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UNIFORM DENSITY SPHERES IN GENERAL RELATIVITY

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Abstract : Adiabatic collapse/expansion solutions of uniform density spheres have been known for about three decades. An analysis of these solutions has been done by considering the baryonic conservation law and the no-heat-flow condition. It is shown that if the fluid is isentropic or (and) the sphere's surface temperature remains constant during the expansion/collapse, the pressure inside the sphere cannot be finite (it must vanish).

If the fluid is not isentropic and also the surface temperature varies during the collapse/expansion the results obtained by earlier author (i.e. Bondi, 1969) are found to be inconsistent with the baryonic conservation law and the no heat flow condition.

† The work was completed during author's visit to IUCAA.

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1. Introduction :

Radial adiabatic motion of perfect fluid spheres of uniform density $E = E(t)$, but non-uniform pressure were discussed by Bonnor and Faulkes (1967), Thompson and Whitrow (1967, 1968) and Bondi (1969) under various assumed relationships between central pressure and density. These authors discussed the problem of collapse or bounce under two assumptions : first, that the motion is isotropic or shear-free; and second, that the density is uniform. But Misra and Srivastava (1972) showed that the condition of uniform density and regularity at the centre necessarily leads to the isotropic motion. The above authors did not consider the baryan conservation during the adiabatic motion.

The theme of this paper is rather different from that of the other authors. Unlike earlier authors, no arbitrary relation is chosen between central energy and central pressure. Instead the author has examined whether the pressure can remain finite or not. The author has already shown that under isentropic condition the pressure vanishes (Durgapal, 1994). Here the author has examined the no-heat-flow condition and baryon-conservation law during the collapse/expansion. It is shown that if the fluid is isentropic or (and) the surface temperature remains constant during the collapse the pressure cannot remain finite (it vanishes). On the other hand if the fluid is neither isentropic nor the surface temperature remains constant during the collapse, then the results obtained by earlier authors (Bondi, 1969) are found to be inconsistent with the baryonic conservation law and no-heat-flow condition.

2. The metric and uniform density sphere

Vanishing shear implies that one can simultaneously introduce isotropic and co-moving coordinates

$$ds^2 = y^2 dt^2 - R^2(dr^2 + r^2 d\Omega^2) \tag{1}$$

$$y = y(r,t), \;\; R = R(r,t), \;\; d\Omega^2 = d\theta^2 + Sin^2\theta d\phi^2$$

It is assumed that the fluid's viscosity vanishes, and no-heat-flow condition, makes $\Im_{10}$ components of energy-momentum tensor vanish in the co-moving coordinates. The energy-momentum tensor can thus be written as,

$$\Im_{\alpha\beta} = (P + E)U_\alpha U_\beta - P g_{\alpha\beta} \tag{2}$$

where E and P are energy density and pressure, respectively; and the four velocity,

$$U_\alpha \equiv (y, 0, 0, 0) \tag{3}$$

Writing Bondi's results (1969) in the present notations, one gets

$$R = \alpha/(1 - \beta r^2) \tag{4}$$

$$y = FR^\circ/R \tag{5}$$

$$R^\circ/R = (\alpha^\circ/\alpha) + \beta^\circ r^2/(1 - \beta r^2) \tag{6}$$

$$y_b/y = (P + E)/E \tag{7}$$

and

$$P/E = \frac{\beta^\circ(r_b^2 - r^2)}{(1 - \beta r_b^2)[\alpha^\circ + r^2(\alpha\beta^\circ - \beta\alpha^\circ)]} \tag{8}$$

$\alpha = \alpha(t)$, $\beta = \beta(t)$, $F = F(t)$, $y_b = y(r = r_b)$, $r_b \equiv r$ at the boundary.

The hydrodynamic equations, $T^{\alpha\beta}_{;\beta} = 0$ and $U_\alpha T^{\alpha\beta}_{;\beta} = 0$, and the equation of baryon conservation, $(nU^\alpha)_{;\alpha} = 0$, give us [Misner and Sharp, 1964; Demianski, 1985]

$$(y'/y) = -P'/(P + E) \tag{9}$$

$$nR^3 = N(r) \equiv \textit{an arbitrary function of } r \tag{10}$$

and

$$U^{\alpha} s_{,\alpha} = 0 \quad or \quad s^{\circ} = 0 \quad and \quad s' \neq 0 \tag{11}$$

$(')$ ≡ partial differentiation w.r.t. r; $(\circ)$ ≡ partial differentiation w.r.t. t.

3. **Thermodynamic relation :**

The basic law of thermodynamic change is

$$Tds = dU + Pd(1/n) \tag{12}$$

or

$$nTds = dE - hdn$$

U ≡ specific internal energy, s ≡ specific entropy, and $h = (P + E)/n$ ≡ specific enthalpy. The units of n are chosen so that, $P \longrightarrow 0$, $E \longrightarrow n$, and $h \longrightarrow 1$.]

