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Large–scale Structure without N–body Simulations: The Legacy of Ya. B. Zel'dovich

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The task of understanding how gravitational instability might be responsible for the formation of large–scale structures in the Universe is one in which computer simulations have played a leading role. Recently, however, a growing theoretical effort has been focussed upon analytical studies of structure formation. Foremost amongst the analytical methods being explored are techniques based on the Zel'dovich approximation, which has been shown to be remarkably accurate when compared with full N–body calculations. In this paper, we illustrate the power of these analytic methods which enable us to study many aspects of the structure formation process without recourse to N–body computations and show that, 25 years after it was first suggested, the Zel'dovich approximation is still opening up new avenues towards an understanding of the origin of large–scale structure in the Universe.

On very large scales the structure of the Universe is composed of giant conglomerations of galaxies called *superclusters*, separated by vast empty regions known as *voids*[1]. On the other hand, the near–uniformity of the Cosmic Microwave Background tells us that the early Universe was nowhere near as lumpy as the Universe we observe around us today. One of the most important challenges in modern cosmology is therefore to understand how today's lumpy Universe evolved from an apparently smooth initial state.

of nature with the longest range is gravity. It therefore makes sense to look towards gravitational physics for a theory of the origin of the largest structures in the Universe. From this line of argument has emerged a *standard* model of structure formation, which is based on two fundamental assumptions. The first is that the galaxies and clusters of galaxies we observe today began as small fluctuations in the density of a predominantly homogeneous Universe. The second assumption of the standard model is that these initial fluctuations constitute a *Gaussian random field*. Fluctuations of this statistical type are predicted by most versions of the inflationary Universe scenario[2]. These initially small fluctuations get transformed into clusters of galaxies by the progressive action of gravitational instability, the theory of which was pioneered by Sir James Jeans, who showed that a uniform distribution of matter in a static background is unstable to small fluctuations in its density[3]. This argument was subsequently generalised to relativistic fluids in an expanding Universe by Lifshitz[4].

The growth of density fluctuations can be characterised by two essentially distinct epochs. In the first, the dimensionless density contrast

$$\delta(\mathbf{x},t) = [\rho(\mathbf{x},t) - \rho_0]/\rho_0 \tag{1}$$

(where ρ is the matter density, ρ_0 is the average matter density and $\mathbf{x}$ is a spatial coordinate) grows according to a universal function of time: $\delta(\mathbf{x},t) = D_+(t)\Delta(\mathbf{x})$. In this *linear* regime each Fourier mode of $\delta(\mathbf{x},t)$ evolves independently and the initial statistical properties of $\delta(\mathbf{x},t)$ are preserved. The second epoch is characterised by the development of non–linear features such as mode–mode coupling, for which a full analytic theory is lacking. Nevertheless, there are various analytic approximations which bring out the essential aspects of these non–linearities in a physically informative way. Perhaps the most interesting such approximation is the famous approximation devised by Zel'dovich[5,6]. The great idea of Zel'dovich was to consider perturbations to the trajectories of fluid elements, rather than perturbations to the density at each point in space. This formulation of the problem, following fluid elements, is essentially Lagrangian in character, in contrast to the Eulerian approach adopted by Jeans and Lifshitz, in which one looks at quantities defined at fixed spatial positions.

In the Eulerian approach, one constructs a pertubation series of the form

$$\delta(\mathbf{x},t) = \sum_{n=1}^{\infty} \delta^{(n)}(\mathbf{x},t) = \sum_{i=1}^{\infty} D_+^n(t)\Delta^{(n)}(\mathbf{x}) \tag{2}$$

(and analagous expansions for the velocity field and gravitational potential). The linear Eulerian theory mentioned above then consists of taking the first term in this

series and assuming it is small compared to the background quantities which satisfy the unperturbed equations of motion.

In the Lagrangian approach, one considers the displacements of fluid elements rather than perturbations to δ: if the final (Eulerian) co–ordinate of a fluid element $\mathbf{x}$ differs from its initial (Lagrangian) co–ordinate $\mathbf{q}$ by a displacement term $\Psi(\mathbf{q}, t)$:

$$\mathbf{x} = \mathbf{q} + \Psi(\mathbf{q}, t) \tag{3}$$

then we simply expand Ψ as a perturbation series:

$$\Psi = \Psi^{(0)} + \Psi^{(1)} + \dots \tag{4}$$

The lowest order (i.e. linear) term is just $\Psi^{(0)} = D_+(t)\mathbf{u}(\mathbf{q})$, where $D_+(t)$ is the growing mode of the linear density contrast in Eulerian theory. The Zel'dovich approximation is obtained by truncating the series (4) at this term, so that:

$$\mathbf{x} = \mathbf{q} + D_+(t)\mathbf{u}(\mathbf{q}). \tag{5}$$

The field $\mathbf{u}(\mathbf{q})$ is the peculiar velocity field induced by the initial density perturbation. The Zel'dovich approximation is therefore a linear Lagrangian perturbation theory.

