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# Connection Variables, the Gravitational Action and Boosted Quasilocal Stress-Energy-Momentum

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# Connection variables, the gravitational action, and boosted quasilocal stress-energy-momentum

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## Abstract

The paper presents a complete set of quasilocal densities which describe the stress-energy-momentum content of the gravitational field and which are built with the canonical Ashtekar variables. The densities are defined on a two-surface $B$ which bounds a generic spacelike hypersurface $\Sigma$ of spacetime. The method used to derive the set of quasilocal densities is a Hamilton-Jacobi analysis of a suitable covariant action principle for the Ashtekar variables. This work also investigates how the quasilocal densities behave under generalized boosts, *i.e.* switches of the $\Sigma$ slice spanning $B$. It is shown that under such boosts the densities behave in a manner which is in full accord with the equivalence principle. The developed formalism is used to discuss the canonical action principle for bounded spacetime regions with "sharp corners".

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## I. INTRODUCTION

The geometric expression for the energy of a nonrelativistic system (the functional form of the Hamiltonian in terms of the coordinates and momenta) can be discerned from the system's action functional. This follows from a basic tenet of Hamilton-Jacobi theory: the classical energy of the system is minus the rate of change of the classical action (the Hamilton-Jacobi principal function) with respect to a unit stretch in the absolute Newtonian time. The ability to define the classical energy in this way rests on the fact that in the conventional variational principle for the system the lapse of absolute time is fixed as boundary data. From a practical standpoint, this means that in order to find the geometric expression for the system's Hamiltonian, one need only consider the general variation of the action (often called the Weiss action principle [1]) in which the endpoints of trajectories in the variational set are not held fixed. Upon inspection of the boundary-term contributions to the variation, one can determine the canonical momenta as the factors which multiply the variations in the endpoint values of the coordinates. Furthermore, after the momenta are determined, careful inspection of the boundary-term factor with multiplies the variation in the absolute time then reveals the functional form of the Hamiltonian.

Recently, Brown and York have proposed a generalization of the Hamilton-Jacobi method, which is applicable to a wide class of generally covariant field theories of a spacetime metric (in any dimension); and they have used this generalized method to discern what geometric expressions play the role of quasilocal stress, energy, and momentum in general relativity. Field theories of a spacetime metric enjoy a crucial feature in common with simple nonrelativistic systems: in the action principle it is possible to fix the time as boundary data. To see that this is indeed the case, consider a spacetime region $\mathcal{M}$ which is topologically the Cartesian product of Riemannian three-manifold $\Sigma$ and a closed connected segment of the real line $I$. The three-manifold $\Sigma$ has a boundary $\partial\Sigma = B$ which need not be simply connected. Therefore, one element of the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ is a three-dimensional timelike hypersurface $\bar{\mathcal{T}}$ ("unbarred" $\mathcal{T}$ is reserved for a more special meaning) which has the topology of $I \times B$ and is a $(\dot{2}+1)$-dimensional spacetime in its own right. The other boundary elements are $t'$, the three-manifold corresponding to the initial point of $I$, and $t''$, the three-manifold corresponding to the final point of $I$.[1] Now suppose that we are given a "suitable" action functional for the metric (and possibly matter) fields on the spacetime region $\mathcal{M}$. By "suitable" we mean that the variational principle associated with the action features fixation of the induced metric on each of the boundary elements $t'$, $t''$, and $\bar{\mathcal{T}}$. In particular, the lapse of proper time between the initial and final hypersurfaces is fixed as boundary data since this information is encoded in the fixed $\bar{\mathcal{T}}$ three-metric. The quasilocal energy is then identified as minus the rate of change of the classical action with respect to a unit stretch in the proper time separation between $t'$ and $t''$. (Therefore, inspection of the boundary-term contributions to the variation of the action can reveal the geometric

[1]One may imagine that $\mathcal{M} \subset \mathcal{U}$, where $\mathcal{U}$ is some ambient spacetime known as the *universe* or sometimes the *heat bath*. The boundary $B$ and its history $\bar{\mathcal{T}}$ are simply collections of points in $\mathcal{U}$ and need not be physical barriers.

expression for the quasilocal energy. This geometric expression is obtained by isolating the factor which multiplies the variation in the lapse function which controls the proper time separation between $B$ slices of $\mathcal{T}$.) However, notice that the $\mathcal{T}$ three-metric provides more than just lapse of proper time between the initial and final slices, since it contains information about all possible spacetime intervals on $\mathcal{T}$. One is free to consider the rate of change in the classical action which corresponds to arbitrary variations in the $\mathcal{T}$ boundary data. A quasilocal surface stress-energy-momentum tensor corresponds to this freedom. For the most relevant case of general relativity, the analysis of Ref. [2] has demonstrated how this tensor leads to quasilocal surface densities for *energy*, *tangential momentum*, and *spatial stress* (all are pointwise tensors defined on $B$) which describe the stress-energy-momentum content of the $\Sigma$ matter and gravitational fields contained within $B$.

This paper uses such a Hamilton-Jacobi type method to derive quasilocal stress-energy-momentum surface densities which are built with the Ashtekar gravitational variables. Since the the Ashtekar version of general relativity is inherently a non-metric formalism, the Hamilton-Jacobi analysis given by Brown and York has to be slightly modified. Nevertheless, the cornerstone of the method used here remains a "suitable" action principle, *i.e.* information about the lapse of proper time must be fixed as boundary data. Now, the usual covariant formulation of the Ashtekar variables is based on the well-known chiral action independently given by Samuel [3] and Jacobson and Smolin [4]. This is a Palatini action which features the independent variation of the spacetime self-dual spin connection and the $SL(2,C)$ soldering form. Applied to our spacetime region $\mathcal{M}$, this action principle does not feature fixation of metric data on $\mathcal{T}$, and hence it is not well-suited for our purposes. Perhaps, one could consider adding the necessary boundary terms to the chiral action in order to obtain a suitable variational principle. However, here we follow another route which is based on a lesser-known covariant formulation of the Ashtekar theory which has been given by Goldberg. [5] Goldberg's action functional is first-order, but in the variational principle the connection is not varied independently from the tetrad. We find that, subject to certain gauge fixation of the tetrad, Goldberg's action is a tetrad version of the action functional used to derive quasilocal stress-energy-momentum in the metric scenario. It should be mentioned now that partial gauge fixation of the tetrad and triad plays a crucial role in what follows. At first sight this may seem objectionable. But one should recall that such gauge fixation is also unavoidable in the triad formulation of Hamiltonian gravity, when one discusses the notions of total energy and momentum in the asymptotically-flat scenario. In that case one must deal with a "fiducial triad at infinity". [5,6] The gauge fixation of the triad in the quasilocal context is of the same nature though less restrictive.

There is a subtle interpretational issue concerning the analysis to follow which deserves so comment at the outset. The Brown-York quasilocal densities are not unique, since one has the freedom to add a *subtraction term* (a functional of the fixed boundary data) to the gravitational action which is used to derive the densities. Brown and York have offered the interpretation that such freedom allows one to set the reference points for the quasilocal densities.[2] Now, the results of gravitational thermodynamics are, in fact, independent of

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[2]This is quite analogous to the situation in nonrelativistic mechanics, where one can affect the

the choice of subtraction term, and, therefore, such freedom seems to be an unnecessary one when examining the statistical mechanics of the strong gravitational field. [7,8] However, the subtraction term plays an important role in several other theoretical contexts. For instance, it must be incorporated into the definition of the quasilocal energy, if in the suitable limit the definition is to agree with the Arnowitt-Deser-Misner (ADM) notion of energy at spacelike infinity. [2] Furthermore, recent research has shown that there is an implicit reference point set in spinor constructions of quasilocal energy based on the Witten-Nester integral. [9] In this paper the passage from the triad ADM variables to the Ashtekar variables is effected by the addition of a purely imaginary boundary term to the action. We formally treat this boundary term as a subtraction term *ala* Brown and York. This allows us to construct the theory in a parallel fashion with the presentations given in Refs. [2,10]. However, though technically this viewpoint is completely satisfactory, it should be realized that it is less satisfactory from an interpretational standpoint. Indeed, if we wish to adopt the Brown-York interpretation for the imaginary subtraction term, then we are confronted with the issue of imaginary reference points for the quasilocal densities. Furthermore, even with the imaginary subtraction term, in the suitable limit the Ashtekar-variable expression for the quasilocal energy as given here does not agree with the ADM notion of energy at spacelike infinity. This seems alarming, but in fact is not a real problem. It merely signifies that it is perhaps better to view the imaginary subtraction term not as true subtraction term, but rather as part of a bigger base action. To derive an expression for the quasilocal energy in terms of the Ashtekar variables which is in agreement with the ADM expression, we would need to begin with an action which differs from this bigger base action by *yet another* subtraction term. In other words, our analysis is actually performed only on a base action (even though we split this base action into two pieces and treat one piece formally as a Brown-York subtraction term), and it should be understood that in some contexts it may be necessary to consider the addition of appropriate subtraction terms to this base action. We discuss these issues in more detail in the concluding section, but remark now that we expect the results of this work to be complete for any potential application of the Ashtekar variables to gravitational thermodynamics.

The organization of this paper is as follows. In §1, the preliminary section, we discuss in detail the geometry of $\mathcal{M}$ in terms of several classes of spacetime foliations. This discussion is the groundwork for the analysis in the main sections. We also collect some notations and conventions in this section. In §2 we derive a complete set of quasilocal densities which are expressed in terms of the Sen connection and triad on $\Sigma$, and thus may be easily rewritten later on in terms of the canonical Ashtekar variables. The geometric forms of these densities are discerned from a careful analysis of the boundary terms which appear in the Goldberg action principle. This analysis is quite analogous to the method described for nonrelativistic systems in the introductory paragraph. In §3 we turn to the issue of how the collection of quasilocal densities behave under generalized boosts. This behavior is similar to the simple boost law for energy-momentum four vectors in special relativity. Also in §3, we consider

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definition of a system's energy and canonical momenta by adding boundary terms to a system's action.

the canonical form of the action principle for spacetime regions with "sharp corners". This analysis supplements recent results from standard metric gravity for such spacetimes. [10,12] The appendices provide some kinematical results necessary for the central discussions. The first three appendices develop the results necessary to write down the boost relations for the quasilocal densities. A forth and final appendix presents a method for dealing with "corner" terms in gravitational actions (such terms are described below).

## II. PRELIMINARIES

### A. Foliations

The boundary structure of $\mathcal{M}$ leads to two classes of spacetime foliations. Our discussion of these foliations is close to one given by Hayward and Wong [12].

*Temporal foliations of* $\mathcal{M}$. The first type of break-up stems from a conventional ADM foliation of $\mathcal{M}$ into a family spacelike hypersurfaces. [13] A foliation of this class, referred to as a *temporal foliation*, is specified by a time function $t : \mathcal{M} \to I$. The leaves of the foliation or *slices* are the level hypersurfaces of this time coordinate $x^0 \equiv t$. Often, the possible time functions are restricted by the requirement that both $t'$ and $t''$ must be level hypersurfaces of coordinate time. The letter $\Sigma$ is used both to denote a foliation of $\mathcal{M}$ and to refer to a generic slice of this foliation, and the $\Sigma$ slice specified by $t = t_*$ ($t_*$ is some constant) is denoted $\Sigma_{t_*}$. If the manifolds $t'$ and $t''$ are level hypersurfaces of coordinate time, then it is convenient to set $t' = \Sigma_{t'}$ and $t'' = \Sigma_{t''}$. The timelike, future-pointing, unit, hypersurface normal of a $\Sigma$ foliation is denoted by $u$.

*Radial foliations of* $\mathcal{M}$. The existence of the timelike boundary $\bar{\mathcal{T}}$ suggests an alternative class of foliations of $\mathcal{M}$. Members of this alternative class are called *radial foliations* and rely on *timelike* hypersurfaces or *sheets* which have the topology of $\bar{\mathcal{T}}$ (informally, sheets are radial leaves while slices are temporal leaves). One assumes that a radial coordinate $x^3 \equiv r$ parameterizes a nested family of such hypersurfaces which extend inward from $\bar{\mathcal{T}}$. This family of timelike sheets may converge on some degenerate sheet, and if this is the case, then there is a coordinate singularity at the degenerate sheet.[3] With a notation similar to the one introduced above, we may represent $\bar{\mathcal{T}}$ by $\bar{\mathcal{T}}_{r''}$, so the level hypersurface specified by $r = r''$ is $\bar{\mathcal{T}}$ (the inner radial sheet is $\bar{\mathcal{T}}_{r'}$). The spacelike, outward-pointing, unit, $\bar{\mathcal{T}}$ hypersurface normal is denoted by $\bar{n}$ (the unprimed letter $n$ is reserved for a related but different vector field introduced below).

*Foliations of* $\Sigma$ *and* $\bar{\mathcal{T}}$. It is of interest to examine how the $\Sigma$ and $\bar{\mathcal{T}}$ spacetime foliations mesh. If a temporal and a radial foliation of $\mathcal{M}$ are simultaneously given, then the intersection $B_{t,r} \equiv \Sigma_t \bigcap \bar{\mathcal{T}}_r$ is a two surface with the topology of $B$. Defining $B_t \equiv B_{t,r''}$, one finds that the family of $B_t$ slices foliates $\bar{\mathcal{T}}$. This foliation of $\bar{\mathcal{T}}$ and its generic leaf are both loosely referred to as $B$. The timelike, future-pointing, unit, hypersurface normal of this

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[3]It should be emphasized that only a "local" radial foliation of an arbitrarily small spacetime region surrounding $\bar{\mathcal{T}}$ is necessary for the analysis in this work. The full radial foliation of $M$ is introduced only to have a closer analogy with temporal foliations.

foliation is $\bar{u}$. In general, the vector fields $u$ and $\bar{u}$ do not coincide on $\bar{\mathcal{T}}$. Fixation of the time gives a family of sheets $B_r \equiv B_{t_*,r}$ which radially foliate the $\Sigma$ hypersurface specified by $t = t_*$. This foliation of $\Sigma$ and its generic leaf are also represented by $B$. The spacelike, outward-pointing, unit normal of this foliation is denoted by $n$, and in general $n$ and $\bar{n}$ do not coincide on $\Sigma$.

*Clamped foliations.* Often in this paper we need to consider a particular subclass of $\Sigma$ foliations, determined by the property that on $\bar{\mathcal{T}}$ the timelike normal $u$ is orthogonal to $\bar{n}$. Such foliations are denoted by $\bar{\Sigma}$ with corresponding normal $\bar{u}$. (So we have $\bar{u} \cdot \bar{n} = 0$ on $\bar{\mathcal{T}}$, where $\bar{u}$ is also the normal for the $B$ foliation of $\bar{\mathcal{T}}$.) We described a $\bar{\Sigma}$ foliation as *clamped.* Note that it may not be possible for a temporal foliation to be clamped over all of $\bar{\mathcal{T}}$, since the $u$ normals of $t'$ and $t''$ may not be orthogonal to $\bar{n}$ (assuming that $t'$ and $t''$ should be members of the family of $\bar{\Sigma}$ slices). We can also consider the locus of points which is the Eulerian history of $B$ with respect to an (in general) unclamped $\Sigma$ foliation. This "boundary", denoted by $\mathcal{T}$ is generated by the integral curves of $u$ and may "crash into" or "emerge from" the actual boundary $\bar{\mathcal{T}}$. Note that by construction the $\Sigma$ foliation is clamped to $\mathcal{T}$, since the outward-pointing spacelike normal of $\mathcal{T}$ is $n$.

We maintain this barred and unbarred notation when it is necessary to deal simultaneously with clamped and unclamped $\Sigma$ foliations. However, in §3, which presents the derivation of the quasilocal densities, we make the *clamping assumption* which means that only clamped $\bar{\Sigma}$ foliations of spacetime $\mathcal{M}$ are considered (or every $\Sigma$ foliation is a $\bar{\Sigma}$ foliation). When the clamping assumption is made, over-bars become redundant, and therefore *in* §3 *we drop all bars from the formalism.* In this section we assume that the $u$ normals of $t'$ and $t''$ are orthogonal to the $\bar{\mathcal{T}}$ normal $\bar{n}$ (in this section denoted simply by $\mathcal{T}$ and $n$). Though this is a limiting assumption, it in no way affects the generality of this paper, as we return to the fully general scenario in the following section. We demonstrate that the clamping assumption is a purely kinematical condition.

### B. Conventions and notation

We adopt the following index notation. Lowercase Greek letters serve as $\mathcal{M}$ spacetime indices. Lowercase Latin indices from the *latter half of the alphabet* serve as $\Sigma$ (and $\bar{\Sigma}$) indices and as $\bar{\mathcal{T}}$ (and $\mathcal{T}$) indices. There is -hopefully- no confusion caused by this dual use of Latin indices. Lowercase Latin letters from the *first half of the alphabet* serve as $B$ indices. Orthonormal (or when appropriate pseudo-orthonormal) labels and indices for each space are represented by the same letters with hats. For example, $\hat{\mu}$ is a spacetime tetrad index and $\hat{a}$ is a $B$ dyad index.

