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MEASUREMENT OF THE TIME-PERIOD OF PULSES EMITTED AND DETECTED BY ACCELERATED OBSERVERS

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We derive expressions for the time period measured by accelerated observers of light pulses emitted at regular proper time intervals by some other accelerated observer. Only when both the emitter and detector belong to the same family of observers having a common event horizon, the intrinsic pulse period can be measured upto a proportionality constant. If the detector is moving at some angle with the emitter, it will never be able to receive more than two pulses, in case all the pulses are emitted in the same direction. In this and all other cases, the measured time-interval has a complicated nonlinear relationship with the intrinsic time-period.

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I. INTRODUCTION

Ever since Einstein put forward the theory of relativity, accelerated observers continued to draw attention of several authors from time to time [1,2]. The Lorentz-invariant theory of relativity is extended to accelerating systems of reference by the hypothesis of locality [3,4] that an accelerated observer at each instant is equivalent to some hypothetical inertial observer moving with the same velocity at the same event. One assumption that is naturally incorporated in this hypothesis is that all the measurements of interest are done instantaneously. In case of a measurement that involves any extension in space and time, the calculation using this hypothesis provides only approximate results.

The measurement of frequency by accelerated observers is an example of such a nonlocal observation. It has, therefore, been shown by various authors [5] that the prediction of the measurement of frequency of a wave in a uniformly accelerated frame of reference, using the hypothesis of locality, yields only approximate results. Although the nonlocal nature of frequency measurements seem to be beyond the access of experiments in near future, it would, however, be important to understand the limitations of such measurements from a conceptual point of view.

In this paper, however, we ask the question how accelerated observers should perform such a measurement (that needs some finite time) of some quantity which is considered by at least one of them to be the 'time-period' of some event occuring at regular intervals of proper time in his or her frame of reference. An answer to such a question will provide us some insight into the problem of comparison among nonlocal observations performed by different accelerated observers.

We consider a uniformly accelerating observer that emits light pulses at regular intervals in its proper time and investigate how other accelerating observers moving in the same or some other direction detect these successive pulses and measure the proper time interval between them. The inverse of the proper time-interval between successive pulses can be defined to be the frequency in case of either emission or detection. Such an intuitive definition of frequency retains the nonlocal property of the frequency measurement process and can be easily extended to the case of uniformly accelerated observers.

In section II, we introduce different characteristics of uniformly accelerated observers and their frames of reference. In sections III and IV, we investigate the problem mentioned above under various situations and discuss different issues encountered in such measurements. We summarize and make some concluding remarks in section V.

II. ACCELERATED OBSERVERS

Let us consider an observer which is accelerating with a constant rate g with respect to its instantaneously comoving inertial frame. Let τ be the proper time along his worldline and I be an inertial reference frame which is comoving with the accelerating observer at time $t = t_0$, $\tau = 0$. The observer is at rest at $x = x_0$ at that instant and accelerating in the positive x-direction.

The hyperbolic worldline of A can then be given by [2]

$$
\begin{aligned}
t &= \frac{\sigma}{c}\sinh(c\tau/\sigma) + t_0, \\
x &= \sigma\cosh(c\tau/\sigma) - \sigma + x_0, \\
y &= y, \\
z &= z,
\end{aligned}
\tag{2.1}
$$

where $\sigma = c^2/g$ and σ/c characterise the motion of the accelerating observers and are called the 'acceleration length' and 'acceleration time' respectively. For example, it was shown [3] that the wavelength and time period of the radiation emitted by an accelerating charged particle following this worldline essentially corresponds to σ and σ/c respectively. The limit $\sigma \to \infty$ corresponds to the inertial observers' worldline, whereas when $\sigma \to 0$, the event should be identified with a singularity. If $t_0 = 0$, the acceleration length σ also represents the distance of the observer A at $t = \tau = 0$ from the point where the asymptote of its hyperbolic path cuts the x-axis.

