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By

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Submitted to General Relativity and Gravitation

Inter-University Centre for Astronomy and Astrophysics

An Autonomous Institution of the University Grants Commission

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Running Title: *Symmetry of the perfect fluid space-times*

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ABSTRACT

The concept of Ricci inheritance has been studied for perfect fluid space - times. The necessary and sufficient conditions for such space-times to admit space-like Ricci inheritance vector has been obtained in terms of the kinematical quantities of the space-like congruences generated by a unit vector n^a orthogonal to the flow velocity vector u^a. Dust and irrotational dust universes have been considered and the conservation laws for space-like Ricci inheritance vector are given. For some special choice the results of this paper reduce to perfect fluid space-times admitting Ricci collineation.

Key Words: Ricci inheritance, Perfect fluid, equations of state, dust and irrotational dust universes

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1. INTRODUCTION

In the general theory of relativity, the curvature tensor describing the gravitational field consists of two parts viz., the matter part and the free gravitational part. The interaction between these two parts is described through the Bianchi identities. For a given distribution of matter,the construction of gravitational potential satisfying Einstein field equations is the principal aim of all investigation in gravitational physics, and this has often been achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of the space-time are expressible through an equation of the form

$$\mathcal{L}_\xi A \; - \; 2\Omega A \; = \; 0 \tag{1}$$

where A represents a geometrical/physical quantity, $\mathcal{L}_\xi$ denotes the Lie derivative of A with respect to the vector field ξ (this vector field may be time-like, space-like or null), and Ω is a scalar.

One of the most simple and widely used example is the metric inheritance symmetry for which $A = g_{ij}$ in equation (1). In this case the vector field ξ^a is called the conformal Killing vector which includes a homothetic vector and a Killing vector according as $\Omega_{,a} = 0$ and $\Omega = 0$, respectively.

In a series of papers, [1] - [5], Katzin, Levine, Davis and collaborators have identified sixteen symmetries for the gravitational field with their interrealtionships and have obtained the corresponding weak conservation laws as the integrals of the geodesic equation. Different types of matter distribution compatible with geometrical symmetries have been the subject of interest of several investigators for quite some time, and in this connection, Oliver and Davis [6], for the space-times filled with perfect fluid, have studied the time-like symmetries with special reference to conformal motion and family of contracted Ricci collineation. The perfect fluid space-times including electromagnetic

field which admit symmetry mapping belonging to the family of contracted Ricci collineation have been studied by Norris et al [7]. The role of metric symmetries in the study of fluid space-times, with an emphasis on conformal collineation, has been explored by Duggal [8,9] (For a comprehensive review on the subject of collineation see [10]).

The curvature and Ricci inheritance symmetries defined through equation (1) when A is replaced, respectively, by the Riemann curvature tensor and Ricci tensor have recently been studied by Duggal [11,12] who obtained a number of results relevant to material curves, proper conformal and proper nonconformal symmetries.

Motivated by the all important place of symmetry inheritances in general relativity and the role played by the Ricci tensor in the study of perfect fluid space-times, in this paper we shall study the Ricci inheritance for the perfect fluid space-times. In fact, we shall find the conditions under which a perfect fluid space-time may admit a space-like Ricci inheritance vector. The main theorem A and its proof is given in section 2 along with some related results. Section 3 contains, in detail, the consequences of theorem A and a discussion of the results has been given in section 4.

2. RICCI INHERITANCE

Let V_4 be the space-time of general relativity with Einstein field equations

$$R_{ab} - \frac{1}{2}g_{ab}R = T_{ab} \tag{2}$$

where T_{ab} is the energy momentum tensor of the perfect fluid, i.e.,

$$T_{ab} = \rho u_a u_b + p\, h_{ab} \tag{3}$$

Here ρ, p and $h_{ab} = g_{ab} + u_a u_b$, respectively, are the energy density, the isotropic pressure relative to the fluid flow velocity vector u_a and the projection tensor which projects into the instantanous rest space of an observer with four velocity u_a.