The spacetime metric is $g_{\mu\nu}$ with associated (metric-compatible and torsion-free) covariant derivative operator $\nabla_\mu$, and $e_{\hat{\mu}}{}^{\sigma}$ denotes a spacetime tetrad. The (pseudo)orthonormal symbol on spacetime is defined by $\epsilon_{\hat{0}\hat{1}\hat{2}\hat{3}} = 1 = -\epsilon^{\hat{0}\hat{1}\hat{2}\hat{3}}$. Respectively, we have $\bar{\gamma}_{ij}$ and $\bar{\mathcal{D}}_j$ ($\gamma_{ij}$ and $\mathcal{D}_j$), $h_{ij}$ and $D_j$ ($\bar{h}_{ij}$ and $\bar{D}_j$), and $\sigma_{ab}$ and $\delta_a$ denoting the metric and intrinsic covariant derivative operators on $\bar{\mathcal{T}}$ ($\mathcal{T}$), $\Sigma$ ($\bar{\Sigma}$), and $B$. We use $\bar{\xi}_{\hat{r}}{}^{j}$ ($\xi_{\hat{r}}{}^{j}$), $E_{\hat{r}}{}^{j}$ ($\bar{E}_{\hat{r}}{}^{j}$), and $\theta_{\hat{a}}{}^{b}$, respectively, to represent a triad on $\bar{\mathcal{T}}$ ($\mathcal{T}$), a triad on $\Sigma$ ($\bar{\Sigma}$), and a dyad on $B$. Respectively, the permutation symbols on $\bar{\mathcal{T}}$ ($\mathcal{T}$), $\Sigma$ ($\bar{\Sigma}$), and $B$ are defined by $\bar{\epsilon}_{\hat{0}\hat{1}\hat{2}} = -1 = \bar{\epsilon}^{\hat{0}\hat{1}\hat{2}}$ ($\epsilon_{\hat{0}\hat{1}\hat{2}} = -1 = \epsilon^{\hat{0}\hat{1}\hat{2}}$), $\epsilon_{\hat{1}\hat{2}\hat{3}} = 1 = \epsilon^{\hat{1}\hat{2}\hat{3}}$ ($\bar{\epsilon}_{\hat{1}\hat{2}\hat{3}} = 1 = \bar{\epsilon}^{\hat{1}\hat{2}\hat{3}}$), and $\epsilon_{\hat{1}\hat{2}} = 1 = \epsilon^{\hat{1}\hat{2}}$. (In Ref. [14] the

convention for the $\mathcal{T}$ orthonormal symbol differs by a sign.)

### C. Spacetime decompositions

The foliations just discussed lead to two decompositions of the spacetime metric. We choose to examine the metric in a frame which has $\partial/\partial t$ and $\partial/\partial r$ as two of the frame legs. To begin with, the temporal foliation $\Sigma$ allows the matrix of metric components to be written as

$$\|g_{\mu\nu}\| = \begin{pmatrix} -N^2 + h_{ij}\,V^i\,V^j & h_{ij}\,V^j \\ h_{ij}\,V^j & h_{ij} \end{pmatrix}, \tag{2.1}$$

where the $\Sigma$ indices run over $(1,2,\mathrm{r})$. The $N$ and $V^j$ are the ordinary ADM lapse and shift. Further, since each of the $\Sigma$ slices is foliated independently by nested sheets with the topology of $B$, the matrix form of the $\Sigma$ three-metric is given by

$$\|h_{ij}\| = \begin{pmatrix} \sigma_{ab} & \sigma_{ab}\,\beta^b \\ \sigma_{ab}\,\beta^b & \alpha^2 + \sigma_{ab}\,\beta^a\,\beta^b \end{pmatrix}. \tag{2.2}$$

Here, $\alpha$ and $\beta^a$ are the "lapse" and "shift" associated with the induced radial foliation of $\Sigma$. The super matrix formed by combining these expressions gives the so-called $(1+2)+1$ form of the metric. The $(1+2)$ indicates that three-space has been split into a radial direction plus a two-space, while the 1 indicates the time direction.

Similarly, beginning with the full radial foliation $\bar{\mathcal{T}}$ of spacetime one has,

$$\|g_{\mu\nu}\| = \begin{pmatrix} \bar{\gamma}_{ij} & \bar{\gamma}_{ij}\,\bar{\beta}^j \\ \bar{\gamma}_{ij}\,\bar{\beta}^j & \bar{\alpha}^2 + \bar{\gamma}_{ij}\,\bar{\beta}^i\,\bar{\beta}^j \end{pmatrix}, \tag{2.3}$$

where the $\bar{\mathcal{T}}$ indices and range over $(\mathrm{t},1,2)$. The $\bar{\alpha}$ and $\bar{\beta}^\alpha$ are the gauge variables associated with this foliation. The submatrix associated with $\bar{\gamma}_{ij}$ is

$$\|\bar{\gamma}_{ij}\| = \begin{pmatrix} -\bar{N}^2 + \sigma_{ab}\,\bar{V}^a\,\bar{V}^b & \sigma_{ab}\,\bar{V}^b \\ \sigma_{ab}\,\bar{V}^b & \sigma_{ab} \end{pmatrix}, \tag{2.4}$$

where $\bar{N}$ and $\bar{V}^a$ are the lapse and shift associated with the induced $B$ foliation of $\bar{\mathcal{T}}$. The super matrix of components for this splitting is the metric in $1+(2+1)$ form.

It is a straightforward exercise to express the "barred" variables in terms of the "unbarred" variables by simply equating the components of the $(1+2)+1$ and $1+(2+1)$ versions of the spacetime metric. First, define

$$v \equiv \frac{V\cdot n}{N} = \frac{\alpha V^{\mathrm{r}}}{N}\ ;\ \bar{v} \equiv \frac{\bar{\beta}\cdot\bar{u}}{\bar{\alpha}} = \frac{\bar{N}\bar{\beta}^{\mathrm{t}}}{\bar{\alpha}}\ \ (\text{where } v = -\bar{v}) \tag{2.5}$$

and the point-dependent boost factor $\gamma = (1-v^2)^{-1/2} = (1-\bar{v}^2)^{-1/2}$. With this boost factor the set of transformation equations may be written as

$$
\begin{aligned}
\bar{N} &= \frac{N}{\gamma} \\
\bar{\alpha} &= \alpha\gamma \\
\bar{V}^b &= V^b + V^{\mathrm{r}}\,\beta^b \\
\bar{\beta}^b &= \gamma^2\left(\beta^b + \frac{v^2}{V^{\mathrm{r}}}\,V^b\right) \\
\bar{\beta}^{\mathrm{t}} &= -\frac{(v\gamma)^2}{V^{\mathrm{r}}}\,.
\end{aligned}
\tag{2.6}
$$

The clamping assumption is tantamount to the $v \to 0$ ($\gamma \to 1$) limit, in which case there is no longer a distinction between barred and unbarred variables. Note that in this case $V^{\vdash} \equiv V \cdot n = \alpha\, V^{\mathrm{r}} = 0$, which, as described in [2], implies that in the canonical form of the theory the $\Sigma$ Hamiltonian can not drive field configurations across the boundary $B$.

## III. QUASILOCAL STRESS-ENERGY-MOMENTUM DENSITIES

### A. Action and variational principle

Before turning to the derivation of the quasilocal densities, we must describe the action principle which is the cornerstone of our approach. Our starting point is the first-order *Goldberg action* [5]

$$
S^{I}\left[e^{\hat{\rho}}{}_{\mu}\right] \equiv \frac{1}{2\kappa}\int_{\mathcal{M}} \Gamma^{\hat{\rho}}{}_{\hat{\tau}} \wedge e^{\hat{\tau}} \wedge \sigma_{\hat{\rho}}\,, \tag{3.1}
$$

where $\kappa = 8\pi$ (in units with $G = c = 1$) and $\Gamma_{\hat{\sigma}\hat{\tau}\mu} = e_{\hat{\sigma}\nu}\,\nabla_{\mu}\, e_{\hat{\tau}}{}^{\nu}$ represent the spacetime connection one-forms which specify the Levi-Civita connection on $\mathcal{M}$ with respect to the tetrad $e^{\hat{\rho}}{}_{\mu}$. Also, the *Sparling two-forms* [5,15] are defined by

$$
\sigma_{\hat{\rho}} \equiv -\frac{1}{2}\,\epsilon_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\mu}}\,\Gamma^{\hat{\sigma}\hat{\tau}} \wedge e^{\hat{\mu}}\,. \tag{3.2}
$$

Therefore, as mentioned, the Goldberg action is not a Palatini action in which tetrad $e^{\hat{\rho}}{}_{\mu}$ and connection $\Gamma^{\hat{\sigma}}{}_{\hat{\tau}\mu}$ are varied independently. As it stands, the action (3.1) possesses superfluous tetrad dependence. However, note that the Goldberg action is invariant under spacetime diffeomorphisms which preserve the boundary, since it is written purely in the language of differential forms. [16]

Our goal is to identify the Goldberg action (3.1) with the familiar "$TrK$" action used in metric gravity. The extrinsic curvature tensor associated with the $\Sigma$ foliation is defined by $K_{\mu\nu} \equiv -h^{\lambda}_{\mu}\,\nabla_{\lambda}\,u_{\nu}$ (with the projection operator $h^{\lambda}_{\mu} = g^{\lambda}_{\mu} + u^{\lambda}\,u_{\mu}$), while the extrinsic curvature tensor associated with the $\bar{\mathcal{T}}$ foliation is defined by $\bar{\Theta}_{\mu\nu} \equiv -\bar{\gamma}^{\lambda}_{\mu}\,\nabla_{\lambda}\,\bar{n}_{\nu}$ (with the projection operator $\bar{\gamma}^{\lambda}_{\mu} = g^{\lambda}_{\mu} - \bar{n}^{\lambda}\,\bar{n}_{\mu}$). The first step towards the desired identification is to note that the action differs from the ordinary Hilbert action by a pure divergence [5,14]

$$
\frac{1}{2\kappa}\int_{\mathcal{M}} \Gamma^{\hat{\rho}}{}_{\hat{\tau}} \wedge e^{\hat{\tau}} \wedge \sigma_{\hat{\rho}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \Re e^{*} - \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}\left(e^{\hat{\rho}} \wedge \sigma_{\hat{\rho}}\right)\,, \tag{3.3}
$$

where $e^*$ is volume form on $\mathcal{M}$. Evidently, all of the action's tetrad dependence resides exclusively in boundary terms,

$$-\frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}\left(e^{\hat{\rho}} \wedge \sigma_{\hat{\rho}}\right) = \frac{1}{\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\, \nabla_\mu \left(e^{\hat{\rho}\mu}\, \nabla_\lambda\, e_{\hat{\rho}}{}^{\lambda}\right) . \tag{3.4}$$

Now, if the time leg of the tetrad $e_{\hat{0}}$ coincides with the future-pointing normals $u$ on both $t'$ and $t''$, then the boundary terms associated with these manifolds are the the desired $TrK$ terms. Likewise, enforcing the condition that the third tetrad leg $e_{\hat{3}}$ coincides with the $\bar{\mathcal{T}}$ normal $\bar{n}$ on $\bar{\mathcal{T}}$ ensures that the one obtains the desired $Tr\bar{\Theta}$ term for the $\bar{\mathcal{T}}$ boundary term. We assume that the variational set of tetrads obey these conditions. However, in general such tetrads are doubled-valued on the *corners* $B'' \equiv t'' \bigcap \bar{\mathcal{T}}$ and $B' \equiv t' \bigcap \bar{\mathcal{T}}$, since $u \cdot \bar{n}$ need not vanish on these two-surfaces. Therefore, to express the action (3.1) in the desired form, relax the second gauge condition on $e_{\hat{3}}$ on a "small" (not simply connected) neighborhood of the corners such that the tetrad is single-valued. Next, take the limit that this small neighborhood "shrinks" to just the corners $B'$ and $B''$. Such a limit procedure is described in *Appendix D*, and it yields the following expression for the action:

$$S^1 = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{-g}\, \Re + \frac{1}{\kappa}\int_{t'}^{t''} \mathrm{d}^3x\, \sqrt{h}\, K - \frac{1}{\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\, \sqrt{-\bar{\gamma}}\, \bar{\Theta} - \frac{1}{\kappa}\int_{B'}^{B''} \mathrm{d}^2x\, \sqrt{\sigma}\, \phi\,, \tag{3.5}$$

where $\phi \equiv \tanh^{-1} v$ is the point-dependent boost parameter on $B''$ and $B'$ associated with the boost velocity $v$ defined in the last section. The corner terms were first given by Hayward and Wong for the metric action. [12] Heuristically, they arise because, though the corners constitute a set of measure zero in the $TrK$ integration over all of $\partial\mathcal{M}$, the trace of extrinsic curvature is infinite on these two-surfaces (as the normal of $\partial\mathcal{M}$ changes discontinuously from $u$ to $\bar{n}$). Note that the corner contributions to the action vanish if the initial and final slices are clamped to $\bar{\mathcal{T}}$. To obtain the variation of (3.5), one may straightforwardly vary the action (3.1) and then apply the limiting procedure. This direct method is sketched in *Appendix D*. However, in the interest of brievity we borrow from results given in Refs. [10,12]. Subject to the chosen internal gauge fixing, the action (3.1) is a tetrad version of the metric action used in Ref. [2] to define quasilocal stress-energy-momentum in general relativity. Hence, for the moment we may regard it as a metric action. Indeed, only the gauge-invariant quantities $\bar{\gamma}_{ij}$, $h'_{ij}$, and $h''_{ij}$ are fixed on the boundary $\partial\mathcal{M}$ in the associated variational principle. Refs. [10,12] have shown that the boundary contributions to the variation of $S^1$ are

$$\left(\delta S^1\right)_{\partial\mathcal{M}} = \int_{t'}^{t''} \mathrm{d}^3x\, p^{ij}\, \delta h_{ij} + \int_{\bar{\mathcal{T}}} \mathrm{d}^3x\, \bar{\pi}^{ij}\, \delta\bar{\gamma}_{ij} - \frac{1}{\kappa}\int_{B'}^{B''} \mathrm{d}^2x\, \phi\, \delta\sqrt{\sigma}\,, \tag{3.6}$$

where the gravitational momenta are given by

$$\begin{aligned} p^{ij} &= \frac{\sqrt{h}}{2\kappa}\left(K\, h^{ij} - K^{ij}\right) \\ \bar{\pi}^{ij} &= -\frac{\sqrt{-\bar{\gamma}}}{2\kappa}\left(\bar{\Theta}\, \bar{\gamma}^{ij} - \bar{\Theta}^{ij}\right) . \end{aligned} \tag{3.7}$$

The variable $p^{ij}$ becomes the standard ADM momenta in the canonical form of metric gravity, and it is conjugate to $h_{ij}$. Likewise, $\bar{\pi}^{ij}$ is the ADM-type momenta conjugate to

$\bar{\gamma}_{ij}$, but now canonical conjugacy is defined with respect to $\bar{\mathcal{T}}$. Note that equation (3.6) includes corner contributions to the variation which feature fixation of intrinsic geometry, in harmony with the fact that the induced metric is fixed on $\partial\mathcal{M}$.

There is a complex-valued action functional, closely related to (3.1), which is based on the *self-dual* (+) (or *anti-self-dual* (−)) *connection forms*

$$\Gamma^{(\pm)\hat{\rho}\hat{\sigma}} = \frac{1}{2}\left(\Gamma^{\hat{\rho}\hat{\sigma}} \mp \frac{i}{2}\epsilon^{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\mu}}\Gamma_{\hat{\tau}\hat{\mu}}\right) . \tag{3.8}$$

This action is referred to as the *complex Goldberg action* and has the form (here we take the self-dual case)

$$S\left[e^{\hat{\rho}}{}_{\mu}\right] = \frac{1}{2\kappa}\int_M \Gamma^{\hat{\rho}}{}_{\hat{\tau}} \wedge e^{\hat{\tau}} \wedge \sigma^{(+)}{}_{\hat{\rho}} , \tag{3.9}$$

where the complex Sparling two-forms are

$$\sigma^{(\pm)}{}_{\hat{\rho}} = -\epsilon_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\mu}}\,\Gamma^{(\pm)\hat{\sigma}\hat{\tau}} \wedge e^{\hat{\mu}} . \tag{3.10}$$

The complex action (3.9) differs from the previous one (3.1) by a purely imaginary boundary term. Indeed, setting

$$S = S^{I} - S^{0} , \tag{3.11}$$

we find that

$$-S^{0} = \frac{1}{2\kappa}\int_M \mathrm{d}\left[e^{\hat{\rho}} \wedge \left(\sigma_{\hat{\rho}} - \sigma^{(+)}{}_{\hat{\rho}}\right)\right] . \tag{3.12}$$

With the gauge choices made above and the limiting procedure described in *Appendix D*, an appeal to Stokes' theorem yields

$$-S^{0} = -\frac{\mathrm{i}}{2\kappa}\int_{t'}^{t''} \mathrm{d}^3x\,\sqrt{h}\,\epsilon^{\hat{r}\hat{s}\hat{p}}\,\omega_{\hat{s}\hat{p}j}\,E_{\hat{r}}{}^{j} + \frac{\mathrm{i}}{2\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\,\sqrt{-\bar{\gamma}}\,\bar{\epsilon}^{\hat{r}\hat{s}\hat{p}}\,\bar{\tau}_{\hat{s}\hat{p}j}\,\bar{\xi}_{\hat{r}}{}^{j} , \tag{3.13}$$

where $\bar{\tau}_{\hat{r}\hat{s}j} = \bar{\xi}_{\hat{r}k}\,\bar{\mathcal{D}}_j\,\bar{\xi}_{\hat{s}}{}^{k}$ and $\omega_{\hat{r}\hat{s}j} = E_{\hat{r}k}\,D_j\,E_{\hat{s}}{}^{k}$ are respectively the triad connection coefficients on $\bar{\mathcal{T}}$ and $\Sigma$. Notice that $-S^0$ contributes no corner terms to the action and that it serves as a *subtraction term* (a functional of the fixed boundary data) [2,10] in the broadest sense (it depends on the boundary data of $\bar{\mathcal{T}}$, $t'$, and $t''$).[4] Because of the triad dependence of the subtraction term, we do not have the option of viewing the action (3.9) as solely a metric action. Furthermore, in order to fully remove the superfluous tetrad dependence associated with the action $S$, one would have to completely specify the triad on each boundary element of $\partial\mathcal{M}$ (though we do not chose to completely do so).