A set of observers having same set of values for t_0 and $(x_0 - \sigma)$ but different values of σ (and so x_0) constitute a uniformly accelerating reference frame called 'linear' Rindler coordinate system [6]. If $t_0 = 0$, $\sigma = x_0$, this coordinate system covers only one quadrant of the inertial frame I ($\mid t \mid \leq x$, $x > 0$). All of these infinite number of Rindler observers have the same asymptotes for their hyperbolic worldlines, i.e. $x = \pm t$ (for $c = 1$), which pass through the origin at $t = \tau = 0$. So, in this case, the event horizon, the boundary of the causal past of these observers is a 45^0 inclined plane which these observers approach asymptotically. For nonzero values of t_0 and $(x_0 - \sigma)$, the origin of the Rindler frame gets displaced from the origin of the inertial frame I, but still all observers constituting the Rindler frame are characterised by common horizon, $x - (x_0 - \sigma) = \pm c(t - t_0)$.

We also consider a spherical array of uniformly accelerated observers described by a 'spherical' Rindler coordinate system [7] :

$$
\begin{aligned}
t &= \frac{\sigma}{c}\sinh(c\tau/\sigma), \\
x &= \sigma\cosh(c\tau/\sigma)\cos\theta, \\
y &= \sigma\cosh(c\tau/\sigma)\sin\theta\sin\phi, \\
z &= \sigma\cosh(c\tau/\sigma)\cos\theta\cos\phi.
\end{aligned}
\tag{2.2}
$$

In this case the event horizon is a light cone since the acceleration is now along radial directions of a sphere, instead of that along one of the cartesian coordinates.

We define two observers A and B to be 'at rest' with respect to (wrt) each other, if the time as measured by any of these observers (say, A) for a light signal to make a round trip from one (A) to the other (B) and back again always remains constant. One can easily verify that the observers described by Eq.(2.1) but with different values of x_0 and/or t_0 and either same or different values of σ do not, in general, remain 'at rest' wrt each other.

Only observers who are characterised by the same horizon line in frame I remain to be at rest wrt each other. So, Rindler observers all of which have same values of $(x_0 - \sigma)$ and t_0 always remain at rest wrt each other as their worldline evolves in time [8]. However, it can be shown (by using equations similar to those to be described in section III) that two observers who were at rest in inertial frame I at different coordinate times t_0 but possess the same horizon also have this property. A set of such observers, however, do not constitute a coordinate frame. In this paper we call all such observers as 'common horizon' observers. Rindler observers are, of course, a subset of 'common horizon' observers, ones who can together form a coordinate frame.

In case of spherical Rindler coordinate system, observers belonging to the same subspace θ = constant, ϕ = constant obviously have this property, but they will not be 'at rest' wrt any other observer outside such a subspace. In general, accelerated observers moving in different directions never remain at rest wrt each other.

III. TIME-PERIOD MEASUREMENT BY OBSERVERS MOVING IN THE SAME DIRECTION

In this paper we consider a uniformly accelerated observer that emits light pulses at regular intervals in its proper time τ. The intrinsic frequency of pulses emitted by the observer will thus be $\nu_0 = 1/\Delta\tau_0$, where $\Delta\tau_o = \tau_2 - \tau_1$; τ_1, τ_2 being the proper time of emission of two successive pulses. In this section, we investigate how other accelerating observers with different sets of values of σ, x_0, t_0 and moving in the same direction measure the time-period of emission of these pulses.

The inertial observer I measures this time difference in its coordinate frame to be

$$
\Delta T_I = t_2 - t_1 = \frac{\sigma}{c}\left[\sinh\left(\frac{\tau_2 c}{\sigma}\right) - \sinh\left(\frac{\tau_1 c}{\sigma}\right)\right]. \tag{3.1}
$$

For any finite value of $\tau c/\sigma$ we obtain by series expansion,

$$\Delta T_I \approx \Delta\tau_0 + \frac{c^2}{6\sigma^2}(\tau_2^3 - \tau_1^3) + O(\frac{c^4}{\sigma^4}). \tag{3.2}$$

So the hypothesis of locality that an accelerating detector is physically equivalent to a momentarily comoving inertial detector remains to be valid and provides accurate result for the measurement of frequency of these pulses by inertial observers only if $\Delta\tau_o c/\sigma \ll 1$. This constraint has been arrived at by various authors [5] while studying the measurement of the frequency of a plane wave by a uniformly accelerating observer travelling through that.

Now we investigate what would be the time-period of these pulses measured by another accelerating observer moving in the same direction. Let us consider two observers A and B characterised by acceleration lengths σ_a and σ_b and initial ($\tau = 0$) positions x_a and x_b at coordinate times t_a and t_b (in Eq.2.1) respectively. The observer A emits two successive pulses at coordinate times t_1 and t_2 respectively and these are received by B at coordinate times t_1' and t_2' respectively.