It is known [12] that the integral curves of a vector field X^a are material curves (a curve that always consists of the same fluid particles) if the fluid lines inherit symmetry properties with respect to X^a. The interest in the material curves arises from the theory of space-like congruences first devaloped by Greenberg [13]. This theory has its applications in relativistic hydrodynamics and relativistic electrodynamics of continous media.

The space-like vector λ^a orthogonal to the flow velocity vector u^a is defined as

$$\lambda^a = \lambda n^a \ , \ \lambda = (\lambda_a \lambda^a)^{1/2} \ > \ 0 \ , \ n_a u^a = 0 \ , \ n_a n^a = 1 \qquad (4)$$

The Ricci inheritance is defined through the relation

$$\mathcal{L}_\lambda \ R_{ij} \ = \ 2\Omega R_{ij} \qquad (5)$$

The deformation of the congruence generated by n^a at any point P can be measured if at P an observer with four velocity u^a orthogonal to n^a is specified. Since $n_a u^a = 0$, an observer comoving with the fluid with four velocity u^a may be employed at P. Once the observer has been specified at any point of the congruence, the observers employed at all other points along the congruence can not be assigned arbitrarily, thier four velocity must satisfy the transport law [13]:

'If a comoving observer with four velocity u^a is chosen at any one given point, then the observers employed at all other points along the congruence can be comoving observers with four velocity if and only if

$$h^a_b \dot{n}^b \ = \ u'^a \ - \ (n_b u'^b) n^a$$

where an overhead dot denotes covariant differentiation along a fluid particle world line, for example, $\dot{X}^a = X^a_{;b} \ u^b$, and a dash denotes the covariant diffrentiation along an integral curve of n^a, for example, $Y'^a = Y^a_{;b} \ n^b$.'

In this section, we shall find the conditions under which a perfect fluid space-time may admit a space-like Ricci inheritance vector λ^a orthogonal to the flow velocity vector u^a in terms of expansion, rotation and shear of the space-like congruences generated by n^a. In fact, we shall prove the following

Theorem A : The necessary and sufficient conditions for a perfect fluid space-time satisfying Einstein field equations to admit a Ricci inheritance vector $\lambda^a = \lambda n^a$ are that

$$(i) \quad (\rho + 3p)\omega_{ac}n^c - \frac{1}{2}(\rho - p)L_a = 0 \tag{6}$$

$$(ii) \quad (\rho - p)(\mathcal{T}_{ab} - 2\Omega\lambda^{-1}) = 0 \tag{7}$$

$$(iii) \quad (\rho - p)[n'_a + (log\lambda)_{,a} - \frac{1}{2}\mathcal{E}n_a] = 0 \tag{8}$$

$$(iv) \quad (\rho - p)[\frac{1}{2}\mathcal{E} + n_c\dot{u}^c - 2\Omega\lambda^{-1}] = 0 \tag{9}$$

$$(v) \quad [(\rho - p)\lambda^a]_{;a} - 4(\rho - p)\Omega\lambda^{-1} = 0 \tag{10}$$

We shall first devalope some results that are necessary for the proof of theorem A as follows:

We define

$$L^a = h^a_b\dot{n}^b - u'^a + (n_b u'^b)n^a \tag{11}$$

It can be shown that [14] $L_a = 0$ is the necessary and sufficient condition for the integral curve of n^a ($n_a u^a = 0$, $n_a n^a = 1$) to be material curve in the fluid and therefore the comoving observer u^a can be employed all along the $n-$congruence if and only if the curves of the congruence are material lines.