While matter trajectories do not cross, equation (5) represents a one–to–one mapping of Lagrangian space onto Eulerian space. Matter continuity therefore requires that the final Eulerian density at $\mathbf{x}$ must be related to the initial (Lagrangian) density at $\mathbf{q}$ via the Jacobian of the transformation (5) between $\mathbf{x}$ and $\mathbf{q}$. Thus,

$$\rho(\mathbf{x}, t) = \frac{\rho_0}{a^3}[1 - D_+(t)\lambda_1(\mathbf{q})]^{-1}[1 - D_+(t)\lambda_2(\mathbf{q})]^{-1}[1 - D_+(t)\lambda_3(\mathbf{q})]^{-1} \tag{6}$$

where the λ_i are the eigenvalues of the deformation tensor $\partial \mathbf{u}/\partial \mathbf{q}$. For random initial conditions, λ_i will not in general be equal. For a Gaussian random field $\mathbf{u}$, by far the most probable configuration is that one eigenvalue is significantly larger than the other two[7]. If this is the case, collapse will occur locally in one direction, leading to the formation of planar singularities where matter trajectories cross. Singularities thus occur when the Jacobian of the transformation (5) vanishes and the corresponding Eulerian density goes to infinity. The condition for this is just $1 - D_+(t)\lambda_1 = 0$, where λ_1 is the largest of the three eigenvalues. Two–dimensional singularities ("pancakes") and their one–dimensional analogues ("filaments") are known from N–body simulations to be generic features of non–linear gravitational instability.

The Zeldovich approximation can be thought of as *ballistic* model: particle trajectories are completely determined by the initial velocity field. Although this method accurately predicts the formation of caustics (regions of formally infinite density) it does have the drawback that, since particle motion is determined solely by initial conditions, particle trajectories are insensitive to the environment they subsequently encounter. Back–reaction effects in high density regions are ignored leading to an artificial thickening of pancakes not seen in practice. This drawback can be remedied in two distinct ways. One could introduce time-dependent smoothing which would systematically remove power on scales smaller than the one which is currently going non-linear. This method (known as TZA – Truncated Zeldovich approximation) accurately describes large scale clustering, but tends to suppress small scale features otherwise present in a simulation[8] (see Figure 1).

Another innovative approach[6,9] aimed at extending the Zeldovich approximation involves the so-called ***adhesion model*** (AM). In this approach pancakes are kept thin by introducing a mock viscosity term in the equations of motion which mimicks the adhesive effects of gravity within pancakes. The equations of motion for self-gravitating particles are known as the Euler–Poisson system. This set of equations is virtually impossible to solve exactly for generic initial conditions, chiefly because of the Euler equation, which can be written in the form[9]

$$\frac{\partial \mathbf{u}}{\partial a} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} = -\frac{3}{2a}\left[A\nabla_x\phi + \mathbf{u}\right], \tag{7}$$

where $A = 2/3H^2a^3$ and H is the Hubble parameter. The Zel'dovich approximation involves setting the right hand side of equation (7) to zero. If these terms are kept, the equation becomes very difficult to solve.

The adhesion approximation considerably simplifies the situation by converting the Euler equation into Burgers equation[10]

$$\frac{\partial \mathbf{u}}{\partial a} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu\nabla^2\mathbf{u} \tag{8}$$

which has an analytical solution for the velocity field:

$$\frac{\mathbf{u}(\mathbf{x}, a) = \int d^3\mathbf{q}\left(\frac{\mathbf{x}-\mathbf{q}}{a}\right)\exp[-S(\mathbf{x}, a; \mathbf{q})/2\nu]}{\int d^3\mathbf{q}\exp[-S(\mathbf{x}, a; \mathbf{q})/2\nu],} \tag{9}$$

where

$$S(\mathbf{x}, a; \mathbf{q}) = -\phi_0(\mathbf{q}) + \frac{(\mathbf{x}-\mathbf{q})^2}{2D_+(t)}; \tag{10}$$