Now consider the boundary-term contributions to the variation of the action (3.9). Since the plan is to work with the Ashtekar variables in the canonical form of the theory, first

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[4]To avoid confusion, it is crucial to note that in Refs. [2,10] the notation $S^0$ represents an arbitrary subtraction term, while in this paper $S^0$ represents the specific term (3.13).

express the boundary-term contributions (3.6) to the variation of the action $S^t$ in terms of the densitized triads on $\bar{\mathcal{T}}$, $t'$, and $t''$. (This is easily done with the identity (4.7) given below and a similar identity for the $\bar{\mathcal{T}}$ metric and triad.) Adding this result to the variation of (3.13), we find that

$$(\delta S)_{\partial\mathcal{M}} = \mathrm{i}\int_{t'}^{t''} \mathrm{d}^3x\, A^{\hat{r}}{}_j\, \delta\left(\sqrt{h}\, E_{\hat{r}}{}^j\right) + \mathrm{i}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\, \bar{\mathcal{A}}^{\hat{r}}{}_j\, \delta\left(\sqrt{-\bar{\gamma}}\, \bar{\xi}_{\hat{r}}{}^j\right) - \frac{1}{\kappa}\int_{B'}^{B''} \mathrm{d}^2x\, \phi\, \delta\sqrt{\sigma}\,, \tag{3.14}$$

where we have introduced the connections

$$\begin{aligned} A^{\hat{r}}{}_j &= \frac{1}{\kappa}\left(\omega^{\hat{r}}{}_j - \mathrm{i}\, K^{\hat{r}}{}_j\right) \equiv \frac{1}{\kappa}\left(-\frac{1}{2}\,\epsilon^{\hat{r}\hat{s}\hat{p}}\,\omega_{\hat{s}\hat{p}j} - \mathrm{i}\, K^{\hat{r}}{}_j\right) \\ \bar{\mathcal{A}}^{\hat{r}}{}_j &= \frac{1}{\kappa}\left(\bar{\tau}^{\hat{r}}{}_j + \mathrm{i}\,\bar{\Theta}^{\hat{r}}{}_j\right) \equiv \frac{1}{\kappa}\left(\frac{1}{2}\,\bar{\epsilon}^{\hat{r}\hat{s}\hat{p}}\,\bar{\tau}_{\hat{s}\hat{p}j} + \mathrm{i}\,\bar{\Theta}^{\hat{r}}{}_j\right). \end{aligned} \tag{3.15}$$

(With these conventions $\omega^{\hat{r}}{}_{\hat{s}j} = \epsilon^{\hat{r}}{}_{\hat{p}\hat{s}}\,\omega^{\hat{p}}{}_j$ and $\bar{\tau}^{\hat{r}}{}_{\hat{s}j} = \bar{\epsilon}^{\hat{r}}{}_{\hat{p}\hat{s}}\,\bar{\tau}^{\hat{p}}{}_j$.) The connection variable $A^{\hat{r}}{}_j$ is (up to a factor of $\kappa$) the $\Sigma$ *Sen connection*, which becomes the Ashtekar connection in the canonical form of the theory. Likewise, the second connection $\mathcal{A}^{\hat{r}}{}_j$ is the Sen connection associated with $\bar{\mathcal{T}}$. It is a complexified $SO(2,1)$ connection and enjoy properties completely analogous to the well-known ones enjoyed by the $\Sigma$ Sen connection. In particular, in terms of the curvature of $\mathcal{A}^{\hat{r}}{}_j$ one may compactly express the constraints associated the embedding of $\bar{\mathcal{T}}$ in the Einstein space $\mathcal{M}$. [14,17] *Note that here these connections are not the canonical Ashtekar connections.* We have not written down imaginary contributions to the corner terms which presumably arise from integration by parts on $\delta S^0$ terms. In fact, these vanish, and a calculation which demonstrates this is outlined in *Appendix D.*

### B. Quasilocal densities

We now present all of the fundamental $B$ tensors which serve as quasilocal densities describing the stress-energy-momentum content of the $\Sigma$ gravitational fields contained within $B$. We express these densities in terms of the $\Sigma$ Sen connection and triad. In the next section when studying the canonical form of the action principle, we consider the canonical versions of these expressions which are written in terms of the $\Sigma$ Ashtekar variables. To begin with, we collect a set $\{\varepsilon, j_a, s^{ab}\}$ of quasilocal densities which is essentially the same as that described extensively in the original Ref. [2]. This set is comprised of an *energy* surface density $\varepsilon$, a *tangential momentum* surface density $j_a$, and a *spatial stress* surface density $s^{ab}$. We also find the need to introduce a new set $\{j_\vdash, \hat{j}_a, t^{ab}\}$ of quasilocal densities (also considered in [10]), which is comprised of a *normal momentum* surface density $j_\vdash$, a *tangential momentum* surface density $\hat{j}_a$ (which turns out to be the same as $j_a$), and a *temporal stress* surface density $t^{ab}$. Both sets may be derived from the gravitational action (3.9) via the Hamilton-Jacobi method as described in the introduction. Therefore, we adopt the unifying point of view that *any quasilocal stress-energy-momentum quantity is given by the rate of change of the classical action $S_{c\ell}$ corresponding to some variation $\delta S_{c\ell}$ in the fixed boundary data of*

$\partial M = t' \bigcup t'' \bigcup \bar{\mathcal{T}}$. However, we do not explicitly consider the classical action as in Ref. [2], since we prefer to "read off" the geometric expressions for above densities from the boundary contributions (3.14) to the variation of $S$.

In order to "read off" the various quasilocal densities from the boundary terms (3.14), we make two assumptions in this subsection. (i) First, we assume that the $\Sigma$ foliation of $\mathcal{M}$ is clamped, so that $u \cdot \bar{n} = 0$. Again, this means that one may drop all overbars associated with three-boundary quantities from the formalism. Also, this sets $\phi = 0$ on the corners. The clamping assumption is made in this section only for convenience, and we return to the general slicing scenario in the next section. (ii) Second, we enforce partial gauge fixation of the triads on the boundary elements of $\mathcal{M}$. Following Ref. [14], we require that the $\mathcal{T}$ triad is time-gauge. This condition ensures that the $\mathcal{T}$ piece $S^0|_{\mathcal{T}}$ of the subtraction term *is functionally linear in the lapse $N$ and shift $V^a$*. As described in detail in Refs. [2,10] this linearity condition is crucial, because it ensures that the quasilocal energy density $\varepsilon$ and momentum density $j_a$ depend solely on the Cauchy data of $\Sigma$. Similarly, the triads on both $t'$ and $t''$ are required to be "radial-gauge". These restrictions on the $\bar{\mathcal{T}}$, $t'$ and $t''$ triads ensure that the purely imaginary piece of the corner contribution to the variation (3.14) vanishes (indeed we have already seen that this is a condition which follows from how the tetrad has been selected) and ensure that the quasilocal densities to be defined behave appropriately under boosts. These points become clear below. The time-gauge and radial-gauge conditions are defined and discussed in *Appendix A*. Unlike the clamping assumption (i), these boundary gauge restrictions (ii) are absolutely necessary for our formalism. Once we have obtained both the geometric form and a physical interpretation of each quasilocal density, we turn in the next section to the issue of how these densities behave under boosts and also consider the canonical form of the action principle when the $\Sigma$ slicing need not be clamped.

Let us first examine the $\mathcal{T}$ contribution to the variation of the complex Goldberg action with the assumption of a clamped $\Sigma$ slicing. Subject to the time-gauge requirement, the $\mathcal{T}$ triad and cotriad can be expressed (at least locally) in terms of a $B$ dyad $\theta_{\hat{a}}{}^{b}$ and codyad $\theta^{\hat{a}}{}_{b}$,

$$\begin{aligned} \xi_{\perp} &= 1/N\,(\partial/\partial t - V^a\,\partial/\partial x^a) & \xi_{\hat{a}} &= \theta_{\hat{a}}{}^{b}\,\partial/\partial x^b \\ \xi^{\perp} &= N\,\mathrm{d}t & \xi^{\hat{a}} &= \theta^{\hat{a}}{}_{b}\left(\mathrm{d}x^b + V^b\,\mathrm{d}t\right) . \end{aligned} \tag{3.16}$$

(The time-gauge condition has been indicated by replacing the triad label $\hat{0}$ with $\perp$.) The associated time-gauge $\mathcal{T}$ connection coefficients are the following:

$$\begin{aligned} \tau_{\hat{c}\perp\perp} &= \theta_{\hat{c}}\,[\log N] \\ \tau_{\perp\hat{a}\hat{c}} &= 1/N\left[\sigma_{bd}\,\theta_{(\hat{a}}{}^{b}\,\dot{\theta}_{\hat{c})}{}^{d} + \theta_{\hat{c}}{}^{b}\,\theta_{\hat{a}}{}^{d}\,\delta_{(b}\,V_{d)}\right] \\ \tau_{\hat{a}\hat{c}\hat{b}} &= \theta_{\hat{b}}{}^{c}\,\theta_{\hat{a}\,b}\left(\delta_c\,\theta_{\hat{c}}{}^{b}\right) \\ \tau_{\hat{1}\hat{2}\perp} &= 1/N\left[\sigma_{bd}\,\theta_{[\hat{1}}{}^{b}\,\dot{\theta}_{\hat{2}]}{}^{d} - 1/2\,\epsilon^{ac}\,\delta_a V_c - V^b\,\tau_{\hat{1}\hat{2}b}\right] , \end{aligned} \tag{3.17}$$

where for this set the "dot" represents partial time differentiation. Plugging these coefficients into $S^0|_{\mathcal{T}}$, one can verify that $S^0|_{\mathcal{T}}$ is functionally linear in the shift $V^a$ and has no $N$ dependence. Next, applying the identites

$$\partial\tilde{\xi}_{\hat{r}}{}^{j}/\partial N = \sqrt{\sigma}\,\theta_{\hat{a}}{}^{j}\,\eta_{\hat{r}}^{\hat{a}}$$

$$\partial\tilde{\xi}_{\hat{r}}{}^{j}/\partial V^b = -\sqrt{\sigma}\,\eta_{\hat{r}}^{\perp}\,\sigma_b^j \tag{3.18}$$

$$\partial\tilde{\xi}_{\hat{r}}{}^{j}/\partial\theta^{\hat{a}}{}_b = N\sqrt{\sigma}\left(\xi_{\hat{r}}{}^{j}\,\theta_{\hat{a}}{}^{b} - \eta_{\hat{r}}^{\hat{c}}\,\theta_{\hat{c}}{}^{b}\,\theta_{\hat{a}}{}^{j}\right).$$

to the $\mathcal{T}$ piece of the boundary variation (3.14), we write the $\mathcal{T}$ contribution to as

$$(\delta S)_{\mathcal{T}} = -\int_{\mathcal{T}} \mathrm{d}^3x\,\sqrt{\sigma}\left[\varepsilon\,\delta N - j_a\,\delta V^a - \frac{N}{2}\,s^{ab}\,(\delta\theta)_{ab}\right]. \tag{3.19}$$

Here the quasilocal density $s^{ab}$ is defined with respect to

$$s_{\hat{c}}{}^{b} \equiv \frac{1}{\sqrt{-\gamma}}\,\frac{\delta S}{\delta\theta^{\hat{c}}{}_b}\bigg|_{\mathcal{T}}, \tag{3.20}$$

via $s^{ab} = s_{\hat{c}}{}^{b}\,\theta^{\hat{c}a}$, and the expression $(\delta\theta)_{ab}$ is shorthand for $2\,\theta_{\hat{a}a}\,\delta\theta^{\hat{a}}{}_b$. Notice that $(\delta\theta)_{(ab)} = \delta\sigma_{ab}$, while $(\delta\theta)_{[ab]}$ is a pure gauge variation of the $B$ dyad. Also, note that $s^{[ab]}$ is completely determined by the subtraction term $\delta S^0|_{\mathcal{T}}$ ($s^{(ab)}$ is determined by $\delta S^1$ and $\delta S^0$ contributions). Explicitly we have

$$\varepsilon \equiv -\frac{1}{\sqrt{\sigma}}\,\frac{\delta S}{\delta N}\bigg|_{\mathcal{T}} = -\mathrm{i}\,\mathcal{A}^{\hat{a}}{}_b\,\theta_{\hat{a}}{}^{b}$$

$$j_a \equiv \frac{1}{\sqrt{\sigma}}\,\frac{\delta S}{\delta V^a}\bigg|_{\mathcal{T}} = -\mathrm{i}\,\mathcal{A}^{\perp}{}_a \tag{3.21}$$

$$s^{ab} \equiv \frac{1}{\sqrt{-\gamma}}\,\theta^{\hat{c}a}\,\frac{\delta S}{\delta\theta^{\hat{c}}{}_b}\bigg|_{\mathcal{T}} = \mathrm{i}\left(\mathcal{A}^{\hat{r}}{}_j\,\xi_{\hat{r}}{}^{j}\,\sigma^{ab} - \mathcal{A}^{\hat{c}}{}_d\,\sigma^{da}\,\theta_{\hat{c}}{}^{b}\right).$$

We can rewrite these densities in terms of the $\Sigma$ Sen connection. The appendix results (C12) express the time-gauge $\mathcal{T}$ Sen connection $\mathcal{A}^{\hat{r}}{}_j$ in terms of the radial-gauge $\Sigma$ Sen connection $A^{\hat{r}}{}_j$ and other gauge variables. Insertion of the appendix results into the above expressions gives the following:

$$\varepsilon \equiv \epsilon^{\hat{a}\hat{c}}\,A_{\hat{a}b}\,\theta_{\hat{c}}{}^{b}$$

$$j_a \equiv \mathrm{i}\,A^{\vdash}{}_a \tag{3.22}$$

$$s^{ab} \equiv \epsilon^{\hat{a}\hat{c}}\,A_{\hat{a}d}\,\sigma^{da}\,\theta_{\hat{c}}{}^{b} + \left(2\mathrm{i}/\kappa\,\Gamma^{(+)}{}_{\hat{1}\hat{2}\perp} - \epsilon^{\hat{a}\hat{c}}\,A_{\hat{a}d}\,\theta_{\hat{c}}{}^{d}\right)\sigma^{ab}.$$

*Henceforth, we assume that $\varepsilon$, $j_a$, and $s^{ab}$ represent these expressions.* Notice that $\varepsilon$ and $j_a$ are built exclusively from $\Sigma$ Cauchy data ($E_{\hat{r}}{}^{j}$, $K^{\hat{r}}{}_j$). Because of this fact, $\varepsilon$ and $j_a$ can

be interpreted as canonical expressions depending on the Ashtekar variables. Because of the presence of $2\mathrm{i}\,\Gamma_{\hat{1}\hat{2}\perp} = a_j\, n^j + \mathrm{i}\,\tau_{\hat{1}\hat{2}\perp}$, the density $s^{ab}$ does not depend solely on $\Sigma$ Cauchy data. This term contains the spacetime acceleration $a^\mu = u^\nu \nabla_\nu u^\mu$ of $u^\mu$ as well as the $\mathcal{T}$ connection coefficient $\tau_{\hat{1}\hat{2}\perp}$, which describes the rotation of the $B$ dyad under parallel transport along the integral curves of $u$. Both of these terms depend on how the Cauchy data evolve in time. The real parts of the densities in the above set correspond exactly to the quasilocal densities first introduced in Ref. [2]. Indeed, expressed in full detail,

$$\begin{aligned}
\varepsilon &= \frac{1}{\kappa}\, k \\
j_a &= -\frac{2}{\sqrt{h}}\, n_i\, \sigma_{aj}\, p^{ij} - \frac{\mathrm{i}}{\kappa}\, \omega_{\hat{1}\hat{2}a} \\
s^{ab} &= \frac{1}{\kappa}\left[k^{ab} + \left(a_j\, n^j - k\right)\sigma^{ab}\right] + \frac{\mathrm{i}}{\kappa}\left(l_c{}^a\, \epsilon^{bc} + \tau_{\hat{1}\hat{2}\perp}\, \sigma^{ab}\right),
\end{aligned} \tag{3.23}$$

where $k_{ab}$ is the extrinsic curvature of $B$ as embedded in $\Sigma$ (with $k = \sigma^{ab}\, k_{ab}$) and $l_{ab}$ is the extrinsic curvature of $B$ as embedded in $\mathcal{T}$.