Since $E = E(t)$ or $E' = 0$, we write [using equation (12)]

$$(n'/n^2) = -Ts'/(P + E) \tag{13}$$

$$= -(Ty/Ey_b)s' \quad [using\ equation\ (7)] \tag{14}$$

From no-heat-flow condition, $(Ty)' = 0$, it follows that

$$Ty = T_b\ y_b \tag{15}$$

where $T_b = T(r = r_b)$

Using equation (15) in (14) one gets,

$$n'/n^2 = -(T_b/E)s'$$

or

$$(E/n) = 1 + T_b(s - s_b) \tag{16}$$

$s_b = s(r = r_b)$.

It is obvious from equation (16) that the entropy of an adiabatic uniform density sphere is minimum at the boundary.

4. Collapse of uniform density sphere

The collapse of uniform density sphere is discussed under various physical conditions. [$FNt \equiv$ arbitrary function of time t; $FNr \equiv$ arbitrary function of r].

4 (i) **Isentropic case :** Let the entropy be constant throughout the sphere, that is, $s = \text{constant} = s_b$. Equation (16) gives

$$E = n \tag{17}$$

[using equation (10)] $ER^3 = FNr$ or

$$R = FNr \times FNt \tag{18}$$

or

$$\frac{R^\circ}{R} = FNt$$

[from equation (5)] $y = FNt$ or

$$y' = 0 \tag{19}$$

[from equation (9)] $P' = 0$ or

$$P = P(t) \tag{20}$$

Since $P(r = r_b) = 0 = P(t)$, the pressure vanishes within the sphere.

Hence an isentropic uniform density sphere undergoes a collapse with vanishing pressure only.

4 (ii) **Constant surface temperature :** We assume that the surface temperature remains constant during the collapse. This is very likely because there is no

radiation loss to the surrounding from the surface of the sphere. With T_b = constant during collapse one gets

$$n = E/[1 + T_b(s - s_b)] = FNt \times FNr \tag{21}$$

[from equation (10)]

$$R = FNt \times FNr \tag{22}$$

Arguments similar to those in 4(i) show that the pressure vanishes inside the sphere.

Hence, an adiabatic uniform density sphere with constant surface temperature collapses with vanishing pressure only.

4 (iii) **General case :** Neither the fluid is isentropic nor the temperature of the surface remains constant. In this case

$$E = n[1 + T_b(s_b - s)]$$

On differentiating with respect to time we obtain

$$\frac{E^\circ}{E} = \frac{n^\circ}{n} + \frac{T_b^\circ(s - s_b)}{1 + T_b(s - s_b)} \tag{23}$$

For an adiabatic motion the total mass energy is a constant of motion, that is,

$$M = \frac{4\pi}{3} E R_b^3 = constant$$

or

$$\frac{E^\circ}{E} = -\frac{3R_b^\circ}{R_b} \tag{24}$$

Using equations (10), (23), and (24) we get

$$\frac{T_b^\circ(s - s_b)}{1 + T_b(s - s_b)} = \frac{2\beta^\circ r^2}{1 - \beta r^2} - \frac{3\beta^\circ r_b^2}{1 - \beta r_b^2} = \frac{3\beta^\circ(r^2 - r_b^2)}{(1 - \beta r^2)(1 - \beta r_b^2)} \tag{25}$$

or

$$\frac{T_b^\circ(s - s_b)}{1 + T_b(s - s_b)} = \frac{3\beta^\circ(r^2 - r_b^2)}{1 - \beta(r^2 - r_b^2) + \beta^2 r^2 r_b^2}$$

No choice of functions $s = s(r)$, $\beta = \beta(t)$ and $T_b = T_b(t)$ can satisfy this equation. The solutions obtained by various authors for collapsing/expanding uniform density spheres are inconsistent with the conservation law and no-heat flow condition

5. Conclusion

After studying adiabatic collapse/expansion of a uniform density sphere using baryon conservation law and no-heat-flow condition it is concluded that,

(i) If the fluid is isentropic or (and) the sphere's surface temperature remains constant during the collapse/expansion, then the pressure vanishes inside the sphere.

(ii) If neither the fluid is isentropic nor the surface temperature of the sphere remains constant during the collapse/expansion the solution obtained by Bondi (1969) and other authors are found to be inconsistent with the baryonic conservation law and the no-heat-flow condition.

Thus, a uniform density sphere collapses/expands adiabatically with vanishing pressure. Expansion/collapse with pressure will involve violation of either the baryonic conservation law or the no-heat-flow condition.

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