ϕ_0 is the initial gravitational potential. The viscosity coefficient ν has dimensions of $(\text{length})^2$ and determines the thickness of pancakes. (Setting $\nu = 0$ in equation (8) recovers the Zeldovich approximation.) The case when ν is chosen to be infinitesimally small (but not zero) is most interesting since then the main contribution to the integral in equation (9) comes from points which minimise the action S. This results in an interesting geometrical interpretation which is illustrated in Figure 2. The main features of large scale clustering are effectively described by descending a paraboloid having radius of curvature $1/2D_+(t)$ onto the linear gravitational potential ϕ_0. The paraboloid is allowed to touch the potential but not to intersect it. (This follows from solving equation (9) using the method of steepest descents.) The paraboloid then describes a mapping from Lagrangian space (the space describing the initial location of particles) to Eulerian space (describing their final location). At early times D_+ is small and this mapping is one-to-one. At late times the mapping becomes many–to–one, as the paraboloid grows broader and touches the potential in two or more points. The multiple mapping of particles in Lagrangian space to a single point in Eulerian space corresponds to the formation of caustics (filaments or clumps in two dimensions; pancakes, filaments and clumps in three dimensions) whose location is given by the apex of the paraboloid.

The adhesion model predicts that the generic motion of particles will initially be towards two dimensional pancake-like objects. Having fallen into pancakes particles will move along them towards regions of higher density – filaments which form

at the intersection of pancakes, settling finally in clumps which form at the intersections of filaments. Thereafter neighbouring clumps will merge causing gravitational instability to proceed in a hierarchical manner.

A quantitative comparison between the effects of Zel'dovich evolution in the framework of TZA/AM and full N–body dynamics shows remarkable agreement well into the non–linear regime[8,11], so models based on the Zel'dovich approximation really do allow us to understand what otherwise we could only simulate. The Zel'dovich approximation also furnishes a framework within which we can understand the formation of voids and their effect upon surrounding pancakes in a precise analytical fashion[8,11]. Moreover, an important feature of both TZA and the adhesion model is that since both have single-step analytical solutions, one does not need to evolve the fluid equations iteratively (as is done in N-body methods). This can lead to an enormous speeding up in computational time, and turns out to be particularly useful for simulating the very large scale distribution of rich galaxy clusters[12].

The gravitational instability picture thus seems to provide an explanation of the qualitative features of the large–scale distribution of galaxies. But other scenarios have been suggested that rely upon quite different premises to create qualitatively similar structures. For example, vast cosmic explosions could form shocks that sweep up the intergalactic medium into shells, leaving large voids inside[13]. Topological defects such as cosmic strings or textures also produce interesting large–scale density inhomogeneities from strongly non–Gaussian initial fluctuations[14]. We need a quantitative description of the clustering pattern to discern systematic differences

between these different scenarios. Moreover, while the Zel'dovich approximation may reproduce the visual appearance of the matter distribution to the satisfaction of one's eye, astronomers do not compare theory against observation this way. We need to ask the question whether these approximations can be trusted to produce reliable quantitative statistical information about the clustering distribution. In fact, the answer to this question seems to be "yes" and, what's more, they reveal an important connection between the dynamics and statistics of matter fluctuations.

To describe this connection, we need to introduce the traditional tools for describing galaxy clustering statistically: the n–point correlation functions[15]. The simplest of these is the two–point correlation function $\xi_2(r)$, defined as the excess probability, over random, of finding a galaxy at a distance r from a given galaxy. The higher–order functions are simple generalisations of this: the three–point function, $\xi_3(r_1, r_2)$, gives the excess probability of finding two galaxies at distances r_1 and r_2 from a given galaxy, and so on. All order correlations are required for a complete statistical description of a distribution of points in space. The n–point correlation functions possess a dependence on the geometry of the spatial distribution that makes their use rather cumbersome (particularly for large n). To simplify matters, it is convenient to consider the volume integrals of ξ_n over cells of volume V:

$$\bar{\xi}_n = \frac{1}{V^n} \int_V \xi_n d^n V. \tag{11}$$

We call these $\bar{\xi}_n$, and they are simply related to the n–th moment of the count of galaxies in a randomly–selected volume V. There is considerable empirical evidence[16] that the $\bar{\xi}_n$ have the so–called *hierarchical* form

$$\bar{\xi}_n = S_n(\bar{\xi}_2)^{n-1}, \tag{12}$$

where S_n is a constant. Notice that knowledge of the full set of $\bar{\xi}_n$ is sufficient to specify all the statistical properties of the distribution of galaxies, which is why they are so useful and important.