We assume that each density has the same physical interpretation as given in Ref. [2] and review these intepretations now. (For the following interpretations to be valid, one should consider the densities $\varepsilon$, $j_a$, and $s^{ab}$ to be evaluated "on-shell", *i.e.* evaluated on some particular solution of the Einstein field equations.) From its definition $\sqrt{\sigma}\,\varepsilon$ equals minus the time rate of change of the action $S$, where the time separation between the $B$ slices of $\mathcal{T}$ is controlled by the lapse $N$ on $\mathcal{T}$ (fixed as boundary data in the variational principle). Therefore, $\varepsilon$ is interpreted as an energy surface density for the system as measured by the Eulerian observers of $\Sigma$ at $B$. The total *quasilocal energy* associated with the $\Sigma$ gravitational fields is

$$E = \int_B \mathrm{d}^2x\, \sqrt{\sigma}\, \varepsilon\,, \tag{3.24}$$

the integral of the quasilocal energy density over the two-surface $B$. This notion of energy is the value of the on-shell Hamiltonian (associated with a single generic $\Sigma$ slice) which generates unit time translations orthogonal to $B$ (which corresponds to the choice $N = 1$ and $V = 0$ on $B$). In a similar fashion, $j_a$ is intepreted as a tangential-momentum surface density, and the integral

$$J_\phi = \int_B \mathrm{d}^2x\, \sqrt{\sigma}\, \phi^a\, j_a \tag{3.25}$$

represents the total quasilocal tangential momentum (angular momentum) carried by the $\Sigma$ fields. On-shell, the real part is the value of the Hamiltonian which generates spatial diffeomorphisms in the direction of $\phi^a$ (corresponding to the choice $N = 0$ and $V^a = \phi^a$ on the boundary). The form of $j_a$ makes it tempting to identify the imaginary part of $J_\phi$ with the "spin" of the $B$ dyad. $E$ and $J_\phi$ may be interpreted as functionals on the gravitational phase space associated with $\Sigma$. The real part of $s^{ab}$ represents the flux of the $a$ component of momentum in the $b$ direction. [2]

The original set of quasilocal densities have been obtained from a careful analysis of the $\mathcal{T}$ contribution to the variation (3.14) of the action. In a similar fashion we now analyze the $t'$ and $t''$ contributions to the variation. Often we drop the $'$ and $''$ notations with the understanding that all expressions may refer to either the manifold $t'$ or $t''$. Remember that the $t'$ and $t''$ triads are radial-gauge.[5] (The radial gauge is indicated by replacing the triad label $\hat{3}$ by $\vdash$.) Therefore, with may split $\tilde{E}_{\hat{r}}{}^{j}$ into $\alpha$ and $\beta^a$ (the gauge variables associated with the $1+2$ split of $h_{ij}$) as well as the $B$ codyad $\theta^{\hat{a}}{}_b$. With this assumption, we find identities like those in (3.18). Therefore, it easy to write the $t'$ and $t''$ contributions to the variation (3.14) as

$$(\delta S)_{t'}^{t''} = -\int_{t'}^{t''} \mathrm{d}^3x\, \sqrt{\sigma}\left[j_{\vdash}\,\delta\alpha + \hat{j}_a\,\delta\beta^a - \frac{\alpha}{2}\,t^{ab}\,(\delta\theta)_{ab}\right] . \tag{3.26}$$

The new quasilocal densities described at the beginning of this subsection are then

$$\begin{aligned} j_{\vdash} &\equiv -\frac{1}{\sqrt{\sigma}}\frac{\delta S}{\delta\alpha}\bigg|_{t''} = -\mathrm{i}\,A^{\hat{a}}{}_b\,\theta_{\hat{a}}{}^b \\ \hat{j}_a &\equiv -\frac{1}{\sqrt{\sigma}}\frac{\delta S}{\delta\beta^a}\bigg|_{t''} = \mathrm{i}\,A^{\vdash}{}_a \\ t^{ab} &\equiv \frac{1}{\sqrt{h}}\,\theta^{\hat{c}a}\,\frac{\delta S}{\delta\theta^{\hat{c}}{}_b}\bigg|_{t''} = \mathrm{i}\left(A^{\hat{r}}{}_j\,E_{\hat{r}}{}^j\,\sigma^{ab} - A^{\hat{a}}{}_c\,\sigma^{ca}\,\theta_{\hat{a}}{}^b\right) , \end{aligned} \tag{3.27}$$

with the same expressions for the densities associated the manifold $t'$. In full detail these are

$$\begin{aligned} j_{\vdash} &= -\frac{2}{\sqrt{h}}\,n_i\,n_j\,p^{ij} \\ \hat{j}_a &= -\frac{2}{\sqrt{h}}\,\sigma_{ai}\,n_j\,p^{ij} - \frac{\mathrm{i}}{\kappa}\,\omega_{\hat{1}\hat{2}a} \\ t^{ab} &= \frac{2}{\sqrt{h}}\,\sigma_i^a\,\sigma_j^b\,p^{ij} - \frac{\mathrm{i}}{\kappa}\left(\omega_{\hat{1}\hat{2}\vdash}\,\sigma^{ab} + k_c{}^a\,\epsilon^{bc}\right) . \end{aligned} \tag{3.28}$$

Note that the definitions of $j_a$ and $\hat{j}_a$ are identical, and hence $\hat{j}_a$ carries the same physical interpretation as $j_a$. Therefore, from now on we suppress the hat on $\hat{j}_a$. This equivalence of $\hat{j}_a$ with $j_a$ results from the chosen gauge conditions. Also a result of these conditions is the fact that both $j_{\vdash}$ and $\varepsilon$ are real. It turns out that the reality of $j_{\vdash}$ and $\varepsilon$ (or equivalently that the subtraction term $S^0$ has no $\alpha$ or $N$ dependence) is quite crucial, as it ensures that $j_{\vdash}$ and $\varepsilon$ behave well under boosts. Note that even if we had not enforced the radial-gauge condition on the $t'$ and $t''$ triads, then all of the densities listed immediately above would still by construction depend only on $\Sigma$ Cauchy data. As shown in [10], $j_{\vdash}$ is a normal momentum density, and the total normal momentum associated with the $\Sigma$ fields is given by

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[5]Technically we only need this condition to hold at the boundary $B$.

$$J_{\vdash} = \int_B \mathrm{d}^2x \sqrt{\sigma}\, j_{\vdash} \,. \tag{3.29}$$

This expression is the minus the value of the on-shell Hamiltonian which generates unit "dilations" orthogonal to $B$ (corresponding to the choice $N = 0$, $V^a = 0$, and $V^{\vdash} = \alpha V^{\mathrm{r}} = 1$ on $B$). Finally, we refer to $t^{ab}$ as the temporal stress density.

## IV. BOOSTED DENSITIES AND THE CANONICAL ACTION

### A. Boost realtions and invariants

We now demonstrate that our collection of quasilocal densities behave under generalized boosts in a manner which is in accord with the equivalence principle. Fix a spacelike two-surface $B$ in spacetime and also consider an arbitrary spacelike hypersurface $\bar{\Sigma}$ which has boundary $\partial\bar{\Sigma} = B$. The hypersurface normal of $\bar{\Sigma}$ is $\bar{u}$. If we view the $\bar{\Sigma}$ slice as a member of a temporal foliation, then we may define the Eulerian history of $B$ as $\bar{\mathcal{T}}$. By construction $\bar{\Sigma}$ is clamped to $\bar{\mathcal{T}}$ The observers at $B$ who are instantaneously at rest in the $\bar{\Sigma}$ slice (Eulerian observers of $\bar{\Sigma}$) determine the following set of quasilocal densities:

$$\begin{aligned}
\bar{\varepsilon} &= \epsilon^{\hat{a}\hat{c}}\, \bar{A}_{\hat{a}b}\, \theta_{\hat{c}}{}^{b} \\
\bar{j}_{\vdash'} &= -\mathrm{i}\, \bar{A}^{\hat{a}}{}_{b}\, \theta_{\hat{a}}{}^{b} \\
\bar{j}_a &= \mathrm{i}\, \bar{A}^{\vdash'}{}_{a} \\
\bar{s}^{ab} &= \epsilon^{\hat{a}\hat{c}}\, \bar{A}_{\hat{a}d}\, \sigma^{da}\, \theta_{\hat{c}}{}^{b} + \left(2\mathrm{i}/\kappa\, \Gamma^{(+)}{}_{\hat{1}\hat{2}\perp'} - \epsilon^{\hat{a}\hat{c}}\, \bar{A}_{\hat{a}d}\, \theta_{\hat{c}}{}^{d}\right) \sigma^{ab} \\
\bar{t}^{ab} &= \mathrm{i}\left(\bar{A}^{\hat{r}}{}_{j}\, \bar{E}_{\hat{r}}{}^{j}\, \sigma^{ab} - \bar{A}^{\hat{a}}{}_{c}\, \sigma^{ca}\, \theta_{\hat{a}}{}^{b}\right) .
\end{aligned} \tag{4.1}$$

The primes appear in these formulae because in the triad formalisms we set $\bar{\xi}_{\perp'} = \bar{u}$ and $\bar{E}_{\vdash'} = \bar{n}$. Note that here $2\mathrm{i}\,\Gamma_{\hat{1}\hat{2}\perp'} = \bar{a}_j\, \bar{n}^j + \mathrm{i}\, \bar{\tau}_{\hat{1}\hat{2}\perp'}$. Now consider a different hypersurface $\Sigma$ which spans $B$ (so like before $\partial\Sigma = B$). We may view $\Sigma$ as a particular leaf of a temporal foliation which is not clamped to $\bar{\mathcal{T}}$, the Eulerian history of $B$ with respect to $\bar{\Sigma}$. Geometrically, the scenario now is identical to the bounded spacetime region $\mathcal{M}$ that we have considered in the preliminary section. The observers at $B$ who are at rest in the $\Sigma$ hypersurface determine the set of quasilocal densities which are listed in (3.22) and (3.27) (simply the "unbarred" versions of the expressions above). We seek the transformation rules between the "barred" and "unbarred" densities, or, in other words, the behavior of the quasilocal expressions under switches of the hypersurface spanning $B$. With the appendix results (C14) and (C15), it is quite a simple matter to establish that

$$\begin{aligned}
\bar{\varepsilon} &= \gamma\, \varepsilon - v\, \gamma\, j_{\vdash} \\
\bar{j}_{\vdash'} &= \gamma\, j_{\vdash} - v\, \gamma\, \varepsilon
\end{aligned}$$

$$\bar{j}_a = j_a - \frac{\gamma^2}{\kappa}\,\delta_a v \tag{4.2}$$

$$\bar{s}^{ab} = \gamma\, s^{ab} - v\,\gamma\, t^{ab} + \frac{1}{\kappa}\,\sigma^{ab}\,\gamma^3\, u[v] + \frac{1}{\kappa}\,\sigma^{ab}\, v\,\gamma^3\, n[v]$$

$$\bar{t}^{ab} = \gamma\, t^{ab} - v\,\gamma\, s^{ab} - \frac{1}{\kappa}\,\sigma^{ab}\,\gamma^3\, n[v] - \frac{1}{\kappa}\,\sigma^{ab}\, v\,\gamma^3\, u[v]$$

These are precisely the boost relations found in Ref. [10]. Remarkably, the particular form of the subtraction term (3.12), subject to the chosen gauge fixation, does not affect the boost relations of the "bare" densities. It must be stressed that if the gauge conditions (time gauge on $\mathcal{T}$, and radial gauge on $t'$ and $t''$) had not been enforced when defining the set of quasilocal densities, then the above boost relations would not have held. In particular, if $\varepsilon$ and $j_\vdash$ are defined with subtraction-term contributions (which in this paper means they would no longer be real), then the first two boost relations are modified.

Following Ref. [10], define tracefree parts of $s^{ab}$ and $t^{ab}$,

$$\begin{aligned} \eta^{ab} &\equiv s^{ab} - 1/2\, s^c{}_c\, \sigma^{ab} = \frac{1}{\kappa}\left(k^{ab} + \mathrm{i}\, l_c{}^a\, \epsilon^{bc}\right) \\ \zeta^{ab} &\equiv t^{ab} - 1/2\, t^c{}_c\, \sigma^{ab} = -\frac{1}{\kappa}\left(l^{ab} + \mathrm{i}\, k_c{}^a\, \epsilon^{bc}\right)\,, \end{aligned} \tag{4.3}$$

Notice that the *shear stress* $\eta^{ab}$ depends only on $\Sigma$ Cauchy data. Of course, $\theta \equiv s^a{}_a = 1/\kappa\,(2\, a_\mu\, n^\mu - k + 2\mathrm{i}\,\tau_{\hat{1}\hat{2}\perp})$ depends on how $u^\mu$ and the $B$ dyad are extended into the future. Similarly, though $t^{ab}$ depends only on $\Sigma$ Cauchy data, its trace $\vartheta \equiv t^a{}_a = 1/\kappa\,(2\, b_\mu\, u^\mu + l - 2\mathrm{i}\,\omega_{\hat{1}\hat{2}\vdash})$ depends on how $n$ and the $B$ dyad are extended into the interior of $B$. We refer to $\zeta^{ab}$ as the *shear temporal stress.* Each of the densities $\varepsilon$, $j_\vdash$, $j_a$, $\eta^{ab}$ and $\zeta^{ab}$ depend only on the extrinsic and intrinsic geometry of $B$ and the the normal $n^\mu$ *at* $B$. One can easily see that

$$\begin{aligned} \bar{\eta}^{ab} &= \gamma\,\eta^{ab} - v\,\gamma\,\rho^{ab} \\ \bar{\rho}^{ab} &= \gamma\,\rho^{ab} - v\,\gamma\,\eta^{ab}\,. \end{aligned} \tag{4.4}$$

With the set $\{\varepsilon, j_\vdash, j_a, \eta^{ab}, \zeta^{ab}\}$ of quasilocal densities we can construct several invariants. For instance, notice that under boosts the density $j_a$ transforms like a gauge potential, since $\gamma^2\,\delta_a v = \delta_a \phi$. Therefore, the "field strength" or curvature $F_{ab} = 2\,\delta_{[a}\, j_{b]}$ of is an invariant. Borrowing the results from [10], we write down the following list of invariants

$$\begin{aligned} F^2 &= F_{ab}\, F^{ab} \\ m^2 &= \varepsilon^2 - j_\vdash^2 \\ (m_1)^2 &= \eta_{ab}\,\eta^{ab} - \zeta_{ab}\,\zeta^{ab} \\ (m_2)^4 &= 2\,\eta^{ab}\,\zeta^{cd}\,(\eta_{ab}\,\zeta_{cd} - \zeta_{ab}\,\eta_{cd})\,. \end{aligned} \tag{4.5}$$

(We make no claim that these need be positive.) To these we may added any local function $\mathcal{G}(\sigma_{ab})$ of the $B$ metric. Hence, a "mass-squared" that we can construct with our densities is

$$M^2 = \int_B \mathrm{d}^2x\,\sqrt{\sigma}\left[a\,F^2 + b\,m^2 + c\,(m_1)^2 + d\,(m_2)^4 + e\,\mathcal{G}\right], \tag{1.6}$$

where $a, b, c, d$, and $e$ are arbitrary functions on $B$. Again, no claim is made that $M^2$ is positive.

### B. Canonical action

Our goal in this subsection is to consider the variational principle associated with the canonical form of the action $S$ (3.11). In order to express $S$ in canonical form, we first consider the (3+1) form of $S$. Begin by expressing $S^I$ in (3+1) form. To do this, borrow the results from Ref. [10]. In that reference the action $S^I$, viewed as a metric action, has been expressed in canonical form.[6] Therefore, we must "translate" this result into the language of triads. This translation is achieved by assuming that the $\Sigma$ metric is a secondary quantity derived from $\tilde{E}_{\hat{r}}{}^{j}$ and by using the identity

$$\partial h_{ij}/\partial \tilde{E}_{\hat{r}}{}^{k} = (E)^{-1}(h_{ij}\,E^{\hat{r}}{}_{k} - h_{kj}\,E^{\hat{r}}{}_{i} - h_{ik}\,E^{\hat{r}}{}_{j}). \tag{1.7}$$

The result is

$$S^I = \int_{\mathcal{M}} \mathrm{d}^4x\left[\frac{1}{\kappa}\,K^{\hat{r}}{}_{j}\,\dot{\tilde{E}}_{\hat{r}}{}^{j} - N\,\mathcal{H} - V^j\,\mathcal{H}_j\right] + \int_{\bar{T}} \mathrm{d}^3x\left[-\frac{\dot{\varphi}}{\kappa}\sqrt{\sigma} - \bar{N}\,\bar{\mathcal{H}}^I - V^b\,\mathcal{H}^I_b\right]. \tag{1.8}$$

where we have the following:

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2\kappa}\left[h^{-1/2}\left(K^{\hat{r}}{}_{i}\,K^{\hat{s}}{}_{j} - K^{\hat{r}}{}_{j}\,K^{\hat{s}}{}_{i}\right)\tilde{E}_{\hat{r}}{}^{j}\,\tilde{E}_{\hat{s}}{}^{i} - h^{1/2}\,R\right] \\
\mathcal{H}_j &= \frac{1}{\kappa}\,D_k\left(K^{\hat{r}}{}_{j}\tilde{E}_{\hat{r}}{}^{k} - h^k_j\,K^{\hat{r}}{}_{i}\tilde{E}_{\hat{r}}{}^{i}\right) \\
\bar{\mathcal{H}}^I &= \sqrt{\sigma}\left[\gamma\,\varepsilon^I - v\gamma\,(j^I)_{\vdash}\right] \\
\bar{\mathcal{H}}^I_a &= -\sqrt{\sigma}\left[(j^I)_a - \frac{1}{\kappa}\,\delta_a\phi\right].
\end{aligned} \tag{1.9}$$

Here $R$ is the Ricci scalar of $\Sigma$, and $\varepsilon^I$, $(j^I)_{\vdash}$, and $(j^I)_b$ stand for the real parts of the densities in (3.23) and (3.28) (the notation is redundant for $\varepsilon$ and $j_{\vdash}$, as these are purely real). Also, at this stage, the hybrid extrinsic curvature $K^{\hat{r}}{}_{j}$ is merely a short-hand notation

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[6]For the metric action the (3+1) form is obtained from the canonical form by simply assuming that $p^{ij}$ has the form given in (3.7) and that $K_{ij} = -1/2N\left(\dot{h}_{ij} + D_i\,V_j + D_j\,V_i\right)$.

for a complicated function of $N$, $V^j$, $\dot{\tilde{E}}_{\hat{r}}{}^j$, and $\tilde{E}_{\hat{r}}{}^j$. Finally, note that $\bar{\mathcal{H}}^l$ and $\mathcal{H}^l_b$ are not constraints.

