse theory is very interesting [1]. It studies, in particular, the variety of paths $\Omega(\mathcal{M}; p, q)$ connecting the points p and q on a smooth manifold $\mathcal{M}$. On a space of piecewise smooth paths $PS(\mathcal{M})$ where $\gamma : \tau \to \gamma(\tau)$, $0 \leq \tau \leq 1$ the action functional S is given, the value of which on the path $\gamma \in PS(\mathcal{M})$ is determined in the local coordinates $\{x^i\}$ by the formula

$$S(\gamma) = \int_0^1 g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \, d\tau \quad . \tag{1}$$

*Poster report at IX International Conference on Mathematical Physics, Swansea, Wales, July 1988.

At the same time, the extremals of the functional S (i.e. path $\gamma \in PS(\mathcal{M})$ for which the linear functional S_* on the space T_γ determined by the variation of the functional δS is equal zero) coincide with geodesic metrics g_{ij}; (extremals of the length functional) in their natural parametrization.

Let us choose a section of the unit interval $0 = \tau_0 < \tau_1 \dots \tau_k = 1$ and denote, by $\Omega^{PS}(\mathcal{M}; \tau_0 \dots \tau_k)$, the space in $\Omega(\mathcal{M}; p, q)$ consisting of paths $\gamma : [0,1] \to \mathcal{M}$ so that

1. $\gamma(0) = 0, \quad \gamma(1) = q$
2. $\gamma/[\tau_{i-1}, \tau_i]$ is the geodesics at each $i = 1 \dots k$, then for each $l \in R$ the manifold

$$\Omega_l = \Omega_l^{PS}(\mathcal{M}; \tau_0 \dots \tau_k) = S^{-1}[0, l] \cap \Omega^{PS}(\mathcal{M}; \tau_0 \dots \tau_k) \qquad (2)$$

is determined.

If the manifold $\mathcal{M}$ is full, then $\Omega_l^0 = E^{-1}[0, l) \cap \Omega_l$ (where Ω is the interior of a set) and it is retracted on a smooth manifold B the points of which are "broken geodesics" with a fixed numbers of links connecting p and q.

Proceeding to the limit $l \to \infty$ we can prove [1] that $\Omega(\mathcal{M}; p, q)$ as the space of "broken geodesics" connecting the points p and q is homotopically equivalent to the space $\Omega(\mathcal{M}; p, q)$ of all continued paths connecting the same points.

In connection with this, one may consider a possibility to compute the continual integral by using the Morse theory. Furthermore, we shall use the methods of geometrical quantization [2] the principle idea of which is that under certain conditions the two-form simplectic manifold exists as a form of the linear fibre bundle curvature with connection, the space of states being constructed by sections of the given bundle space .

Let us consider the bundle space $\mathcal{L}$ over $\mathcal{M}$ with the fibre C (a complex plane) through introducing the connection [3]

$$\nabla_\xi f = \frac{i}{\hbar} \mathcal{Q}(\xi) f \quad , \qquad (3)$$

where $\xi \in \mathcal{M}$; f is the section of the bundle space and $\mathcal{Q} = \mathcal{Q}_i dx^i$ is the 1-form on $\mathcal{M}$. Using the connection one can determine the covariant derivative

$$\frac{\partial f}{\partial \tau} = \frac{df}{d\tau} - \frac{i}{\hbar} \mathcal{Q}_i \frac{dx^i}{d\tau} f \quad . \qquad (4)$$

The equality of this derivative to zero gives the condition for translation of a differentiable path from the basic space $\mathcal{M}$ into bundle space $\mathcal{L}$ and thus $\mathcal{Q}$ defines the linear connection in it:

$$df - \frac{i}{\hbar}\mathcal{Q}_i dx^i f = 0 \quad . \tag{5}$$

A solution to this equation for the case $\mathcal{Q} = p_i dx^i$ is a flat wave

$$f_p(x) = ce^{-\frac{i}{\hbar}px} \tag{6}$$

describing the movement of a free particle.

For a given linear connection one may define the connection coefficients using the formula [4]

$$\nabla_{\frac{d}{dx^i}} \cdot f = \Gamma^{\alpha}_{i\beta} f_\alpha \quad . \tag{7}$$

In the case of the connection being defined by (3) one has

$$\Gamma^{\alpha}_{i\beta} = \frac{1}{\hbar} p_i \delta^{\alpha}_{\beta} \quad .$$

The spaces with these connection coefficients are called "projective-euclidean", and the Ricci tensor, in our case, has the form

$$R_{ij} = R^{\alpha}_{ij\alpha} = \frac{p_i p_j}{\hbar^2} \quad . \tag{8}$$

For the action $S(\gamma)$ defined by formula (1) the continual integral will be of the form

$$J_{pq} = \int_p^q \left\{ \exp \frac{i}{\hbar} \int_0^1 S(\gamma)\, d\tau \right\} \mathcal{D}\gamma \quad , \tag{9}$$

Since

$$R = g_{ij} R_{ij} > 0 \quad . \tag{10}$$

The given space has the positive Gaussian curvature along any two-dimensional direction. In this space there are focal points over any geodesic. Let the distance between focal points be $a = \hbar/mc$. Then, it follows from the principal Morse theorem that the space $\Omega(\mathcal{M}; p, q)$ of all paths connecting p and q is homotopically equivalent to the cell space, the dimensions of which λ are in a bijective correspondence with the geodesics of the index λ connecting p and q.