The $\bar{\xi}_n$ are particularly interesting in the context of theories based on Gaussian initial data because $\bar{\xi}_n \equiv 0$ for $n > 2$ for a Gaussian distribution. However, the initial statistics are preserved only during linear evolution; during non–linear evolution the distribution of fluctuations becomes skewed by the generation of non–zero higher–order moments. Until recently, the growth of moments of cosmological density fluctuations with time was poorly understood because of the lack of a full analytical theory. Some arguments based on scaling properties of the BBGKY hierarchy[17] had suggested a hierarchical form, at least for relatively small n. in the strongly non–linear regime characterised by $\xi_2 \gg 1$. Perturbative calculations had also suggested hierarchical behaviour in the weakly non–linear regime[18–20] where $\xi_2 < 1$. Using an ingenious Tree–summation method, Bernardeau[21] subsequently showed that the problem of calculating the coefficients S_n in equation (12) using Eulerian perturbation theory can be cast in the same form as the problem of solving the evolution of an isolated spherical inhomogeneity in an expanding background, a problem that can be solved exactly. It has further been shown that the calculation

of the S_n in any approximation is equivalent to solving the spherical inhomogeneity problem in the approximation in question[22]. We now know, therefore, how to calculate the S_n in Eulerian perturbation theory (at least within Bernardeau's Tree–level approximation), the Zel'dovich approximation and, in fact, in higher–order Lagrangian theories obtained by including higher–order terms in Ψ; see equation (4). Remarkably, an m–th order Lagrangian theory turns out to produce a series S_n that agrees with the exact one, up to and including S_{m+1}: a second–order Lagrangian theory produces an exact S_3 (skewness), a third–order theory will give an exact S_3 and S_4 (kurtosis), and so on. One can therefore custom–build a high–order variant of the Zel'dovich approximation which can match exact non–linear theory to any desired order of statistical completeness. Because so many properties of the matter distribution can be expressed in terms of low–order moments $\bar{\xi}_n$, this gives cosmologists plenty of scope to calculate interesting quantities analytically. More importantly, these considerations reveal a fascinating connection between formal perturbation methods and practical statistical descriptors, a connection which has long been the goal of structure formation theorists.

The wide–ranging success of the Zel'dovich approximation is a remarkable testament to the physical insight of its originator. Ya. B. Zel'dovich died in 1987; last year (1994) would have seen his 80th birthday. Without him we would have simulated more, and understood less.

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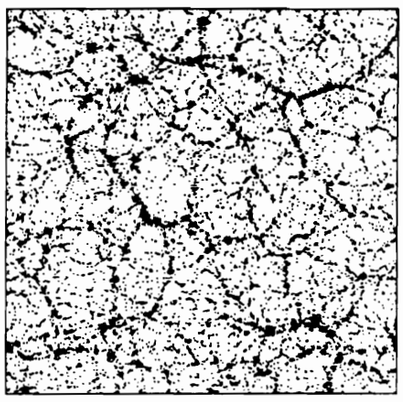
Figure Captions

Figure 1

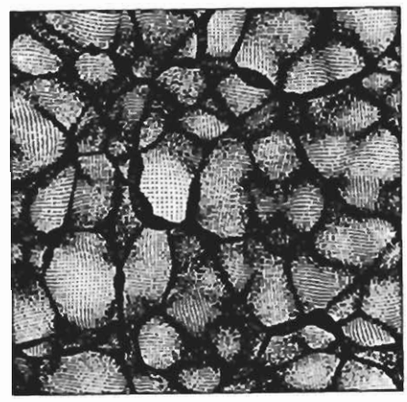
A comparison of the Zel'dovich approximation against full N–body evolution of the same initial data[23]. A two–dimensional example is shown for simplicity. Both pictures were generated from the same initial conditions, one (TZA) using a method based on the Zel'dovich approximation (with some filtering of the initial conditions to suppress shell–crossing) and the other (NB) using an N–body simulation of 512^2 particles. Although the Zel'dovich technique does not reproduce all the small–scale structure present in the N–body example, the large–scale network of filaments and voids is reproduced with remarkable accuracy.