The next step is to calculate the (3+1) form of the subtraction term, which can be written as

$$-S^0 = -\frac{\mathrm{i}}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\, \partial_t \left(\epsilon^{\hat{r}\hat{s}\hat{p}}\, \omega_{\hat{s}\hat{p}j}\, \tilde{E}_{\hat{r}}{}^j\right) + \frac{\mathrm{i}}{2\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\, \sqrt{-\bar{\gamma}}\, \epsilon^{\hat{r}\hat{s}\hat{p}}\, \bar{\tau}_{\hat{s}\hat{p}\hat{r}}\,. \tag{4.10}$$

Tedious but straightforward manipulations yield

$$\begin{aligned} -S^0 = &-\frac{\mathrm{i}}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\, \epsilon^{\hat{r}\hat{s}\hat{p}}\, \omega_{\hat{s}\hat{p}j}\, \dot{\tilde{E}}_{\hat{r}}{}^j \\ &+\frac{\mathrm{i}}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{h}\, D_i \left(\epsilon^i{}_{jk}\, \underset{\sim}{E}^{\hat{r}j}\, \dot{\tilde{E}}_{\hat{r}}{}^k\right) + \frac{\mathrm{i}}{2\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\, \sqrt{-\bar{\gamma}}\, \epsilon^{\hat{r}\hat{s}\hat{p}}\, \bar{\tau}_{\hat{s}\hat{p}\hat{r}}\,. \end{aligned} \tag{4.11}$$

where $\underset{\sim}{E}^{\hat{r}j} = h^{-1/2}\, E^{\hat{r}j}$. For the middle integral on the right-hand side, we now use Stokes' theorem for each $\Sigma$ slice and enforce the radial-gauge condition at the boundary $B$ of each $\Sigma$. Also, we expand the integrand in the final integral subject to the assumption that the $\bar{\mathcal{T}}$ triad is time-gauge with respect to $\bar{u}$. The result of these calculations is

$$\begin{aligned} -S^0 = &-\frac{\mathrm{i}}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\, \epsilon^{\hat{r}\hat{s}\hat{p}}\, \omega_{\hat{s}\hat{p}j}\, \dot{\tilde{E}}_{\hat{r}}{}^j \\ &-\frac{\mathrm{i}}{2\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\, \sqrt{\sigma}\, \sigma_{bd}\, \epsilon^{\hat{a}\hat{c}}\, \theta_{[\hat{a}}{}^b\, \dot{\theta}_{\hat{c}]}{}^d + \frac{\mathrm{i}}{2\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\, \sqrt{-\bar{\gamma}}\, \epsilon^{\perp'\hat{a}\hat{c}}\, \bar{\tau}_{\hat{a}\hat{c}\perp'}\,. \end{aligned} \tag{4.12}$$

Next, "barring" the last formula in (3.17), one finds

$$\sigma_{bd}\, \theta_{[\hat{1}}{}^b\, \dot{\theta}_{\hat{2}]}{}^d = \bar{N}\, \bar{\tau}_{\hat{1}\hat{2}\perp'} + 1/2\, \epsilon^{ac}\, \delta_a \bar{V}_c + \bar{V}^b\, \bar{\tau}_{\hat{1}\hat{2}b}\,. \tag{4.13}$$

Insertion of this formula into (4.12) gives the desired (3+1) form of the subtraction term,

$$-S^0 = -\frac{\mathrm{i}}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\, \epsilon^{\hat{r}\hat{s}\hat{t}}\, \omega_{\hat{s}\hat{t}}\, \dot{\tilde{E}}_{\hat{r}}{}^j - \frac{\mathrm{i}}{\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\, \sqrt{\sigma}\, \omega_{\hat{1}\hat{2}b}\, \bar{V}^b\,, \tag{4.14}$$

where we have also used $\bar{\tau}_{\hat{a}\hat{c}b} = \omega_{\hat{a}\hat{c}b}$.

We now turn to the canonical form of the action principle. We shall avoid the issue of the reality conditions by working first with canonical form of the real action $S^l$. Therefore, upon adding the pure imaginery boundary term (4.14) to $S^l$, we merely introduce a complex chart on the real phase space. The canonical form of the action $S^l$ is the following:

$$S^l = \int_{\mathcal{M}} \mathrm{d}^4x \left[P^{\hat{r}}{}_j\, \dot{\tilde{E}}_{\hat{r}}{}^j - N\,\mathcal{H} - V^j\,\mathcal{H}_j - 1/2\, \phi_{\hat{r}\hat{s}}\, J^{\hat{r}\hat{s}}\right] + \int_{\bar{\mathcal{T}}} \mathrm{d}^3x \left[-\frac{\phi}{\kappa}\sqrt{\sigma} - \bar{N}\,\bar{\mathcal{H}}^l - V^b\,\mathcal{H}^l_b\right]. \tag{4.15}$$

In general $P^{\hat{r}}{}_j \neq 1/\kappa\, K^{\hat{r}}{}_j$. Indeed, setting $P_{ij} = P^{\hat{r}}{}_j\, E_{\hat{r}i}$, one has that

$$P_{ij} \approx \frac{1}{\kappa} K_{ij}\,, \tag{4.16}$$

where $\approx$ stands for modulo the *rotation constraint*,

$$J^{\hat{r}\hat{s}} \equiv 2\, P^{[\hat{r}}{}_{j}\, \delta^{\hat{s}]\hat{t}}\, \tilde{E}_{\hat{t}}{}^{j} \,. \tag{4.17}$$

Furthermore, in (4.15) $\mathcal{H}$, $\mathcal{H}_j$, $\bar{\mathcal{H}}^1$, and $\bar{\mathcal{H}}^1_b$ have the same forms as given in (4.9) but now are built with $P^{\hat{r}}{}_{j}$ rather than $1/\kappa\, K^{\hat{r}}{}_{j}$. In particular, in the canonical action $S^I$

$$\begin{aligned} \bar{\mathcal{H}}^I &= \sqrt{\sigma}\left[\frac{1}{\kappa}\,\gamma\, k + v\gamma\, \sigma_{ij}\, P^{ij}\right] \\ \bar{\mathcal{H}}^I_a &= -\sqrt{\sigma}\left[n_i\, P^i{}_j\, \sigma^j_a - \frac{1}{\kappa}\,\delta_a \phi\right] \end{aligned} \tag{4.18}$$

(with the radial-gauge condition at the boundary $B$ of $\Sigma$, one can write $k = -\omega^{\hat{a}}{}_{\vdash\hat{a}}$). As is well-known, the anticommuting Lagrange multiplier $\phi_{\hat{r}\hat{s}}$ associated with the rotation constraint can be geometrically interpreted as the time component of the connection forms, [18]

$$\phi_{\hat{r}\hat{s}} = -\Gamma_{\hat{r}\hat{s}t} = -\,\langle \Gamma_{\hat{r}\hat{s}}\,,\, \partial/\partial t\rangle \,. \tag{4.19}$$

Enforcement of the radial-gauge condition at the boundary $B$ of each $\Sigma$ slice places a boundary condition on $\phi_{\hat{r}\hat{s}}$. This boundary condition is the canonical version of setting the connection coefficient $\Gamma_{\hat{a}\vdash\mu}\,\bar{u}^\mu = 0$, where we are working in the RT-gauge described in the appendix. The coefficient $\Gamma_{\hat{a}\vdash\mu}\,\bar{u}^\mu$ describes the rotation of $n$ as it is parallel transported along the integral curves of $\bar{u}$. To see what the required boundary condition on $\phi_{\hat{r}\hat{s}}$ is, first recall that in the triad formalism the vector constraint is not the generator of diffeomorphisms, rather $\mathcal{H}^{diff}_j = \mathcal{H}_j - 1/2\, J^{\hat{r}\hat{s}}\, \omega_{\hat{r}\hat{s}j}$ so the vector constraint generates rotation of the triad. [18] Therefore, the boundary condition

$$-\,1/\bar{N}\left(\phi_{\hat{a}\vdash} + \omega_{\hat{a}\vdash b}\,\bar{V}^b\right)\Big|_{\mathcal{T}} = 0 \tag{4.20}$$

ensures consistency between the selection of the radial-gauge condition for the $\Sigma$ triad at $B$ and the evolution of the triad as obtained from the variation of the canonical action.

We now add the boundary term (4.14) to the canonical action (4.15) and get

$$S = \int_{\mathcal{M}} \mathrm{d}^4x \left[\mathrm{i}\, A^{\hat{r}}{}_{j}\, \dot{\tilde{E}}_{\hat{r}}{}^{j} - \underset{\sim}{N}\,\tilde{\mathcal{H}} - V^j\,\mathcal{H}_j - 1/2\,\phi_{\hat{r}\hat{s}}\, J^{\hat{r}\hat{s}}\right] + \int_{\mathcal{T}} \mathrm{d}^3x \left[-\frac{\phi}{\kappa}\,\dot{\sqrt{\sigma}} - \bar{N}\,\bar{\mathcal{H}} - \bar{V}^b\,\bar{\mathcal{H}}_b\right] , \tag{4.21}$$

where in anticipation of dealing with the Ashtekar version of the canonical constraints, we have written $\underset{\sim}{N} \equiv h^{-1/2}\, N$ and $\tilde{\mathcal{H}} \equiv h^{1/2}\,\mathcal{H}$. Here $\bar{\mathcal{H}} = \bar{\mathcal{H}}^I$, while $\bar{\mathcal{H}}_b = \bar{\mathcal{H}}^I_b + \mathrm{i}/\kappa\,\sqrt{\sigma}\,\omega_{\hat{1}\hat{2}b}$. *Furthermore, for the rest of this section $A^{\hat{r}}{}_{j}$ is the canonical Ashtekar connection*

$$A^{\hat{r}}{}_{j} = \frac{1}{\kappa}\,\omega^{\hat{r}}{}_{j} - \mathrm{i}\, P^{\hat{r}}{}_{j} \,. \tag{4.22}$$

As usual, one may replace the rotation constraint with the *Gauss constraint*,

$$J_{\hat{r}} = -\frac{\mathrm{i}}{\kappa}\, {}^{A}\mathcal{D}_j\, \tilde{E}_{\hat{r}}{}^{j} \equiv -\frac{\mathrm{i}}{\kappa}\left(D_j\, \tilde{E}_{\hat{r}}{}^{j} - \kappa \tilde{E}_{\hat{s}}{}^{j}\, \epsilon^{\hat{s}}{}_{\hat{p}\hat{r}}\, A^{\hat{p}}{}_{j}\right) = -\frac{1}{2}\, \epsilon_{\hat{r}\hat{s}\hat{t}}\, J^{\hat{s}\hat{t}}\,. \tag{4.23}$$

where ${}^{A}\mathcal{D}_j$ is the derivative operator associated with the Ashtekar connection. Moreover, using the Ashtekar curvature,

$$\begin{aligned} F^{\hat{r}}{}_{jk} &\equiv 2D_{[j} A^{\hat{r}}{}_{k]} + \kappa\, \epsilon^{\hat{r}\hat{s}\hat{t}}\, A_{\hat{s}j}\, A_{\hat{t}k} \\ &= \frac{1}{\kappa}\left(2D_{[j}\omega^{\hat{r}}{}_{k]} + \epsilon^{\hat{r}\hat{s}\hat{t}}\, \omega_{\hat{s}j}\, \omega_{\hat{t}k}\right) - \kappa\, \epsilon^{\hat{r}\hat{s}\hat{t}}\, P_{\hat{s}j}\, P_{\hat{t}k} - 2\mathrm{i}\, E^{\hat{r}}{}_{i}\, D_{[j}\, P^{i}{}_{k]}\,. \end{aligned} \tag{4.24}$$

one can build the standard $\Sigma$ constraints:

$$\begin{aligned} \mathcal{C} &\equiv \frac{1}{2}\, \epsilon^{\hat{r}\hat{s}\hat{t}}\, \tilde{E}_{\hat{r}}{}^{i}\, \tilde{E}_{\hat{s}}{}^{j}\, F_{\hat{t}ij} = \tilde{\mathcal{H}} - \mathrm{i}\, D_j\left(\tilde{E}_{\hat{r}}{}^{j}\, J^{\hat{r}}\right) \\ \mathcal{C}_j &\equiv i\, \tilde{E}_{\hat{r}}{}^{i}\, F^{\hat{r}}{}_{ij} = \mathcal{H}_j - \mathrm{i}\, \kappa\, J_{\hat{r}}\, P^{\hat{r}}{}_{j}\,. \end{aligned} \tag{4.25}$$

With this machinery, one may rearrange terms in the expression (4.21) to find

$$S = \int_{\mathcal{M}} \mathrm{d}^4x \left\{\mathrm{i}\, A^{\hat{r}}{}_{j}\, \dot{\tilde{E}}_{\hat{r}}{}^{j} - \underset{\sim}{N}\, \mathcal{C} - V^j \mathcal{C}_j - \varphi^{\hat{r}}\, J_{\hat{r}}\right\} + \int_{\bar{T}} \mathrm{d}^3x \left[-\frac{\phi}{\kappa}\, \dot{\sqrt{\sigma}} - \underset{\sim}{\bar{N}}\, \bar{\mathcal{C}} - V^b \bar{\mathcal{C}}_b\right]\,. \tag{4.26}$$

The Lagrange multiplier associated with the Gauss constraint here has the explicit form

$$\varphi^{\hat{r}} = -1/2\, \epsilon^{\hat{r}\hat{s}\hat{p}}\, \phi_{\hat{s}\hat{p}} - \mathrm{i}\, \delta^{\hat{r}\hat{s}}\, \tilde{E}_{\hat{s}}{}^{j}\, D_j\, \underset{\sim}{N} + \mathrm{i}\kappa\, P^{\hat{r}}{}_{j}\, V^j\,. \tag{4.27}$$

Furthermore, now we have

$$\begin{aligned} \mathcal{C} &= \sqrt{h}\, \tilde{\mathcal{H}} - \mathrm{i}\, \gamma\, (\mathrm{d}r)_i\, \epsilon_{\hat{r}}{}^{\hat{s}\hat{p}}\, P^{\hat{r}}{}_{j}\, \tilde{E}_{\hat{s}}{}^{j}\, \tilde{E}_{\hat{p}}{}^{i} = \alpha\sigma\, (\gamma\, \varepsilon - v\gamma\, j_{\vdash}) \\ \bar{\mathcal{C}}_a &= \bar{\mathcal{H}}_a = -\sqrt{\sigma}\left[j_a - \frac{1}{\kappa}\, \delta_a \phi\right]\,, \end{aligned}$$

where $\alpha = h^{1/2}\sigma^{-1/2}$. (Again, $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}}_b$ are not constraints.) At this point the densities $\varepsilon$, $j_{\vdash}$, and $j_a$ have the same forms as in (3.22) and (3.27) but are constructed with the canonical Ashtekar connection. Therefore, off the constraint surface in phase space defined by the Gauss constraint, the energy density $\varepsilon$ is no longer manifestly real. Notice that $\mathcal{C}$ has been defined as a density of weight one, because it is paired with the boundary Lagrange multiplier $\bar{\underset{\sim}{N}}$, which we have taken as a density of weight minus one. We also remark that the kinematical torsion which is present in the Ashtekar connection modifies the boost relations. Therefore, for instance, it is not true that $\bar{\varepsilon} = \gamma\, \varepsilon - v\gamma\, j_{\vdash}$ in the canonical picture.

Before considering the variation of the action (4.26), we find it convenient to rewrite the Lagrange parameter $\varphi^{\hat{r}}$ in the following way. Take

$$\phi_{\hat{r}\hat{s}} = -\Gamma_{\hat{r}\hat{s}\perp} = -N\Gamma_{\hat{r}\hat{s}\perp} - \omega_{\hat{r}\hat{s}j}\, V^j\,, \tag{4.28}$$

and also write

$$-\mathrm{i}\,\delta^{\hat{r}\hat{s}}\tilde{E}_{\hat{s}}{}^{j}\,D_j\,\underset{\sim}{N} = -\mathrm{i}\,N\,a^{\hat{r}} = -\mathrm{i}\,N\Gamma^{\hat{r}}{}_{\perp\perp}\,, \tag{4.29}$$

where $a_{\hat{r}} = E_{\hat{r}}[\log N]$ are the triad components of the spacetime acceleration of $u$. With these relations one can set

$$\varphi^{\hat{r}} = \epsilon^{\hat{r}\hat{s}\hat{p}}\,N\Gamma^{(+)}{}_{\hat{s}\hat{p}\perp} - \kappa\,A^{\hat{r}}{}_j\,V^j\,, \tag{4.30}$$

which is, of course, essentially the well-known result that $\varphi^{\hat{r}} = \epsilon^{\hat{r}\hat{s}\hat{p}}\,\Gamma^{(+)}{}_{\hat{s}\hat{p}t} \equiv -A^{\hat{r}}{}_t$. We shall need the expression for $\varphi^{\vdash}$ when the radial gauge condition is enforced,

$$\varphi^{\vdash} = 2N\Gamma^{(+)}{}_{\hat{1}\hat{2}\perp} - \kappa\,A^{\vdash}{}_j\,V^j\,. \tag{4.31}$$

Using $V^j = V^{\vdash}\,n^j + \bar{V}^b\,\sigma^j_b$, one can put this result in the handy form

$$\varphi^{\vdash} = 2N\Gamma^{(+)}{}_{\hat{1}\hat{2}\perp} - \kappa\,A^{\hat{r}}{}_j\,E_{\hat{r}}{}^j\,V^{\vdash} + \mathrm{i}\kappa\left(j_{\vdash}\,V^{\vdash} + j_c\,\bar{V}^c\right)\,. \tag{4.32}$$

Direct calculation yields the following for the variation of the canonical action (4.26):

$$\begin{aligned}\delta S = {} & (\text{terms which give the constraints and equations of motion}) + \mathrm{i}\int_{t'}^{t''}\mathrm{d}^3x\,A^{\hat{r}}{}_j\,\delta\tilde{E}_{\hat{r}}{}^j \\ & - \int_{\mathcal{T}}\mathrm{d}^3x\,\sqrt{\sigma}\left[\bar{\mathcal{C}}\,\delta\bar{N} - \bar{\mathcal{C}}_b\,\delta\bar{V}^b - \bar{N}/2\left(\gamma\,s^{ab} - v\gamma\,t^{ab} + 1/\kappa\,\bar{u}[\phi]\,\sigma^{ab} + \Delta\,\sigma^{ab}\right)(\delta\theta)_{ab}\right] \\ & + \frac{1}{\kappa}\int_{\mathcal{T}}\mathrm{d}^3x\,\sqrt{\sigma}\left[\kappa\bar{N}\,(\gamma\,j_{\vdash} - v\gamma\,\varepsilon) + \delta_a\,\bar{V}^a - 1/2\,\sigma^{ab}\,\dot{\sigma}_{ab}\right]\delta\phi - \frac{1}{\kappa}\int_{B'}^{B''}\mathrm{d}^2x\,\phi\,\delta\sqrt{\sigma}\,,\end{aligned} \tag{4.33}$$

where here $\bar{N}$ is $h^{1/2}\,\underset{\sim}{N}$, and it would perhaps be better to express $(\delta\theta)_{ab}$ as a variation in terms of the densitized dyad (as is certainly possible). Also above,

$$\Delta = -\mathrm{i}\,A^{\hat{r}}{}_j\,E_{\hat{r}}{}^j\,V^{\vdash} - j_{\vdash}\,V^{\vdash} - j_c\,\bar{V}^c + \mathrm{i}/\kappa\left(2N\,\Gamma^{(+)}{}_{\hat{1}\hat{2}\perp} - \varphi^{\vdash}\right)\,. \tag{4.34}$$