As described in section II, A and B, in general, do not remain to be at rest with respect to each other. We can also observe that A is always able to send light to B only if it is moving ahead of B, i.e. $x_a > x_b$. However, we should remember that if it is moving behind B, i.e. $x_a < x_b$, and if their horizons are different, as shown in Fig.1, it cannot send pulses to B once it passes the point P where the asymptote ('horizon') of B crosses its worldline.

We can write the following equations for the propagation of two pulses

$$(x_a - \sigma_a) + [\sigma_a^2 + c^2(t_i - t_a)^2]^{1/2} + c(t_i' - t_i) = (x_b - \sigma_b) + [\sigma_b^2 + c^2(t_i' - t_b)^2]^{1/2}; \quad i = 1, 2. \tag{3.3}$$

Rearranging and substituting for t_i and t_i' using Eq.(2.1),

$$\sigma_b[\cosh(c\tau_i'/\sigma_b) - \sinh(c\tau_i'/\sigma_b)] = \sigma_a[\cosh(c\tau_i/\sigma_a) - \sinh(c\tau_i/\sigma_a)] + K_b, \tag{3.4}$$

where

$$K_b = (x_a - \sigma_a) - (x_b - \sigma_b) + c(t_b - t_a). \tag{3.5}$$

Combining two equations in (3.4), we obtain the expression for the time difference measured by B to be

$$\Delta T_B = \frac{\sigma_b}{c} \ln\left[\frac{\sigma_a \exp(c\tau_2/\sigma_a) + K_b}{\sigma_a \exp(c\tau_1/\sigma_a) + K_b}\right]. \tag{3.6}$$

If $\sigma_a \exp(c\tau_2/\sigma_a) \geq -K_b$, by doing series expansion of the right hand side [9], one can get rid of K_b, however, even then the expression for ΔT_B remains to be complicated. An exact linear relation can be obtained only if $K_b = 0$ which is satisfied by all 'common horizon' observers. It is easy to prove, using equations similar to (3.3) for the onward and backward journey of a particular pulse, that two observers remain to be at rest wrt each other only when $K_b = 0$. Thus, all (Rindler or non-Rindler) 'common horizon' observers can be characterised by this equation, $K_b = 0$.

So if both A and B are 'common horizon' observers, we obtain

$$\Delta T_B = \frac{\sigma_b}{\sigma_a}\Delta\tau_0. \tag{3.7}$$

The relation (3.7) is valid irrespective of whether A is behind or ahead of B. So all observers having the same event horizon as that of A measure the time period upto a constant of proportionality.

IV. TIME-PERIOD MEASUREMENT BY OBSERVERS MOVING AT AN ANGLE WITH THE EMITTER

In this section, we discuss two situations on the basis of the direction of pulse emission. First we discuss in details the case when all pulses are emitted in the same direction. The calculation presented in this context also reveals how complicated would be the second situation when the pulses are emitted as light flashes that propagate in all directions along a spacetime cone.

We, therefore, consider some other observer D that is moving in the r direction at an angle θ with the observer A as shown in Fig.2. The observer D is characterised by an acceleration length σ_d and its initial position in frame I at $t = \tau = 0$ is r_d. The observer A emits successive pulses while at positions $X_1, X_2, \ldots$etc. at coordinate times

$t_1, t_2, \ldots$etc. respectively at an angle $\beta = \beta_1 = \beta_2 = \cdots$ with the negative x axis. These reach D while at $R_1, R_2, \ldots$ at coordinate times $t'_1, t'_2, \ldots$ respectively.

Fig.3 shows the coordinate times of arrival of the first three successive pulses in $t-r$ plane of the spacetime diagram. In all of the vertical lines $\overline{R_jP_j}$, $j = 1, 2, \ldots$ represents the coordinate time of the observer A at his or her proper time of emission of the j-th pulse, τ_j, and $\overline{P_jQ_j}$ represents the propagation time of the same pulse. So, the coordinate time of detection of the j-th pulse by observer D is

$$\overline{R_jQ_j} = \frac{\sigma_a}{c}\sinh\left(\frac{c\tau_j}{\sigma_a}\right) + t_a + \frac{\kappa}{c}\left[\sigma_a\cosh\left(\frac{c\tau_j}{\sigma_a}\right) + x_a - \sigma_a\right], \quad j = 1, 2, \ldots \tag{4.1}$$

where $\kappa = \sin\theta/\sin\alpha$.