Figure 2

The geometrical model illustrating the adhesion approximation is shown for two dimensional gravitational clustering. The paraboloid maps a particle originally located at $\mathbf{q}$ to its final position at $\mathbf{x}$. Filaments and clumps, which form when the paraboloid touches the potential in two or three points respectively, are also shown.



NB



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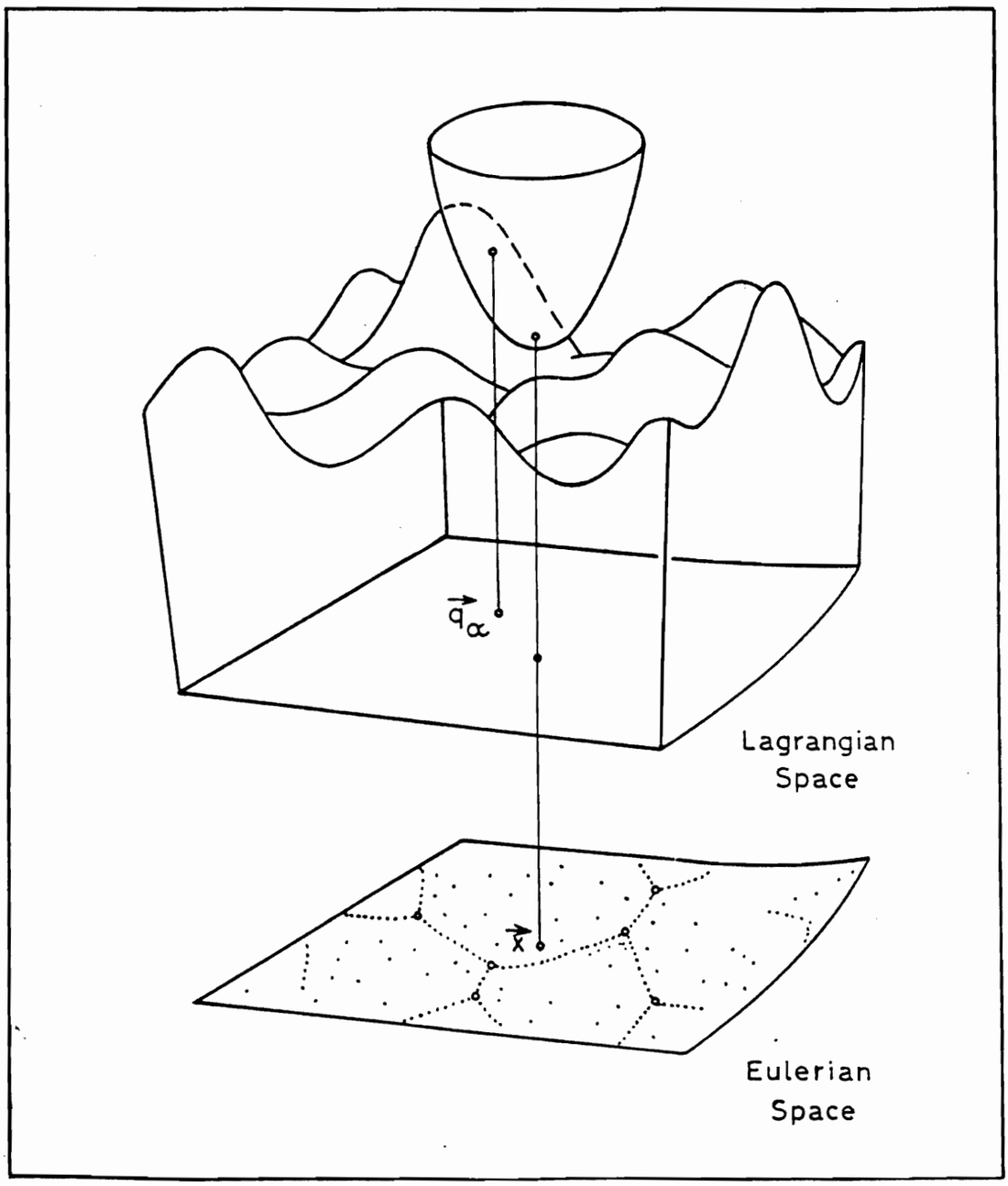
$\vec{q}_\alpha$
Lagrangian Space
$\vec{x}$
Eulerian Space