With the interpretation (4.32) $\Delta$ vanishes. Modulo the Gauss contraint the boost relations (4.2) are valid, and, therefore, one finds that

$$\begin{aligned}(\delta S)_{\mathcal{T}} \approx {} & -\int_{\mathcal{T}}\mathrm{d}^3x\,\sqrt{\sigma}\left[\bar{\varepsilon}\,\delta\bar{N} - \bar{j}_{\vdash'}\,\delta\bar{V}^b - \bar{N}/2\;\bar{s}^{ab}\,(\delta\theta)_{ab}\right] \\ & + \frac{1}{\kappa}\int_{\mathcal{T}}\mathrm{d}^3x\,\sqrt{\sigma}\left(\kappa\bar{N}\,\bar{j}_{\vdash'} + \delta_a\,\bar{V}^a - 1/2\,\sigma^{ab}\,\dot{\sigma}_{ab}\right)\delta\phi - \frac{1}{\kappa}\int_{B'}^{B''}\mathrm{d}^2x\,\phi\,\delta\sqrt{\sigma}\,,\end{aligned} \tag{4.35}$$

where now one must again consider the quasilocal densities to be expressed in terms of the Sen connection. The density $\bar{j}_{\vdash'} = -1/\kappa\,\sigma^{ab}\,\bar{l}_{ab}$, and in the non-canonical picture $\bar{l}_{ab} = -1/2\bar{N}(\dot{\sigma}_{ab} + \delta_a\bar{V}_b + \delta_b\bar{V}_a)$, so the middle integral on the right-hand side vanishes in this case. With the $\bar{\mathcal{T}}$ Sen connection expression $\bar{j}_{\vdash'} = \epsilon^{\hat{a}\hat{c}}\,\bar{\mathcal{A}}_{\hat{a}b}\,\theta_{\hat{c}}{}^b$ and the results (3.21), we then have

$$(\delta S)_{\mathcal{T}} \approx \mathrm{i}\int_{\mathcal{T}}\mathrm{d}^3x\,\bar{\mathcal{A}}^{\hat{r}}{}_j\,\delta\left(\sqrt{-\bar{\gamma}}\,\bar{\xi}_{\hat{r}}{}^j\right) - \frac{1}{\kappa}\int_{B'}^{B''}\mathrm{d}^2x\,\phi\,\delta\sqrt{\sigma}\,, \tag{4.36}$$

in agreement with the variation (3.14) of the non-canonical action.

## V. DISCUSSION

We conclude by briefly commenting on several technical matters. These are the interpretation of the imaginary boundary term $-S^0$, the relationship of our formalism with the Sparling two-forms, and problems encountered in the attempt to extend the Brown-York notion of gravitational charge to the Ashtekar-variable construction.

As mentioned in the introduction, we have chosen to formally treat the imaginary boundary term $-S^0$ as a true subtraction term *ala* Brown and York. However, we now argue that in some contexts it is necessary to consider the freedom to append to the action $S = S^I - S^0$ an additional arbitrary subtraction term $-S^0_{general}$. In the interest of economy we restrict our argument to matters concerning the quasilocal energy surface density $\varepsilon$, though much of this discussion also pertains to the other quasilocal densities. Consider first the Brown-York expression

$$\varepsilon = \frac{1}{\kappa}\left(k - k^0\right) . \tag{5.1}$$

In the metric formalism, as in this paper, $k$ represents the trace of the extrinsic curvature of $B$ as embedded in $\Sigma$ and comes from an $S^I$ action in the derivation. The $k^0$ term represents the trace of the extrinsic curvature of a two-surface which has the same metric as $B$ but which is uniquely embedded in a three-dimensional manifold different than $\Sigma$. In the Brown-York formalism it arises from a subtraction-term $-S^0_{general}$ contribution to the action, where now $-S^0_{general}$ is real and unspecified. When possible, $-S^0_{general}$ is typically chosen such that the different three-space is $R^3$, and hence the $k^0$ term references the energy against flat-space. For a given asymptotically-flat spacetime, the presence of the appropriate $k^0$ term is crucial if the quasilocal energy,

$$E = \frac{1}{\kappa}\int_B \mathrm{d}^2x\,\sqrt{\sigma}\left(k - k^0\right) , \tag{5.2}$$

is to agree with the ADM notion of energy in the suitable limit. [2,19]

Though we have added the imaginary boundary term $-S^0$ term to $S^I$ in this paper, the resulting quasilocal energy,

$$E = \int_B \mathrm{d}^2x\,\sqrt{\sigma}\,\epsilon^{\hat{a}\hat{c}}\,A_{\hat{a}b}\,\theta_{\hat{c}}{}^b \approx \frac{1}{\kappa}\int_B \mathrm{d}^2x\,\sqrt{\sigma}\,k\,, \tag{5.3}$$

is really only the "unreferenced" energy. If we wish to put the the Ashtekar-variable expression for the quasilocal energy into fuller accord with the ADM notion of energy, then we should allow for the freedom to append to the action *yet another* subtraction term $-S^0_{general}$. Use of the larger action $S' = S - S^0 - S^0_{general}$ in our analysis would yield

$$E = \int_B \mathrm{d}^2x\,\sqrt{\sigma}\left(\epsilon^{\hat{a}\hat{c}}\,A_{\hat{a}b}\,\theta_{\hat{c}}{}^b - \epsilon^{\hat{a}\hat{c}}\,A^0_{\hat{a}b}\,\theta_{\hat{c}}{}^b\right) \tag{5.4}$$

for the Ashtekar-variable quasilocal energy. The new reference-point contribution $\epsilon^{\hat{a}\hat{c}}\,A^0_{\hat{a}b}\,\theta_{\hat{c}}{}^b$ stems from $-S^0_{general}$. In this case $-S^0_{general}$ is an arbitrary functional of $\bar{T}$ data. With this new freedom, we could define the quasilocal energy in such a way that it agreed with the ADM expression in the suitable limit for a particular asymptotically-flat spacetime.

As is well-known, the the real and complex Sparling two-forms obey the *Sparling relation*

$$\mathrm{d}\sigma_{\hat{\mu}} = \mathrm{d}\sigma^{(+)}{}_{\hat{\mu}} = \tau_{\hat{\mu}} + G_{\hat{\mu}}{}^{\hat{\sigma}}\, e^{*}_{\hat{\sigma}}\,, \tag{5.5}$$

where $e^{*}_{\hat{\sigma}}$ is a basis for three-forms, $G_{\hat{\mu}}{}^{\hat{\sigma}}$ is the Einstein tensor, and $\tau_{\hat{\mu}}$ are the *Sparling three-forms*. The explicit form for $\tau_{\hat{\mu}}$ (which is real) is not needed here but may be found in, for example, Ref. [15]. The Sparling relation suggests that $\tau_{\hat{\mu}}$ (when pulled-backed to a three-dimensional slice $\Sigma$ of spacetime) may be interpreted as a tetrad-dependent local energy-momentum density for the gravitational field. [15,5] The corresponding frame-dependent potential can be taken either as $\sigma_{\hat{\mu}}$ or $\sigma^{(+)}{}_{\hat{\mu}}$. If we fix a two-surface $B$ and its spanning three-slice $\Sigma$ in spacetime, then the boundary structure of our selection provides a natural (almost-unique) frame at $B$. Namely, the radial time-gauge tetrad of *Appendix A*, which has the $\Sigma$ hypersurface normal $u$ as its time leg and $n$, the normal of $B$ in $\Sigma$, as its third space leg. With this frame choice, the pullbacks $s^{*}(\sigma^{(+)}{}_{\hat{\mu}})$ to $B$ ($s$ is the inclusion mapping $s : B \to \mathcal{M}$ and $\hat{\mu}$ runs over $(\perp, \vdash, \hat{a})$) are precisely the densities $-\kappa\,\varepsilon\,\sqrt{\sigma}\,\mathrm{d}^2x$, $\kappa\, j_{\vdash}\,\sqrt{\sigma}\,\mathrm{d}^2x$, and $\kappa\,\theta_{\hat{a}}{}^{b}\, j_b\,\sqrt{\sigma}\,\mathrm{d}^2x$ (here expressed in terms of the $\Sigma$ Sen connection and triad). Further, the pullbacks $s^{*}(\sigma_{\hat{\mu}})$ of the real Sparling two-forms are the real parts $-\kappa\,\varepsilon^{I}\,\sqrt{\sigma}\,\mathrm{d}^2x$, $\kappa\,(j^{I})_{\vdash}\,\sqrt{\sigma}\,\mathrm{d}^2x$, and $\kappa\,\theta_{\hat{a}}{}^{b}\,(j^{I})_b\,\sqrt{\sigma}\,\mathrm{d}^2x\;\sigma_{\hat{\mu}}$ of these quasilocal densities.

The Brown-York notion of gravitational charge is based on the $\bar{\mathcal{T}}$ momentum constraint,

$$-2\,\bar{\mathcal{D}}_i\left(\bar{\pi}^{i}{}_{j} - (\bar{\pi}^{0})^{i}{}_{j}\right) = -1/\kappa\,\sqrt{-\bar{\gamma}}\,\bar{\gamma}^{\mu}_{j}\,\bar{n}^{\lambda}\,G_{\mu\lambda}\,, \tag{5.6}$$

where $\bar{\pi}^{ij}$ is given in (3.7) and, in the metric formalism, $(\bar{\pi}^{0})^{ij} \equiv \delta S^{0}_{general}/\delta\bar{\gamma}_{ij}$ depends only on $\bar{\gamma}_{ij}$ (and so it annihilated by $\bar{\mathcal{D}}_i$). Now we work on-shell and in vacuum so this expression vanishes. Brown and York define a "stress tensor" $\bar{\tau}^{ij} = 2/\sqrt{-\bar{\gamma}}\,(\bar{\pi}^{ij} - (\bar{\pi}^{0})^{ij})$. Assume that $\bar{\mathcal{T}}$ possesses a Killing field $\zeta^{j}$, and so $\bar{\mathcal{D}}_i\,\bar{\tau}^{ij}\,\zeta_j = 0$. Therefore, since $-\bar{u}_i\,\tau^{ij} = \bar{\varepsilon}\,\bar{u}^{j} + \bar{j}_b\,\sigma^{bj}$, one has the following conserved charge: [2]

$$Q_{\zeta}(B) = \int_B \mathrm{d}^2x\,\sqrt{\sigma}\left(\bar{\varepsilon}\,\bar{u}^{j} + \bar{j}_b\,\sigma^{bj}\right)\zeta_j\,. \tag{5.7}$$

When attempting to introduce such a notion of charge into our formalism, we run into some difficulty since our subtraction term $S^{0}$ is triad-dependent. The natural way around this difficulty is the following. First define

$$(\bar{\Pi}^{0})^{\hat{r}}{}_{j} = \delta S^{0}/\delta\left(\sqrt{-\bar{\gamma}}\,\bar{\xi}_{\hat{r}}{}^{j}\right)\,. \tag{5.8}$$

In our situation $(\bar{\Pi}^{0})_{ij} = \bar{\xi}_{\hat{r}i}\,(\bar{\Pi}^{0})^{\hat{r}}{}_{j}$ is not annihilated by $\bar{\mathcal{D}}_j$, though $\bar{\mathcal{D}}_j\,(\bar{\Pi}^{0})_{(ik)} = 0$. Therefore, set $(\bar{\pi}^{0})^{ij} = \sqrt{-\bar{\gamma}}/2\kappa\left((\bar{\Pi}^{0})^{k}{}_{k}\,\bar{\gamma}^{ij} - (\bar{\Pi}^{0})^{(ij)}\right)$ and use it in the above construction. The charge $Q_{\zeta}$ may now be imaginary, but, subject to the assumptions made above, it is conserved.

## VI. ACKNOWLEDGMENTS

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## APPENDIX: KINEMATICAL FRAMEWORK

*Appendices A*, *B*, and *C* outline a kinematical framework for examining how the intrinsic and extrinsic geometry of spacetime as foliated by a family $\bar{\mathcal{T}}$ hypersurfaces is related to the intrinsic and extrinsic geometry of spacetime as foliated by a family of $\Sigma$ hypersurfaces. With this framework one can express objects such as the $\bar{\mathcal{T}}$ extrinsic curvature $\bar{\Theta}_{ij}$ or the time-gauge $\bar{\mathcal{T}}$ Sen connection $\bar{\mathcal{A}}^{\hat{r}}{}_j$ in terms of the intrinsic and extrinsic geometry of spacetime as foliated by $\Sigma$ hypersurfaces. Such a "splitting" of the $\bar{\mathcal{T}}$ Sen connection is needed in order to derive a similar splitting of the $\bar{\Sigma}$ radial-gauge Sen connection $\bar{A}^{\hat{r}}{}_j$ in terms of the geometry of the $\Sigma$ foliation. The splitting of $\bar{A}^{\hat{r}}{}_j$ is used to obtain the boost laws (4.2) for the quasilocal densities. The kinematical framework consists of (i) two distinct spacetime tetrads (one adapted to the $\Sigma$ foliation and one adapted to the $\bar{\mathcal{T}}$ foliation), (ii) the transformation equations between these tetrads, and (iii) the inhomogeneous transformation law between the sets of associated connection coefficients. The relevant spacetime tetrads are constructed in *Appendix A*, and their associated connection coefficients are tabulated in *Appendix B*. *Appendix C* outlines the splitting procedure by applying it to $\bar{\Theta}_{ij}$, the simplest example. We then quote the splitting results for the $\bar{\mathcal{T}}$ time-gauge connection coefficients $\bar{\tau}_{\hat{r}\hat{s}j}$, $\bar{\mathcal{A}}^{\hat{r}}{}_j$, and $\bar{A}^{\hat{r}}{}_j$. The final *Appendix D* applies some of this formalism to explain the origin of the corner terms in the action (3.5).

## APPENDIX A: ADAPTED TETRADS

The boundary structure of $\mathcal{M}$ suggests two natural classes of spacetime tetrads. The first class is a subclass of time-gauge tetrads determined by the boundary structure of $\Sigma$. The second class is a subclass of "radial-gauge" tetrads determined by the $B$ foliation of $\bar{\mathcal{T}}$. It should be mentioned that these tetrads need only be defined on some small spacetime neighborhood surrounding $\bar{\mathcal{T}}$. We do not address the issue of whether or not either of these tetrads can be extended globally over all of $\mathcal{M}$.

### 1. RT-gauge tetrads

Enforcement of the *time gauge* condition locks the time leg of the tetrad to the $\Sigma$ foliation normal $u$. This condition is indicated by replacing the tetrad time label $\hat{0}$ with $\perp$ so that $e_\perp = u$. Because each $\Sigma$ slice has a boundary $B$, a natural subclass of all time-gauge tetrads exists which is determined by an auxiliary condition on $\bar{\mathcal{T}}$. This further requirement is that *on the three-boundary* $\bar{\mathcal{T}}$ one of the space legs of the tetrad, chosen to be $e_\vdash \equiv e_{\hat{3}}$, coincides with $n$. One should note that this correspondence is not made between $e_{\hat{3}}$ and $\bar{n}$ in general. Such a choice of tetrad is said to obey the *radial time-gauge* or *RT-gauge*. RT-gauge indices and labels take the values $(\perp, \hat{1}, \hat{2}, \vdash)$. Now the usual assumption is that the vector field $\partial/\partial t$ points everywhere tangent to the hypersheets of constant $r$. Equivalently, $\langle dr, \partial/\partial t\rangle = 0$ or $\partial r/\partial t = 0$, and the $r$ coordinate is Lie transported along the integral curves of the time vector field. This assumption results in almost no loss of physical generality. It does demand that the integral curves of the time vector field may not emerge from or flow into the three-boundary $\bar{\mathcal{T}}$. However, since the spacetime-filling extension of the three-boundary

$\bar{\mathcal{T}}$ in terms of hypersheets of constant $r$ is completely arbitrary, on the interior of $\Sigma$ these integral curves can be chosen to flow in any direction (as long as the tangent field $\partial/\partial t$ lies at each point within the future light-cone). Subject to the requirement $\partial r/\partial t = 0$, one can write the most general *radial vector field* mapped to unity by d$r$ as $\partial/\partial r = \alpha\, n + \beta$, which is similar to the familiar formula $\partial/\partial t = N\, u + V$. As seen earlier, the variables $\alpha$ and $\beta^a$ are respectively the kinematical "lapse" and "shift" associated with the induced radial foliation of the $\Sigma$ slices. Therefore, we have the following explicit formulae for the RT-gauge tetrad and cotetrad:

$$
\begin{aligned}
e_{\perp} = u = \frac{1}{N}\left(\frac{\partial}{\partial t} - V^{\hat{a}}\, E_{\hat{a}} - V^{\vdash}\, E_{\vdash}\right) \qquad & e^{\perp} = N\mathrm{d}t \\
e_{\hat{a}} = E_{\hat{a}} = \theta_{\hat{a}} \qquad & e^{\hat{a}} = \theta^{\hat{a}} + V^{\hat{a}}\mathrm{d}t + \beta^{\hat{a}}\mathrm{d}r \\
e_{\vdash} = n = E_{\vdash} = \frac{1}{\alpha}\left(\frac{\partial}{\partial r} - \beta^{\hat{a}}\, \theta_{\hat{a}}\right) \qquad & e^{\vdash} = \alpha\, \mathrm{d}r + V^{\vdash}\mathrm{d}t\,.
\end{aligned}
\tag{A1}
$$