We show below that it would never be possible for an accelerated observer moving in r direction to detect more than two pulses. To detect two successive pulses, j-th and $(j+1)$-th, the constants, σ_d, t_d and r_d of the observer D should satisfy the following equation

$$(\overline{OR_j} + \sigma_d - r_d)^2 - c^2(\overline{R_jQ_j} - t_d)^2 = (\overline{OR_{j+1}} + \sigma_d - r_d)^2 - c^2(\overline{R_{j+1}Q_{j+1}} - t_d)^2. \tag{4.2}$$

Obviously, for different values of j, the set of the values of the three quantities, σ_d, t_d and r_d should be different to satisfy the above equation, thus proving that not more than two pulses will ever be detected by any accelerated observer moving in r direction.

Now to find what would be the difference in the time of detection of two successive pulses by observer D, using the simple geometry of Fig.2, we can write equations for the propagation of pulses as

$$[[\sigma_a^2 + c^2(t_i - t_a)^2]^{1/2} - \sigma_a + x_a]\cos\theta + c(t'_i - t_1)\cos\alpha_i = [\sigma_d^2 + c^2(t'_i - t_r)^2]^{1/2} - \sigma_d + r_d; \quad i = 1, 2 \tag{4.3}$$

where $\alpha_i = \alpha = \pi - (\beta + \theta)$. The two equations above, after getting rearranged, become

$$\sigma_d\left[\cosh\left(\frac{c\tau'_i}{\sigma_d}\right) - \cos\alpha\sinh\left(\frac{c\tau'_i}{\sigma_d}\right)\right] = \sigma_a\left[\cos\theta\cosh\left(\frac{c\tau_i}{\sigma_a}\right) - \cos\alpha\sinh\left(\frac{\tau_i}{\sigma_a}\right)\right] + K_d; \quad i = 1, 2 \tag{4.4}$$

where

$$K_d = (x_a - \sigma_a)\cos\theta - (r_d - \sigma_d) + c(t_d - t_a)\cos\alpha. \tag{4.5}$$

To solve these equations we need to use the following two identities

$$X\sinh\delta + Y\cosh\delta = \begin{cases} \sqrt{Y^2 - X^2}\cosh\left(\delta + \frac{1}{2}\ln\frac{Y+X}{Y-X}\right), & \text{when } Y > |X| \\ \sqrt{X^2 - Y^2}\sinh\left(\delta + \frac{1}{2}\ln\frac{X+Y}{X-Y}\right), & \text{when } Y < |X| \end{cases} \tag{4.6}$$

So, to apply these identities to the right hand side of Eq.(4.4), we have to consider two cases separately : (1) when $\cos\theta > |\cos\alpha|$ and (2) when $\cos\theta < |\cos\alpha|$. We also investigate case (3) when $\cos\theta = \cos\alpha$.

For case (1) Eq.(4.4) turns out to be

$$\sigma_d\sin\alpha\cosh\left[\frac{c\tau'_i}{\sigma_d} + \frac{1}{2}\ln\frac{1-\cos\alpha}{1+\cos\alpha}\right] = \sigma_a\sqrt{\cos^2\theta - \cos^2\alpha}\cosh\left[\frac{c\tau_i}{\sigma_a} + \frac{1}{2}\ln\frac{\cos\theta - \cos\alpha}{\cos\theta + \cos\alpha}\right] + K_d; \quad i = 1, 2. \tag{4.7}$$

Subtracting $i = 2$ equation from that for $i = 1$ and after some algebraic manipulation, we obtain

$$\Delta T_{D1} = \frac{2\sigma_d}{c}\sinh^{-1}\left[S_1\sinh\left(\frac{c\Delta\tau_0}{2\sigma_d}\right)\right], \tag{4.8}$$

where

$$S_1 = \frac{\sigma_a\sqrt{\cos^2\theta - \cos^2\alpha}}{\sigma_d\sin\alpha}\frac{\sinh\left(\frac{\tau_1+\tau_2}{2\sigma_a} + \frac{1}{2}\ln\frac{\cos\theta-\cos\alpha}{\cos\theta+\cos\alpha}\right)}{\sinh\left(\frac{\tau'_1+\tau'_2}{2\sigma_d} + \frac{1}{2}\ln\frac{1-\cos\alpha}{1+\cos\alpha}\right)}. \tag{4.9}$$