Consider the continual integral for a one 3-dimensional cell [3] $x_1^2 + x_2^2 + x_3^2 \leq a^2$. Let $\bar{\gamma}(\tau)$ be the classical trajectory along which the action is extremal. The variation of the trajectory can be represented as

$$\gamma(\tau) = \bar{\gamma}(\tau) + \delta\eta(\tau) \quad , \tag{11}$$

where $\delta\eta(\tau)$ is the deviation from the classical trajectory (see Fig.1) in the configurational 3-dimensional space $\{\vec{x}\}$.

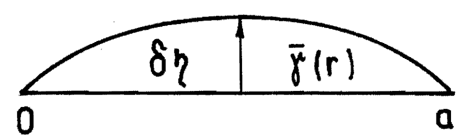

Fig.1

In this case the action function can be represented in the form[1]

$$S(\gamma) = S_{kl}(\bar{\gamma}) + \delta S + \delta^2 S \quad . \tag{12}$$

As we have considered the segments of the geodesics, $\delta S = 0$. Using (12) we can write expression (9) as follows:

$$J_{0a} = \exp \frac{i}{\hbar} S_{kl}(\gamma) \int \left\{ \exp \frac{i}{\hbar} \int_0^a \delta^2 S \, d\tau \right\} \mathcal{D}\gamma \quad . \tag{13}$$

Consider the integral

$$\begin{aligned} G_{0a} &= \int \left\{ \exp \tfrac{i}{\hbar} \int_0^a \delta^2 S \, d\tau \right\} \mathcal{D}\gamma = \\ &= \int \left(\exp \tfrac{i}{\hbar} \int_0^a \left\{ \delta\eta_i, \tfrac{d^2(\delta\eta^i)}{d\tau^2} + R_\gamma \delta\eta^i \right\} d\tau \right) \mathcal{D}\gamma \quad , \end{aligned} \tag{14}$$

[1] The second order limitation supposes that under certain conditions the geodesic transfers to the geodesic.

where R_γ is defined by formula (10); $R_\gamma \simeq 1/a^2$. Assuming $\delta\eta(\tau)$ to be a periodic function of τ with the period a and integrating with the properties of the second variation [1] we get

$$G_{0a} \simeq \frac{\pi}{a}\left(\frac{1}{m^2+\bar{p}^2}\right)^{1/2} \quad . \tag{15}$$

Substituting (15) in to (13) we obtain

$$J_{0a} \simeq \frac{\pi}{a}\frac{1}{\sqrt{m^2+\bar{p}^2}}e^{ip(x_0-x_a)} \quad . \tag{16}$$

If a configurational space is represented as a 3-dimensional cell manifold, then the integral (9), in accordance with the principle theorem of the Morse theory, will be represented as a sum over cells. Using (16) it can be represented as follows;

$$J_{pq} = \left(\frac{2\pi}{a}\right)^3 \sum_{n_1,n_2,n_3} \exp\left\{\frac{2\pi i}{a}(n_0x_0 - \bar{n}\bar{x})\right\}\frac{a}{2\pi n_0} \quad , \tag{17}$$

$$n_0 = \sqrt{n_1^2+n_2^2+n_3^2+\frac{a^2m^2}{4\pi^2}} \quad .$$

This expression for the continual integral corresponds to that obtained in [5], i.e., a great number of pahts contributing to the integral may be approximated with the help of a certain projective operator which project the paths onto a certain definite group of paths contributing to the integral, in our case, on the edge of the cubic lattice. Since according to the Morse theory, the broken geodesics approximates a great number of continual paths $\Omega(\mathcal{M};p,q)$. Formula (17) can be written as a finite result $(2\pi/a) \to d^3p$ at $n_i \to \infty$. Therefore

$$J_{pq} = \mathcal{D}(x_p - x_q) = \int \frac{1}{\sqrt{\bar{p}^2+m^2}} \exp\left\{-\frac{i}{\hbar}(px - p_0x_0)\right\} = \int \frac{e^{\frac{i}{\hbar}}px}{p^2+m^2}\,d^4p\,. \tag{18}$$

Generally spiking, this expression is indeterminate since the rules [6] for $p^2 = m^2$ poles are not determined. These rules being given, we obtain different expressions for the Green functions of the Klein-Gordon equation.

The principal goal of this work is to confirm the validity of the method of continual integral computation on the lattice in the case when the Ricci tensor R_γ on $\mathcal{M}$ depending on the potentials contributing to the action formula, is positively determined everywhere. Otherwise, particular solutions appear due to the connection between topology and curvature [1]. In the solution of these problem the works by S. P. Novikov [7, 8] on the generalization of the Morse theory for multiple-valued functionals are very promising.

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