### 2. Radial-gauge tetrads

The *radial-gauge condition* requires that one of the space legs of the tetrad, taken to be $e_{\vdash'} \equiv e_{\hat{3}}$ , coincides with the $\bar{\mathcal{T}}$ normal $\bar{n}$. A natural further requirement can be placed on radial-gauge tetrads. Namely, the time leg $e_{\perp'} \equiv e_{\hat{0}}$ can be tied to the $B$ timelike normal $\bar{u}$, so the indices and labels associated with this class of tetrads run over $(\perp', \hat{1}, \hat{2}, \vdash')$. Such a tetrad is referred as *time radial-gauge* or *TR-gauge*. Now the radial vector field is written as $\partial/\partial r = \bar{\alpha}\, \bar{n} + \bar{\beta}$, though it still points tangent to the $\Sigma$ slices. The variables $\bar{\alpha}$ and $\bar{\beta}^{\alpha}$ are associated with the $\bar{\mathcal{T}}$ foliation of $\mathcal{M}$. On $\bar{\mathcal{T}}$ one can express the time vector field as $\partial/\partial t = \bar{N}\bar{u} + \bar{V}$, where $\bar{N}$ and $\bar{V}^a$ are the gauge variables associated with the $B$ foliation of $\bar{\mathcal{T}}$. The RT-gauge tetrad and cotetrad is

$$
\begin{aligned}
e_{\perp'} = \bar{u} = \bar{\xi}_{\perp'} = \frac{1}{\bar{N}}\left(\frac{\partial}{\partial t} - \bar{V}^{\hat{a}}\, \theta_{\hat{a}}\right) \qquad & e^{\perp'} = \bar{N}\mathrm{d}t + \bar{\beta}^{\perp'}\mathrm{d}r \\
e_{\hat{a}} = \bar{\xi}_{\hat{a}} = \theta_{\hat{a}} \qquad & e^{\hat{a}} = \theta^{\hat{a}} + \bar{V}^{\hat{a}}\mathrm{d}t + \bar{\beta}^{\hat{a}}\mathrm{d}r \\
e_{\vdash'} = \bar{n} = \frac{1}{\bar{\alpha}}\left(\frac{\partial}{\partial r} - \bar{\beta}^{\hat{a}}\, \bar{\xi}_{\hat{a}} - \bar{\beta}^{\perp'}\, \bar{\xi}_{\perp'}\right) \qquad & e^{\vdash'} = \bar{\alpha}\mathrm{d}r\,.
\end{aligned}
\tag{A2}
$$

## APPENDIX B: ASSOCIATED CONNECTION COEFFCIENTS

For the special tetrads considered above, certain of the corresponding connection coefficients have notable geometric meanings. This subsection is a glossary of various connection coefficients and their geometric interpretations. RT-gauge connection coefficients are represented as $\Gamma^{\hat{\rho}}{}_{\hat{\sigma}\hat{\mu}}$, while TR-gauge connection coefficients are represented by $\Gamma^{\hat{\rho}'}{}_{\hat{\sigma}'\hat{\mu}'}$. Note

that inspection of the indices allows one to discern which set of connection coefficients is being dealt with. In following lists, since the geometry of $\mathcal{M}$ is torsion-free (*i.e.* the torsion two-form of Cartan vanishes [16]), all of the extrinsic curvature tensors are symmetric.

### 1. RT-gauge connection coefficients

The RT-gauge connection coefficients are tailored to $B$ as embedded in $\Sigma$. We have the following correspondences:

$$
\begin{aligned}
K^{\hat{s}}{}_{\hat{r}} &\equiv -e^{\hat{s}}{}_{\mu}\, e_{\hat{r}}{}^{\nu}\, \nabla_{\nu}\, e_{\perp}{}^{\mu} = -\,\Gamma^{\hat{s}}{}_{\perp\hat{r}} \\
a^{\hat{r}} &\equiv e^{\hat{r}}{}_{\mu}\, e_{\perp}{}^{\nu}\, \nabla_{\nu}\, e_{\perp}{}^{\mu} = \Gamma^{\hat{r}}{}_{\perp\perp} \\
k^{\hat{a}}{}_{\hat{c}} &\equiv -e^{\hat{a}}{}_{\mu}\, e_{\hat{c}}{}^{\nu}\, \nabla_{\nu}\, e_{\vdash}{}^{\mu} = -\Gamma^{\hat{a}}{}_{\vdash\hat{c}} = -\omega^{\hat{a}}{}_{\vdash\hat{c}} \\
b^{\hat{r}} &\equiv e^{\hat{r}}{}_{\mu}\, e_{\vdash}{}^{\nu}\, \nabla_{\nu}\, e_{\vdash}{}^{\mu} = \Gamma^{\hat{r}}{}_{\vdash\vdash}\,,
\end{aligned}
\tag{B1}
$$

Note that the formulas for $K^{\hat{r}}{}_{\hat{s}}$ and $a^{\hat{r}}$ are general time-gauge expressions. Also, $b^{\hat{r}}$ are the tetrad components of the spacetime "acceleration" of $n$, while the $\Sigma$ "acceleration" of $n$ has componets $b^{\hat{a}}$. For $b^{\hat{r}}$ the $\hat{r}$ is a $\mathcal{T}$ index and can take the values $(\perp, \hat{a})$, while for $K^{\hat{r}}{}_{\hat{s}}$ and $a^{\hat{r}}$ the $\hat{r}$ and $\hat{s}$ are $\Sigma$ indices taking the values $(\hat{1}, \hat{2}, \vdash)$.

### 2. TR-gauge connection coefficients

The TR-gauge connection coefficients are tailored to $B$ as embedded in $\bar{\mathcal{T}}$. We have the following correspondences:

$$
\begin{aligned}
\bar{\Theta}^{\hat{r}}{}_{\hat{s}} &\equiv -e^{\hat{r}}{}_{\mu}\, e_{\hat{s}}{}^{\nu}\, \nabla_{\nu}\, e_{\vdash'}{}^{\mu} = -\Gamma^{\hat{r}}{}_{\vdash'\hat{s}} \\
\bar{b}^{\hat{r}} &\equiv e^{\hat{r}}{}_{\mu}\, e_{\vdash'}{}^{\nu}\, \nabla_{\nu}\, e_{\vdash'}{}^{\mu} = \bar{\Gamma}^{\hat{r}}{}_{\vdash'\vdash'} \\
\bar{l}^{\hat{a}}{}_{\hat{c}} &\equiv -e^{\hat{a}}{}_{\mu}\, e_{\hat{c}}{}^{\nu}\, \nabla_{\nu}\, e_{\perp'}{}^{\mu} = -\Gamma^{\hat{a}}{}_{\perp'\hat{c}} = -\bar{\tau}^{\hat{a}}{}_{\perp'\hat{c}} \\
\bar{a}^{\hat{r}} &\equiv e^{\hat{r}}{}_{\mu}\, e_{\perp'}{}^{\nu}\, \nabla_{\nu}\, e_{\perp'}{}^{\mu} = \Gamma^{\hat{r}}{}_{\perp'\perp'}\,.
\end{aligned}
\tag{B2}
$$

Like before, the formulas for $\bar{\Theta}^{\hat{r}}{}_{\hat{s}}$ and $\bar{b}^{\hat{r}}$ are general radial-gauge expressions. For $\bar{\Theta}^{\hat{r}}{}_{\hat{s}}$ and $\bar{b}^{\hat{r}}$ in this list the $\hat{r}$ and $\hat{s}$ are $\bar{\mathcal{T}}$ indices taking the values $(\perp', \hat{a})$. The $\bar{a}^{\hat{r}}$ ($\hat{r}$ can take the values $(\hat{a}, \vdash')$) are the tetrad components of the spacetime acceleration of $\bar{u}$, while the $\bar{\mathcal{T}}$ acceleration of $\bar{u}$ has componets $\bar{a}^{\hat{c}} = \bar{\tau}^{\hat{c}}{}_{\perp'\perp'}$.

## APPENDIX C: SPLITTING PROCEDURE

### 1. Transformation equations

The set (2.6) of transformations for the metric variables can be used to express the transformations between the RT-gauge tetrad (A1) and the TR-gauge tetrad (A2). For example,

$$\begin{aligned} e_{\perp'} &= \frac{1}{\bar{N}}\left(\frac{\partial}{\partial t} - \bar{V}^b\,\theta_b\right) \\ &= \frac{\gamma}{N}\left(\frac{\partial}{\partial t} - V^b\,\theta_b - V^{\mathrm{r}}\,\beta^b\,\theta_b\right) \\ &= \frac{\gamma}{N}\left(\frac{\partial}{\partial t} - V^b\,\theta_b - V^{\mathrm{r}}\,\frac{\partial}{\partial r} + V^{\mathrm{r}}\,\frac{\partial}{\partial r} - V^{\mathrm{r}}\,\beta^b\,\theta_b\right) \\ &= \gamma\, e_{\perp} + v\gamma\, e_{\vdash}\,. \end{aligned} \tag{C1}$$

The complete set of transformations is

$$\begin{aligned} e_{\perp'} &= \gamma\, e_{\perp} + v\gamma\, e_{\vdash} & e^{\perp'} &= \gamma\, e^{\perp} - v\gamma\, e^{\vdash} \\ e_{\vdash'} &= v\gamma\, e_{\perp} + \gamma\, e_{\vdash} & e^{\vdash'} &= -v\gamma\, e^{\perp} + \gamma\, e^{\vdash}\ . \\ e_{\hat{a}} &= e_{\hat{a}} & e^{\hat{a}} &= e^{\hat{a}} \end{aligned} \tag{C2}$$

Notice that the $B$ legs of both the tetrads are the same, which is why the notation can be compressed so that TR-gauge tetrad indices like $\hat{\rho}'$ run over $(\perp', \hat{a}, \vdash')$.

The inhomogeneous transformation rule describing the behavior of the spacetime connection coefficients under the above tetrad transformation is the following:

$$\Gamma^{\hat{\rho}'}{}_{\hat{\sigma}'\hat{\tau}'} = e^{\hat{\rho}'}{}_{\hat{\sigma}}\,\Gamma^{\hat{\sigma}}{}_{\hat{\rho}\hat{\mu}}\,e_{\hat{\sigma}'}{}^{\hat{\rho}}\,e_{\hat{\tau}'}{}^{\hat{\mu}} + e^{\hat{\rho}'}{}_{\hat{\sigma}}\,e_{\hat{\tau}'}{}^{\hat{\mu}}\,e_{\hat{\mu}}\left[e_{\hat{\sigma}'}{}^{\hat{\sigma}}\right]\ . \tag{C3}$$

This law provides the bridge between the TR-gauge connection coefficients (B2) and the RT-gauge connection coefficients (B1).

### 2. Geometric link between $\bar{\mathcal{T}}$ and $\Sigma$

As an example, we apply the developed formalism and derive the splitting result for the three-boundary extrinsic curvature $\bar{\Theta}_{ij}$. This result has been obtained via ordinary tensor methods with projection operators in Ref. [10]. However, the ordinary projection-operator method is not sufficient for calculating the analogous split of the $\bar{\mathcal{T}}$ Sen connection. We provide the splitting calculation for $\bar{\Theta}_{ij}$ here as a simple demonstration of how such calculations are performed. Beginning with the first expression of (B2), one uses the rule (C3) in tandem with the set (C2) to find

$$\bar{\Theta}^{\hat{r}}{}_{\hat{s}} = -e^{\hat{r}}{}_{\hat{\sigma}}\,e_{\hat{s}}{}^{\hat{\mu}}\left(v\gamma\,\Gamma^{\hat{\sigma}}{}_{\perp\hat{\mu}} + \gamma\,\Gamma^{\hat{\sigma}}{}_{\vdash\hat{\mu}}\right) - e^{\hat{r}}{}_{\perp}\,e_{\hat{s}}\,[v\gamma] - e^{\hat{r}}{}_{\vdash}\,e_{\hat{s}}\,[\gamma]\ . \tag{C4}$$

(Note that in this equation $\hat{r}$ and $\hat{s}$ are $\bar{\mathcal{T}}$ triad indices which take the values $(\perp', \hat{a})$.) A bit of work and the relations (B1) yield the set of $\bar{\Theta}^{\hat{r}}{}_{\hat{s}}$ triad componets,

$$
\begin{aligned}
\bar{\Theta}^{\perp'}{}_{\perp'} &= -\gamma\, a^{\vdash} + v\gamma\, K^{\vdash}{}_{\vdash} - \gamma^3\, e_{\perp}[v] - v\gamma^3\, e_{\vdash}[v] \\
\bar{\Theta}^{\perp'}{}_{\hat{a}} &= K^{\vdash}{}_{\hat{a}} - \gamma^2\, e_{\hat{a}}[v] \\
\bar{\Theta}^{\hat{a}}{}_{\hat{c}} &= \gamma\, k^{\hat{a}}{}_{\hat{c}} + v\gamma\, K^{\hat{a}}{}_{\hat{c}}\,.
\end{aligned}
\tag{C5}
$$

With the set (C5), construction of the sought-for splitting of $\bar{\Theta}_{ij}$ is not difficult. For convenience work in spacetime coordinates. The boundary three-metric may be written as

$$
\bar{\gamma}_{\mu\nu} = \sigma_{\mu\nu} - \bar{u}_{\mu}\, \bar{u}_{\nu}\,, \tag{C6}
$$

where the two-metric $\sigma_{\mu\nu} = g_{\mu\nu} - \bar{n}_{\mu}\, \bar{n}_{\nu} + \bar{u}_{\mu}\, \bar{u}_{\nu}$ here serves as the projection operator into the $B$ slices. Wiring the above form of $\bar{\gamma}^{\mu}_{\nu}$, the identity operator on $\bar{\mathcal{T}}$, on each of the free indices of $\bar{\Theta}_{\mu\nu}$, one obtains

$$
\bar{\Theta}_{\mu\nu} = \bar{u}_{\mu}\, \bar{u}_{\nu}\, \bar{\Theta}_{\perp'\perp'} - 2\, \bar{u}_{(\mu}\, \sigma^{\lambda}_{\nu)}\, \bar{\Theta}_{\perp'\lambda} + \sigma^{\lambda}_{\mu} \sigma^{\kappa}_{\nu}\, \bar{\Theta}_{\lambda\kappa}\,, \tag{C7}
$$

where an appeal to the symmetry of $\bar{\Theta}_{\mu\nu}$ has been made. Plugging $\bar{u}_{\mu} = \gamma\, u_{\mu} + v\gamma\, n_{\mu}$ and the results from (C5) into (C7), one arrives at the following split of the three-boundary extrinsic curvature:

$$
\begin{aligned}
\bar{\Theta}_{\mu\nu} = {} & \gamma\, k_{\mu\nu} + v\gamma\, K_{ij}\, \sigma^{i}_{\mu}\, \sigma^{j}_{\nu} \\
& + \left\{ \gamma^2\, u_{\mu}\, u_{\nu} + 2\, v\gamma^2\, u_{(\mu}\, n_{\nu)} + (v\gamma)^2\, n_{\mu}\, n_{\nu} \right\} \\
& \times \left\{ \gamma\, n^{i}\, a_{i} - v\gamma\, n^{i}\, n^{j}\, K_{ij} + \gamma^3\, u[v] + v\gamma^3\, n[v] \right\} \\
& + 2\, \left\{ \gamma\, u_{(\mu}\, \sigma^{i}_{\nu)} + v\gamma\, n_{(\mu}\, \sigma^{i}_{\nu)} \right\} \left\{ n^{j}\, K_{ij} - \gamma^2\, D_{i} v \right\}
\end{aligned}
\tag{C8}
$$

Enforcement of the clamping condition $v \to 0$ recovers equation (A.16) of Ref. [2],

$$
\Theta_{\alpha\beta} = k_{\alpha\beta} + u_{\alpha}\, u_{\beta}\, n^{i}\, a_{i} + 2\, u_{(\alpha}\, \sigma^{i}_{\beta)}\, n^{j}\, K_{ij}\,. \tag{C9}
$$

The set of $\bar{\mathcal{T}}$ time-gauge connection coeffcients is $\left\{ \bar{\tau}^{\hat{a}}{}_{\perp'\perp'}\,,\ \bar{\tau}^{\hat{a}}{}_{\perp'\hat{c}}\,,\ \bar{\tau}^{\hat{a}}{}_{\hat{c}\perp'}\,,\ \bar{\tau}^{\hat{a}}{}_{\hat{c}\hat{b}} \right\}$, where the first two have been considered in the set (B2). The spittings of these expressions are

$$
\begin{aligned}
\sigma^{\rho}_{\mu}\, \bar{a}_{\rho} &= \gamma^2\, \sigma^{\rho}_{\mu}\, a_{\rho} + (v\gamma)^2\, b_{\mu} - v\gamma^2\, \sigma^{\tau}_{\mu}\, n^{\rho}\, K_{\tau\rho} + v\gamma^2\, \sigma^{\tau}_{\mu}\, u^{\lambda}\, \nabla_{\lambda}\, n_{\tau} \\
\bar{l}_{\mu\nu} &= \gamma\, K_{\tau\rho}\, \sigma^{\tau}_{\mu}\, \sigma^{\rho}_{\nu} + v\gamma\, k_{\mu\nu} \\
\bar{\tau}_{\hat{a}\hat{c}\perp'} &= \gamma\, \tau_{\hat{a}\hat{c}\perp} + v\gamma\, \omega_{\hat{a}\hat{c}\vdash} \\
\bar{\tau}_{\hat{a}\hat{c}b} &= \omega_{\hat{a}\hat{c}b}\,.
\end{aligned}
\tag{C10}
$$