Similarly, for case (2), we obtain

$$\Delta T_{D2} = \frac{2\sigma_d}{c} \sinh^{-1}\left[S_2 \sinh\left(\frac{c\Delta\tau_0}{2\sigma_d}\right)\right], \tag{4.10}$$

where

$$S_2 = \frac{\sigma_a\sqrt{\cos^2\alpha - \cos^2\theta}}{\sigma_d \sin\alpha} \frac{\cosh\left(\frac{\tau_1+\tau_2}{2\sigma_a} + \frac{1}{2}\ln\frac{\cos\alpha-\cos\theta}{\cos\alpha+\cos\theta}\right)}{\sinh\left(\frac{\tau_1'+\tau_2'}{2\sigma_d} + \frac{1}{2}\ln\frac{1-\cos\alpha}{1+\cos\alpha}\right)}. \tag{4.11}$$

Expressions (4.8) and (4.10) are not valid when either $\alpha = 0$, π or when $\theta = \alpha$. The former case corresponds to Eqs.(3.6) and (3.7), whereas the latter is what we call case (3). In this case we obtain

$$\Delta T_{D3} = \frac{2\sigma_d}{c} \sinh^{-1}\left[S_3\left(\exp\left(\frac{c\tau_2}{2\sigma_d}\right) - \exp\left(\frac{c\tau_1}{2\sigma_d}\right)\right)\right], \tag{4.12}$$

where

$$S_3 = \frac{\sigma_a}{2\sigma_d \sin\alpha} \frac{1}{\sinh\left(\frac{\tau_1'+\tau_2'}{2\sigma_d} + \frac{1}{2}\ln\frac{1-\cos\alpha}{1+\cos\alpha}\right)}. \tag{4.13}$$

We can easily see that the Eqs.(4.8),(4.10) and (4.12) are also valid for a set of observers belonging to a $2+1$ dimensional spherical Rindler coordinate system. However, in this case, the existence of an expanding conical event horizon may, in some situations, prevent communication between two observers moving in different directions. An extension of these results to $3+1$ dimensional spherical Rindler system (Eq.2.2) is not being attempted here since we wish to concentrate only on the effect of acceleration on the measurement process, thus keeping away other unnecessary complications.

In the second situation when pulses are emitted in the form of light flashes which propagate in all directions, the observer D would obviously be able to detect more than two pulses. But in this case, all α_js (and so all β_js) would have values different from each other. From Eq.(4.3) one can guess that in this case it would be very difficult to calculate the measured time intervals which have a complicated nonlinear relationship with the intrinsic time-period. The measured interval is also different for different times of emission.

V. CONCLUDING REMARKS

We observe that the measurement (as described in previous sections) of time-period, or frequency, has a strong relationship with the fact whether or not the emitter and detector are 'at rest' with respect to each other. If these two uniformly accelerated observers remain 'at rest' with respect to each other, or equivalently, if they are members of a set of observers having a common event horizon, the time-period can be measured upto a constant of proportionality. If they do not belong to the same set of 'common horizon' observers, in some cases, it may not even be possible to detect more than one or two number of pulses. Even if it is possible to detect successive pulses, the measured time-period would have a complicated nonlinear relationship with the intrinsic time-period of emission.

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[1] A. Einstein and N. Rosen, Phys. Rev., **48**, 73 (1935); P.G. Bergmann, Phys. Rev. Lett., **12**, 139 (1964).

[2] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973) ch.6; C. Moller, *The Theory of Relativity* (Clarendon, Oxford, 1972) 2nd ed., p 253; W. Rindler, *Essential Relativity* (Springer, NY, 1977) 2nd ed., p 49,156; F.W. Sears and R.W. Brehme, *Introduction to The Theory of Relativity* (Addison Wesley, MA, 1968) ch.5, etc...

[3] B. Mashhoon, Phys. Lett., **A122**, 67 (1987); *ibid*, **A126**, 393 (1988).