For the space-like congruences generated by n^a as measured by an observer moving with four velocity u^a, the expansion $\mathcal{E}$, the rotation tensor $\mathcal{R}_{ab}$ and the shear tensor $\mathcal{T}_{ab}$ are given by [13]

$$\mathcal{E} = q^{ab} n_{a;b} \tag{12}$$

$$\mathcal{R}_{ab} = q^c_a q^d_b n_{[c;d]} \tag{13}$$

$$\mathcal{T}_{ab} = q^c_a q^d_b n_{(c;d)} - \frac{1}{2}\mathcal{E} q_{ab} \tag{14}$$

where q^{ab} is the projection tensor that projects onto the two-space orthogonal to u^a and n^a :

$$q^{ab} = g^{ab} + u^a u^b - n^a n^b \quad , \quad q^{ab} u_b = 0 \quad , \quad q^{ab} n_b = 0 \tag{15}$$

The covariant derivative of n_a may be decomposed as ([13])

$$n_{a;b} = \mathcal{R}_{ab} + \frac{1}{2}\mathcal{E} q_{ab} + \mathcal{T}_{ab} + n'_a n_b - \dot{n}_a n_b + u_a (n^c u_{c;b})$$
$$+ (n^c \dot{u}_c) u_a u_b - (n_c u'^c) u_a u_b \tag{16}$$

where $\mathcal{E}$, $\mathcal{R}_{ab}$ and $\mathcal{T}_{ab}$ are given by equations (12) - (14) respectively.

From the definition of Lie derivative, we have

$$\mathcal{L}_{\lambda^a = \lambda n^a} R_{ab} = \lambda [R'_{ab} + 2 n^c R_{c(a} (log\lambda)_{,b)} + 2 R_{c(a} n^c_{;b)}] \tag{17}$$

where $R'_{ab} = R_{ab;c} n^c$.

With the assumption that equation (5) hold with $\lambda^a = \lambda n^a$ as Ricci inheritance vector and Einstein field equations are satisfied for a perfect fluid, equation (17) takes the form

$$\frac{1}{2}(\rho' + 3p') + \frac{1}{2}(\rho' - p') h_{ab} + 2(\rho + p)[\rho'_{(a} u_{b)} - u_{(a} u^c_{;b)} n_c]$$
$$+ (\rho - p)[n_{(a;b)} + n_{(a} (log\lambda)_{,b)}] = 2\Omega\lambda^{-1} R_{ab} \tag{18}$$

Contracting equation (18)in turn with $u^a u^b, u^a n^b, u^a q^{bc}, n^a n^b, n^a q^{bc}, q^{ab}$ and $q^{ac} q^{bd} - \frac{1}{2} q^{ab} q^{cd}$, we get

$$u^a u^b \ : \ \rho' + 3p' + 2(\rho + 3p)[n_a \dot{u}^a - \Omega\lambda^{-1}] = 0 \tag{19}$$

$$u^a n^b \ : \ (\rho - p)[(log\lambda)^{\cdot} + u_a n'^a] = 0 \tag{20}$$

$$u^a q^{bc} \ : \ \frac{1}{2}(\rho - p)h^b_a \dot{n}^a - (\rho + p)(u'_a - n_b u'^b n_a)$$
$$+\frac{1}{2}(\rho + 3p)q^b_a n^c u_{c;b} = 0 \tag{21}$$

$$n^a n^b \ : \ \rho' - p' + 2(\rho - p)[(log\lambda)' - \Omega\lambda^{-1}] = 0 \tag{22}$$

$$n^a q^{bc} \ : \ (\rho - p)q^b_a[n'_b + (log\lambda)_{,b}] = 0 \tag{23}$$

$$q^{ab} \ : \ \rho' - p' + (\rho - p)[\mathcal{E} - 2\Omega\lambda^{-1}] = 0 \tag{24}$$

$$q^{ac}q^{bd} - \frac{1}{2}q^{ab}q^{cd} \ : \ (\rho - p)[\mathcal{T}_{ab} - 2\Omega\lambda^{-1}] = 0 \tag{25}$$

The momentum conservation equation for perfect fluid is ([15])

$$(\rho + p)\dot{u}_a \ = \ -p_{,b}h^b_a \tag{26}$$

Contract equation (26) with n^a to get

$$(\rho + p)\dot{u}_a n^a \ = \ -p_{,b}h^b_a n^a \ = \ -p' \tag{27}$$

With all the above considerations we are now in a position to prove theorem A.