Using this set and (C8), one finds the following split of the time-gauge $\bar{\mathcal{T}}$ Sen connection in terms of the radial-gauge $\Sigma$ Sen connection and other gauge variables:

$$
\begin{aligned}
\bar{\mathcal{A}}^{\perp'}{}_{\mu} = {} & (\gamma u_{\mu} + v\gamma n_{\mu}) \left( v\gamma\, A^{\vdash}{}_{j}\, n^{j} - (2\gamma/\kappa) \Gamma_{\hat{1}\hat{2}\perp} + (\mathrm{i}\gamma^3/\kappa)\, u[v] + (\mathrm{i}v\gamma^3/\kappa)\, n[v] \right) \\
& - \sigma^{\lambda}_{\mu} \left( A^{\vdash}{}_{\lambda} + (i\,\gamma^2/\kappa)\, \nabla_{\lambda} v \right) \\
\bar{\mathcal{A}}^{\hat{a}}{}_{\mu} = {} & (\gamma u_{\mu} + v\gamma n_{\mu}) \left( v\gamma^2 A^{\hat{a}}{}_{j}\, n^{j} + \mathrm{i}v\gamma^2 \epsilon^{\hat{a}\hat{c}} A_{\hat{c}j}\, n^{j} + (\mathrm{i}2\gamma^2/\kappa) \Gamma^{(+)\hat{a}}{}_{\vdash\perp} - (2v\gamma^2/\kappa) \epsilon^{\hat{a}\hat{c}}\, \Gamma^{(+)}{}_{\hat{c}\vdash\perp} \right) \\
& - \sigma^{\lambda}_{\mu} \left( v\gamma\, A^{\hat{a}}{}_{\lambda} + i\,\gamma\, \epsilon^{\hat{a}}{}_{\hat{c}}\, A^{\hat{c}}{}_{\lambda} \right)\,,
\end{aligned}
\tag{C11}
$$

where explicity one has $2\Gamma^{(+)}{}_{\hat{1}\hat{2}\perp} = \tau_{\hat{1}\hat{2}\perp} - \mathrm{i}\, a^j\, n_j$ and $2\Gamma^{(+)}{}_{\hat{a}\vdash\perp} = \Gamma_{\hat{a}\vdash\perp} + \mathrm{i}\, \epsilon_{\hat{a}\hat{c}}\, a^{\hat{c}}$. Taking the $v \to 0$ limit, one finds the clamped result

$$\begin{aligned} \mathcal{A}^{\perp}{}_{\mu} &= -u_\mu\,(2/\kappa)\,\Gamma_{\hat{1}\hat{2}\perp} - \sigma^\lambda_\mu\, A^{\vdash}{}_\lambda \\ \mathcal{A}^{\hat{a}}{}_{\mu} &= u_\mu\,(\mathrm{i}2/\kappa)\,\Gamma^{(+)\hat{a}}{}_{\vdash\perp} - \mathrm{i}\,\sigma^\lambda_\mu\,\epsilon^{\hat{a}}{}_{\hat{c}}\, A^{\hat{c}}{}_\lambda\,, \end{aligned} \tag{C12}$$

To find the splitting of the radial-gauge $\bar{\Sigma}$ Sen connection in terms of the radial-gauge $\Sigma$ Sen connection and other gauge variables, first find the split of $\bar{A}^{\hat{r}}{}_j$ in terms of the $\bar{\mathcal{T}}$ foliation variables,

$$\begin{aligned} \bar{A}^{\vdash'}{}_{\mu} &= -\bar{n}_\mu\,(2/\kappa)\,\Gamma^{(+)}{}_{\hat{1}\hat{2}\vdash'} - \sigma^\lambda_\mu\,\bar{\mathcal{A}}^{\perp'}{}_\lambda \\ \bar{A}^{\hat{a}}{}_{\mu} &= \bar{n}_\mu\,(\mathrm{i}\,2/\kappa)\,\Gamma^{(+)\hat{a}}{}_{\perp'\vdash'} - \mathrm{i}\,\sigma^\lambda_\mu\,\epsilon^{\hat{a}\hat{c}}\,\bar{\mathcal{A}}_{\hat{c}\lambda}\,, \end{aligned} \tag{C13}$$

where $2\,\Gamma^{(+)}{}_{\hat{1}\hat{2}\vdash'} = \bar{\omega}_{\hat{1}\hat{2}\vdash'} + \mathrm{i}\,\bar{b}_j\,\bar{u}^j$ and $2\,\Gamma^{(+)\hat{a}}{}_{\perp'\vdash'} = \Gamma^{\hat{a}}{}_{\perp'\vdash'} + \mathrm{i}\,\epsilon^{\hat{a}\hat{c}}\,\hat{b}_{\hat{c}}$. Combination of this result with (C11) gives

$$\begin{aligned} \bar{A}^{\vdash'}{}_{\mu} &= -\bar{n}_\mu\,(2/\kappa)\,\Gamma^{(+)}{}_{\hat{1}\hat{2}\vdash'} + \sigma^\lambda_\mu\,\left(A^{\vdash}{}_\lambda + (i\,\gamma^2/\kappa)\,\nabla_\lambda v\right) \\ \bar{A}^{\hat{a}}{}_{\mu} &= \bar{n}_\mu\,(\mathrm{i}\,2/\kappa)\,\Gamma^{(+)\hat{a}}{}_{\perp'\vdash'} + \mathrm{i}\,\sigma^\lambda_\mu\,\left(v\gamma\,A^{\hat{a}}{}_\lambda + i\,\gamma\,\epsilon^{\hat{a}}{}_{\hat{c}}\,A^{\hat{c}}{}_\lambda\right)\,. \end{aligned} \tag{C14}$$

The boost relations (XX) for $\bar{\varepsilon}$, $\bar{\jmath}_{\vdash'}$, and $\bar{\jmath}_a$ can be derived with these expressions. To derive the boost results for $\bar{s}^{ab}$ and $\bar{t}^{ab}$, one must use these expressions and also the result

$$2\,\Gamma^{(+)}{}_{\hat{1}\hat{2}\perp'} = 2\,\Gamma^{(+)}{}_{\hat{1}\hat{2}\perp} + \Gamma^{(+)}{}_{\hat{1}\hat{2}\vdash} - \mathrm{i}\,\gamma^2\,\bar{u}[v]\,. \tag{C15}$$

Note that on the left-hand side the selfdual coefficient is TR-gauge, while those on the right-hand side are RT-gauge.

## APPENDIX D: CORNER TERMS IN THE GRAVITATIONAL ACTION

This appendix presents a simple tetrad method for analyzing "sharp-corner" terms in the gravitational action principle. We show how the corner terms in the action (3.5) arise. As mentioned, the Goldberg action differs from the Hilbert action by the pure divergence

$$-\frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}\left(e^{\hat{\rho}}\wedge\sigma_{\hat{\rho}}\right) = \frac{1}{\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,\nabla_\mu\left(e^{\hat{\rho}\mu}\,e_{\hat{\sigma}}{}^{\nu}\,\Gamma^{\hat{\sigma}}{}_{\hat{\rho}\nu}\right)\,. \tag{D1}$$

To ensure that, upon the use of Stokes' theorem, this divergence gives the desired "$TrK$" and "$Tr\Theta$" terms on the boundary elements, tie $e_{\hat{0}}$ to the $u = e_\perp$ hypersurface normals on $t'$ and $t''$ and tie $e_{\hat{3}}$ to the normal $\bar{n} = e_{\vdash'}$ on $\bar{\mathcal{T}}$. However, if these gauge conditions are enforced simultaneously, then in general the tetrad is doubled-valued on the corners $B'$ and $B''$. Therefore, in order to both retain the desired "$TrK$" and "$Tr\Theta$" terms yet avoid double-valuedness on the corners, use a limit procedure in which the condition on $e_{\hat{3}}$ is

relaxed in a small neighborhood of the corners. Next, consider the limit as this neighborhood "shrinks" to the corners.

The precise procedure is as follows. Suppose that $e_{\hat{0}}$ does indeed coincide with $u$ on $t'$ and $t''$, but that on $\bar{\mathcal{T}}$ the tetrad has the form

$$\begin{aligned} e_{\hat{0}} &= \psi\,\bar{u} - w\psi\,\bar{n} \\ e_{\hat{3}} &= \psi\,\bar{n} - w\psi\,\bar{u}\,, \end{aligned} \tag{D2}$$

where $\psi \equiv (1-w^2)^{-1/2}$. For each $\delta \in [0,1]$ $w = w(x;\delta)$ is a suitably continuous and differentiable point-dependent boost velocity defined on $\bar{\mathcal{T}}$. Further, for each $\delta$ assume that $w(x;\delta) = 0$ except on a "small" neighborhood $\mathcal{N}_\delta$ of the corners $B'$ and $B''$. For each $\delta$ the set $\mathcal{N}_\delta$ is not connected, but is comprised of the disjoint union of two connected pieces $\mathcal{N}'_\delta$ and $\mathcal{N}''_\delta$. The set $\mathcal{N}'_\delta$ is "small" region of $\bar{\mathcal{T}}$ which contains $B'$, and in the limit $\delta \to 0$ we have that $(B' - \mathcal{N}'_\delta) \to \emptyset$. Similarly, the set $\mathcal{N}''_\delta$ is "small" region of $\bar{\mathcal{T}}$ which contains $B''$, and in the limit $\delta \to 0$ we have that $(B'' - \mathcal{N}''_\delta) \to \emptyset$. Finally, for each $\delta$ demand that $w(x;\delta) = v(x)$ whenever $x \in B' \bigcup B''$. This ensures that on the corner two-surfaces $e_{\hat{0}} = u$ and $e_{\hat{3}} = n$. Our construction provides us with a family of tetrads parametrized by $\delta$. By construction the member tetrad corresponding to each value of $\delta$ is TR-gauge on most of $\bar{\mathcal{T}}$, however, as the corners are approached, each member is continuously boosted until it is RT-gauge on the corners. Hence, each $\delta$ tetrad is single-valued on the corners and everywhere else. The idea is to use a $\delta$ tetrad in our divergence expression and consider

$$\frac{1}{\kappa}\int_{\partial\mathcal{M}} \mathrm{d}^3x\,\sqrt{{}^3g}\,TrK = \lim_{\delta\to 0}\frac{1}{\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,\nabla_\mu\left(e^{\hat{\rho}\mu}\,e_{\hat{\sigma}}{}^{\nu}\,\Gamma^{\hat{\sigma}}{}_{\hat{\rho}\nu}\right)\,, \tag{D3}$$

where the expression on the left-hand side symbolically represents the integral of the trace of the extrinsic curvature of $\partial\mathcal{M}$ as embedded in $\mathcal{M}$ over all of $\partial\mathcal{M}$ (which picks up finite corner contributions, since the normal of $\partial\mathcal{M}$ changes discontinuously from $u$ to $\bar{n}$ on these two-surfaces). We can use Stokes' theorem to find

$$\frac{1}{\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,\nabla_\mu\left(e^{\hat{\rho}\mu}\,e_{\hat{\sigma}}{}^{\lambda}\,\Gamma^{\hat{\sigma}}{}_{\hat{\rho}\lambda}\right) = \frac{1}{\kappa}\int_{t'}^{t''} \mathrm{d}^3x\,\sqrt{h}\,K + \frac{1}{\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\,\sqrt{-\bar{\gamma}}\,\bar{n}_\mu\,\left(e^{\hat{\rho}\mu}\,e_{\hat{\sigma}}{}^{\lambda}\,\Gamma^{\hat{\sigma}}{}_{\hat{\rho}\lambda}\right)\,, \tag{D4}$$

Focus attention on the $\bar{\mathcal{T}}$ boundary term,

$$\begin{aligned} &\frac{1}{\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\,\sqrt{-\bar{\gamma}}\,\bar{n}_\mu\left(e^{\hat{\rho}\mu}\,e_{\hat{\sigma}}{}^{\nu}\,\Gamma^{\hat{\sigma}}{}_{\hat{\rho}\nu}\right) = \\ &\qquad \frac{1}{\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\,\sqrt{-\bar{\gamma}}\,\bar{n}_\mu\,e^{\hat{\rho}\mu}\left(e^{\hat{\sigma}}{}_{\hat{\sigma}'}\,e_{\hat{\rho}}{}^{\hat{\rho}'}\,\Gamma^{\hat{\sigma}'}{}_{\hat{\rho}'\nu}\,e_{\hat{\sigma}}{}^{\nu} + e^{\hat{\sigma}}{}_{\hat{\sigma}'}\,e_{\hat{\sigma}}\left[e_{\hat{\rho}}{}^{\hat{\sigma}'}\right]\right)\,. \end{aligned} \tag{D5}$$

We have used the inhomogeneous transformation rule for connection coefficients to express the $\delta$ connection coefficients in terms of the connection coefficients $\Gamma^{\hat{\kappa}'}{}_{\hat{\tau}'\nu}$ determined by the TR-gauge tetrad. Using the the boost relations (D2), we find after some algebra that

$$\frac{1}{\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\,\sqrt{-\bar{\gamma}}\,\bar{n}_\mu\left(e^{\hat{\rho}\mu}\,e_{\hat{\sigma}}{}^{\nu}\,\Gamma^{\hat{\sigma}}{}_{\hat{\rho}\nu}\right) = -\frac{1}{\kappa}\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\,\sqrt{-\bar{\gamma}}\left(\bar{\Theta} + \bar{u}[\varphi]\right)\,, \tag{D6}$$

where $\bar{\Theta} = -\Gamma^{\hat{\sigma}'}{}_{\perp'\hat{\sigma}'} = -\Gamma^{\hat{s}}{}_{\perp'\hat{s}}$ ($\hat{s}$ runs over $(\perp', \hat{1}, \hat{2})$) and $\varphi = \varphi(x;\delta) = \tanh^{-1}(w(x;\delta))$. Next, since $\bar{u} = 1/\bar{N}(\partial/\partial t - \bar{V})$, with some integrations by parts the final integral on the right-hand side becomes

$$-\frac{1}{\kappa}\int_{\bar{T}} \mathrm{d}^3x\,\sqrt{-\bar{\gamma}}\,\bar{u}[\varphi] = -\frac{1}{\kappa}\int_{B'}^{B''} \mathrm{d}^2x\,\sqrt{\sigma}\,\varphi + \frac{1}{\kappa}\int_{\bar{T}} \mathrm{d}^3x\,\varphi\,\dot{\sqrt{\sigma}} - \frac{1}{\kappa}\int_{\bar{T}} \mathrm{d}^3x\,\sqrt{\sigma}\,\varphi\,\left(\delta_b\,\bar{V}^b\right) . \quad \text{(D7)}$$

We have that $\lim_{\delta\to 0}\varphi(x;\delta) = 0$ everywhere on $\bar{T}$ except for corner points where $\lim_{\delta\to 0}\varphi(x;\delta) = \phi(x)$. Therefore, in this limit only the first corner-term integrals on the right-hand side survive. Hence we have the main result

$$\frac{1}{\kappa}\int_{\partial\mathcal{M}} \mathrm{d}^3x\,\sqrt{{}^3g}\,TrK = \frac{1}{\kappa}\int_{t'}^{t''} \mathrm{d}^3x\,\sqrt{h}\,K - \frac{1}{\kappa}\int_{\bar{T}} \mathrm{d}^3x\,\sqrt{-\bar{\gamma}}\,\bar{\Theta} - \frac{1}{\kappa}\int_{B'}^{B''} \mathrm{d}^2x\,\sqrt{\sigma}\,\phi\,, \quad \text{(D8)}$$

which justifies (3.5). Since the action $S^1$ in (3.5) is essentially a metric action, we have borrowed the results from Ref. [10] to obtain the variation (3.6). However, it is not difficult to use the $\delta$ tetrad method to obtain this result. To perform this calculation it helps to assume that $\delta w = 0$, or, in other words, the variations of the $\delta$ tetrad and TR-gauge tetrad are "locked" together. We note in passing that a straightforward though somewhat lengthy calculation shows that variation of the action (3.1) is

$$\begin{aligned}\delta S^1 = &-\frac{1}{\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,G^{\mu\nu}\,e_{\hat{\rho}\mu}\,\delta e^{\hat{\rho}}{}_{\nu} \\ &-\frac{1}{\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,\nabla_\mu\left[\left(2\Gamma^{\hat{\rho}\hat{\sigma}}{}_{\hat{\sigma}}\,e_{\hat{\rho}}{}^{[\mu}\,e_{\hat{\tau}}{}^{\nu]} - \Gamma^{\hat{\rho}\hat{\sigma}}{}_{\hat{\tau}}\,e_{\hat{\rho}}{}^{\mu}\,e_{\hat{\sigma}}{}^{\nu}\right)\delta e^{\hat{\sigma}}{}_{\nu}\right] .\end{aligned} \quad \text{(D9)}$$

One must insert the $\delta$ tetrad into this expression and then take the limit $\delta \to 0$.

The pure imaginary boundary term (3.12) added to the Goldberg action may also be expressed as

$$-S^0 = \frac{\mathrm{i}}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,\nabla_\sigma\left(e_{\hat{\sigma}}{}^{\sigma}\,\epsilon^{\hat{\sigma}\hat{\rho}\hat{\mu}\hat{\tau}}\,\Gamma_{\hat{\rho}\hat{\mu}\hat{\tau}}\right) . \quad \text{(D10)}$$

The variation of this expression is

$$\begin{aligned}-\delta S^0 = &-\frac{\mathrm{i}}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,\nabla_\sigma\,\nabla_\lambda\left(\epsilon^{\sigma\lambda\tau\mu}\,e_{\hat{\rho}\tau}\,\delta e^{\hat{\rho}}{}_{\mu}\right) \\ &+\frac{\mathrm{i}}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,\nabla_\sigma\left[\epsilon^{\hat{\sigma}\hat{\rho}\hat{\mu}\hat{\tau}}\,\Gamma_{\hat{\rho}\hat{\mu}\lambda}\left(e_{\hat{\tau}}{}^{\lambda}\,\delta e_{\hat{\sigma}}{}^{\sigma} + e_{\hat{\sigma}}{}^{\sigma}\,\delta e_{\hat{\tau}}{}^{\lambda} - e_{\hat{\sigma}}{}^{\sigma}\,e_{\hat{\tau}}{}^{\lambda}\,e^{\hat{\kappa}}{}_{\kappa}\,\delta e_{\hat{\kappa}}{}^{\kappa}\right)\right]\end{aligned} \quad \text{(D11)}$$

Clearly, the first integral vanishes. Using the $\delta$ tetrad in each of the expressions, one can take the $\lim_{\delta\to 0}$ and verify that $-S^0$ and $-\delta S^0$ contribute no corner terms.

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