[4] B. Mashhoon, Phys. Lett., **A143**, 176 (1990); Phys. Lett., **A145**, 147 (1990).

[5] B. Mashhoon, Phys. Lett., **A122**, 67 (1987); *ibid* Phys. Lett., **A139**, 103 (1989); W. Moreau, Am. J. Phys., **60**, 561 (1992); R. Neutze and W. Moreau, Phys. Lett., **A179**, 389 (1993).

[6] W. Rindler, Am. J. Phys., **34**, 1174 (1976).

[7] U.H. Gerlach, Phys. Rev., **D28**, 761 (1983).

[8] *see, for example*, E.A. Deslodge and R.J. Philpott, Am. J. Phys., **55**, 252 (1987).

[9] *Handbook of Mathematical Functions*, eds. M. Abramowitz and I.A. Stegun (Dover Publications, NY, 1972) Eq. 4.1.29.

FIG. 1. The accelerated observer A can not send a pulse to B once it passes the junction point P of its worldline with the horizon of B (shown as the dashed line that is asymptote to the worldline of B).

FIG. 2. The points X_j ($j = 1, 2, \ldots$) represent the spatial points of emission of pulses on the x-axis, whereas R_j represent the corresponding spatial points where the paths of the pulse propagation cross the r-direction.

FIG. 3. The points Q_j ($j = 1, 2, \ldots$) represent the spacetime points of arrival of pulses in the $t - r$ plane.