Proof of the Theorem A

(i) From the definition, we have

$$n^c u_{c;b} = 2n^c u_{[c;b]} + u'_b = -2\omega_{bc}n^c - (n_c\dot{u}^c)u_b + u'_b \tag{28}$$

where ω_{ab} is the vorticity tensor which determines the rigid rotation of the fluid. The vorticity pseudo-vector ω^a, dual to ω_{ab}, is defined as ([16])

$$\omega^a = \frac{1}{2}\eta^{abcd}u_b\omega_{cd} \tag{29}$$

which gives $\omega_{ab} = \eta_{abcd}\omega^c u^d$ so that

$$\omega^a u_a = 0 = \omega^{ab}u_b \ , \quad \omega^a\omega_{ab} = 0 \tag{30}$$

Clearly this vector is space-like and its magnitude

$$\omega = (\omega^a \omega_a)^{1/2} = (\frac{1}{2}\omega^{ab}\omega_{ab})^{1/2}$$

is the vorticity of the fluid.

Substitution of equation (28) in equation (21) leads to the condition (i) of the theorem.

(ii) This has been proved by equation (25).

(iii) Expand equation (23) to get

$$(\rho - p)[n'_a + g^b_a(log\lambda)_{,b} + u^b u_a(log\lambda)_{,b} - n^b n_a(log\lambda)_{,b}] = 0$$

which from equation (20) yields

$$(\rho - p)[n'_a + (log\lambda)_{,a} - (log\lambda)' n_a] = 0 \tag{31}$$

But from equations (22) and (24), we have

$$\rho' - p' + 2(\rho - p)[(log\lambda)' - \Omega\lambda^{-1}] - [\rho' - p' + (\rho - p)(\mathcal{E} - 2\Omega\lambda^{-1})] = 0$$

which gives

$$(\rho - p)[2(log\lambda)' - \mathcal{E}] = 0 \tag{32}$$

This equation together with equation (31) leads to the condition (iii) of the theorem.

(iv) Substituting the value of p' from equation (27) into equation (19), we get

$$\rho' = (\rho - 3p)(n_a \dot{u}^a - 2\Omega\lambda^{-1}) \tag{33}$$

Now using p' from equation (27) into equation(24) we obtain

$$\rho' - (\rho + p)n_a \dot{u}^a + (\rho - p)[\mathcal{E} - 2\Omega\lambda^{-1}] = 0$$

and substituting ρ' from equation (33) in this equation, we get

$$(\rho - p)[\frac{1}{2}\mathcal{E} + n_a \dot{u}^a - 2\Omega\lambda^{-1}] = 0$$

which proves condition (iv).

(v) From equations (27) and (33) substituting the values of p' and ρ', respectively, into the equation (22), we obtain

$$(\rho - p)(log\lambda)' = -(\rho - p)(n_a \dot{u}^a - 2\Omega\lambda^{-1}) = 0 \tag{34}$$

which from equation (32) yields

$$(\rho - p)(log\lambda)' = (\rho - p)(n_a \dot{u}^a + \mathcal{E} - 2\Omega\lambda^{-1}) = 0 \tag{35}$$

Now, from the definition of $\mathcal{E}$ (i.e., equation (12)), we have

$$n_a \dot{u}^a + \mathcal{E} = n^a_{;a} \tag{36}$$

and with this, equation (35) reduces to

$$(\rho - p)(log\lambda)' = (\rho - p)(n^a_{;a} - 2\Omega\lambda^{-1}) = 0 \quad . \tag{37}$$

Also, equation (22) can be written as

$$\rho' - p' + (\rho - p)(log\lambda)' - (\rho - p)\Omega\lambda^{-1} + (\rho - p)(log\lambda)' - (\rho - p)\Omega\lambda^{-1} = 0$$

which when used in equation (37) yields

$$(\rho - p)_{,a}\lambda^a + (\rho - p)(\lambda_{,a}n^a + \lambda n^a_{;a} - 4\Omega\lambda^{-1}) = 0 \tag{38}$$

which gives $[(\rho - p)\lambda^a]_{;a} = 4(\rho - p)\Omega\lambda^{-1}$ and this proves the condition (v).