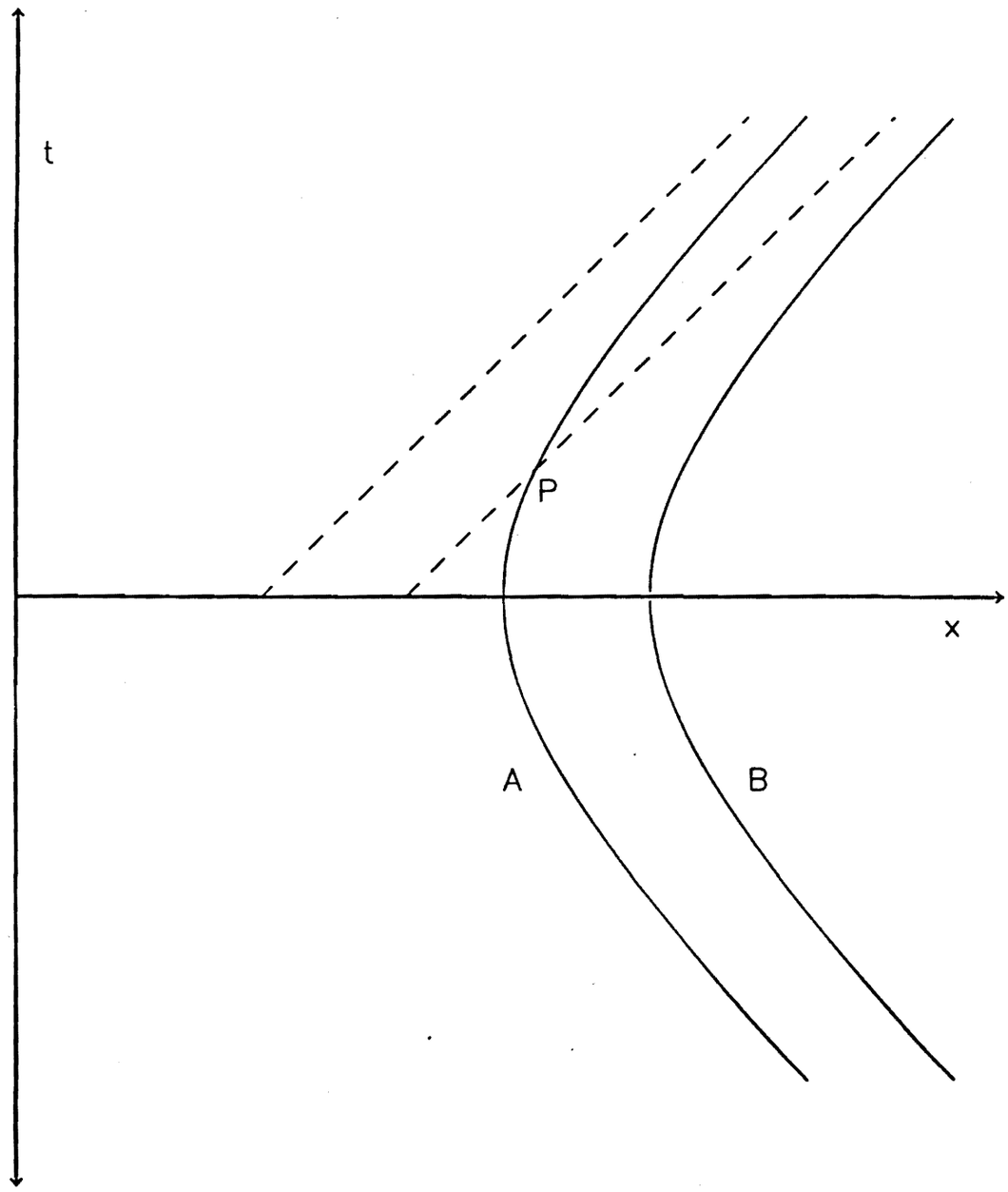

FIG. 1

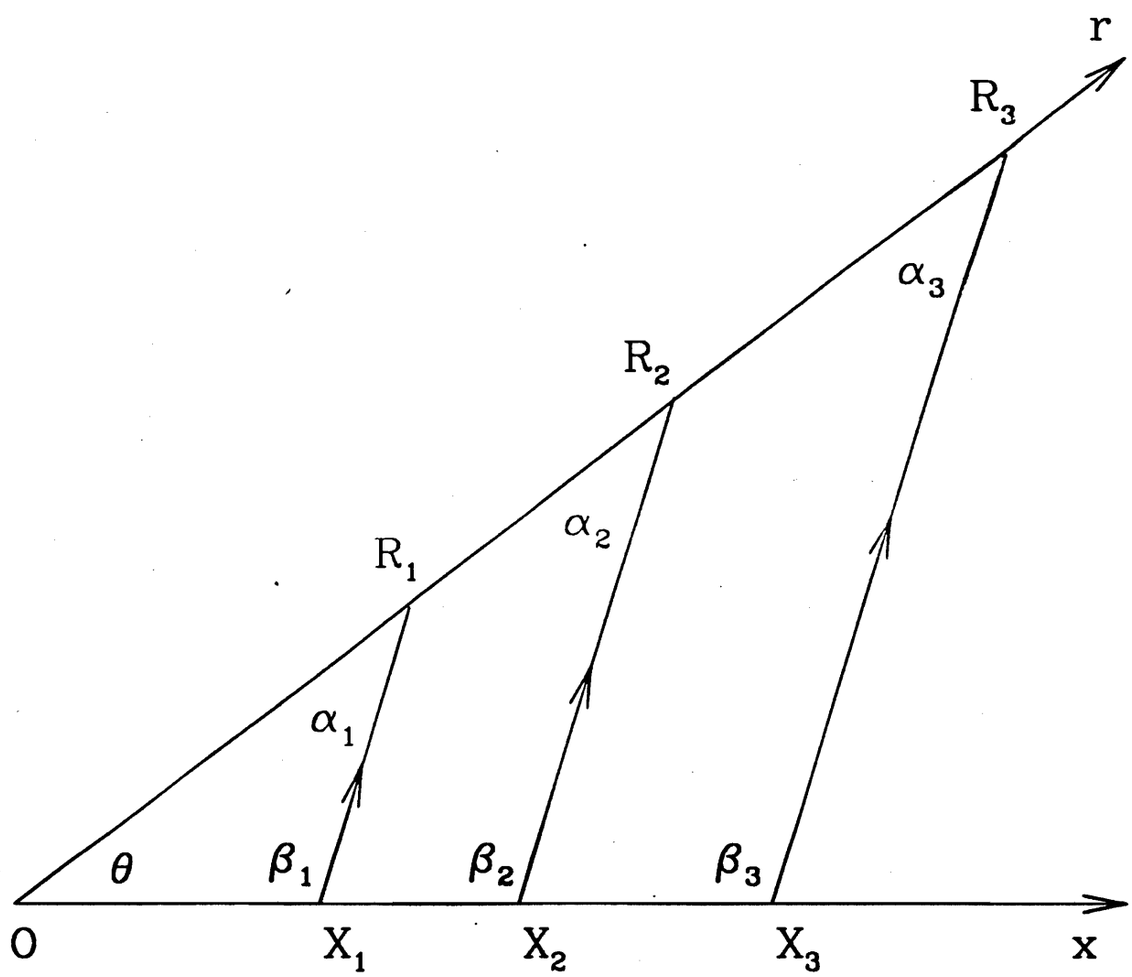

FIG. 2

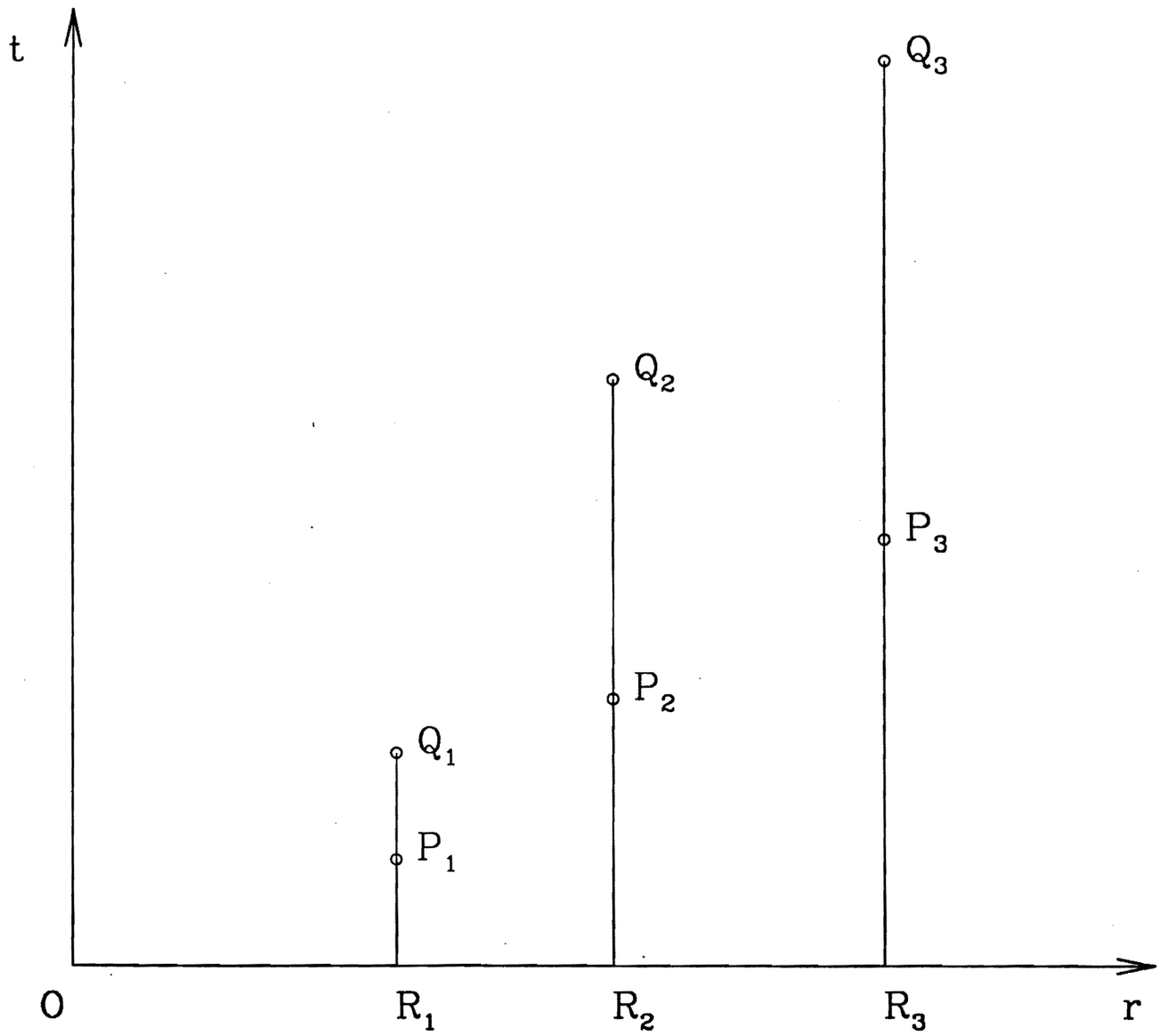

FIG. 3.