Thus, if $\lambda^a = \lambda n^a$ is a Ricci inheritance vector then the conditions (i) - (v) of the theorem are fulfilled.

Conversely, let the conditions (i) - (v) hold alongwith Einstein field equations. Now using equation (16) for $n_{a;b}$, equation (8) for $(\rho - p)(log\lambda)_{,a}$, equation (28) for $n^c u_{c;b}$, equation (6) for $(\rho + 3p)\omega_{ac}n^c$, equation (8) for $\mathcal{E}$, equation (27) for p' and the remaining conditions of the theorem in equation (18), after a lengthy but simple calculations, we get

$$\mathcal{L}_{\lambda^a = \lambda n^a} R_{ij} = 2\Omega R_{ij}$$

and hence $\lambda^a = \lambda n^a$ is a Ricci inheritance vector.

This completes the proof of the theorem.

3. CONSEQUENCES OF THE THEOREM

1. If $\Omega = 0$ then theorem A reduces to the fluid space-times satisfying Einstein field equations to admit Ricci collineation vector.

2. If $\Omega = 0$ then equation (10) reduces to

$$[(\rho - p)\lambda n^a]_{;a} = 0 \tag{39}$$

which serves as a conservation law for the vector λ^a, and for dust model (i.e., $p = 0$) the conservation law for the vector $\lambda^a = \lambda n^a$ is $[\rho\lambda n^a]_{;a} = 0$.

3. If $\omega = 0$ then from equation (6), either $\rho = p$ (stiff equation of state) or $L_a = 0$. When $L_a = 0$, the integral curves of n^a are material curves in the fluid. Thus, theorem A provides a direct relationship between Ricci inheritance vector and material curves.

4. If the fluid satisfies the stiff equation of state $\rho = p$ then equation (6) reduces to $\omega_{ac}n^c = 0$ for $\omega \neq 0$. Now contracting $\omega_{ac}n^c$ with $\eta^{abcd}\omega_c u_d$ and using equation (29), we get

$$n^a = [\omega_c n^c/\omega^2]\omega^a \tag{40}$$

and as n^a and ω^a are nonzero, we have

$$n^a = \pm\omega^a/\omega \tag{41}$$

Thus, n^a may be considered as a unit vector tangent to a vortex line ([13]).

5. From equation (7), either the fluid satisfies the stiff equation of state $\rho = p$ or the shear tensor $\mathcal{T}_{ab}$ is related to the Ricci inheritance vector λ^a through the equation $\mathcal{T}_{ab} = 2\Omega\lambda^{-1}$.

6. From equation (8), either $\rho = p$, or

$$\frac{1}{2}\mathcal{E}n_a = n'_a + (log\lambda)_{,a} \tag{42}$$

while form equation (9), either $\rho = p$, or

$$\frac{1}{2}\mathcal{E} = 2\Omega\lambda^{-1} - n_c\dot{u}^c \tag{43}$$

which may be considered as a kinematical equation. From equations (42) and (43) we have

$$n'_a = n_a n_c \dot{u}^c + 2n_a\Omega\lambda^{-1} - (log\lambda)_{,a} \tag{44}$$

which is a relation between the covariant differentiations along a fluid particle world line and along an integral curve of n^a, and the Ricci inheritance vector.

7. It is known [16] that for a radiative perfect fluid, the resulting universe must be homogeneous and isotropic. Thus, if we take $p = \frac{1}{3}\rho$ then theorem A reduces to

Corollary 1: A homogeneous and isotopic universe satisfying Einstein field equations admits a Ricci inheritance vector $\lambda^a = \lambda n^a$ if and only if

$$(i) \quad \rho(2\omega_{ac}n^c - \frac{1}{3}L_a) = 0$$

$$(ii) \quad \mathcal{T}_{ab} = 3\Omega(\lambda\rho)^{-1}$$

$$(iii) \quad \frac{2}{3}\rho[n'_a + (log\lambda)_{,a} - \frac{1}{2}\mathcal{E}n_a] = 0$$

$$(iv) \quad \frac{2}{3}\rho[\frac{1}{2}\mathcal{E} + n_c\dot{u}^c - 2\Omega\lambda^{-1}] = 0$$

$$(v) \quad [\rho\lambda^a]_{;a} - 4\rho\Omega\lambda^{-1} = 0$$

From (iv) above, since $\rho \neq 0$,

$$2\Omega\lambda^{-1} = \frac{1}{2}\mathcal{E} + n_c\dot{u}^c$$

which when substituted in (ii) of Corollary 1 yields

$$\mathcal{T}_{ab} = \frac{3}{2}[\frac{1}{2}\mathcal{E} + n_c\dot{u}^c]\rho^{-1}$$

which is a relation between the expansion and shear of the space-like congruences generated by n^a.

8. When $p = 0$ then theorem A leads to the following

Corollary 2: A dust universe satisfying Einstein field equations admits a Ricci inheritance vector $\lambda^a = \lambda n^a$ if and only if

$$(i) \quad \rho(\omega_{ac}n^c - \frac{1}{2}L_a) = 0$$

$$(ii) \quad \mathcal{T}_{ab} = 2\Omega(\lambda\rho)^{-1}$$

$$(iii) \quad \rho[n'_a + (log\lambda)_{,a} - \frac{1}{2}\mathcal{E}n_a] = 0$$

$$(iv) \quad \rho[\frac{1}{2}\mathcal{E} + n_c\dot{u}^c - 2\Omega\lambda^{-1}] = 0$$

$$(v) \quad [\rho\lambda^a]_{;a} - 4\rho\Omega\lambda^{-1} = 0$$

9. Irrotational dust space-time have recently been studied by Maartens et al [17] who showed that such universes in general relativity must have both gravito-electric and gravito-magnetic fields. For these universes to admit Ricci inheritance vector we have

Corollary 3: An irrotational dust universe ($p = 0,\ \omega_a = 0,\ \rho > 0$) satisfying Einstein field equations admits a Ricci inheritance vector $\lambda^a = \lambda n^a$ if and only if

$$(i)\ \ \frac{1}{2}\rho L_a = 0$$

$$(ii)\ \ \mathcal{T}_{ab} = 2\Omega(\lambda\rho)^{-1}$$

$$(iii)\ \ \rho[n'_a + (log\lambda)_{,a} - \frac{1}{2}\mathcal{E}n_a = 0$$

$$(iv)\ \ \rho[\frac{1}{2}\mathcal{E} + n_c\dot{u}^c - 2\Omega\lambda^{-1}] = 0$$

$$(v)\ \ [\rho\lambda^a]_{;a} - 4\rho\Omega\lambda^{-1} = 0$$

Since $\rho \neq 0$, it follows from (i) above that $L_a = 0$ and thus the integral curves of n^a are material lines in an irrotational dust space-time.

4. DISCUSSION

The results obtained in this paper are dynamical as they are obtained with the help of Einstein field equations, and they depend upon the nature of the energy momentum tensor and the equation of state. The relationships between the Ricci inheritance vector, the material lines of the fluid space-time, expansion, shear and vorticity of the space-like congruences generated by n^a have been established. The stiff equation of state ($\rho = p$) is seen to be related with the vorticity vector ω_a. For isotropic and homogeneous universe ($p = \frac{1}{3}\rho$), dust universe ($p = 0,\ \omega \neq 0$) and irrotational dust universe ($p = 0,\ \omega_a = 0,\ \rho > 0$) the existence of Ricci inheritance vector is

obtained in terms of the expansion and shear of the congruence generated by n^a, and it is seen that in an irrotational dust universe the integral curves of n^a are material lines.

It may be noted that for the scalar $\Omega = 0$, the results obtained here reduce to the space-time admitting Ricci collineation. Moreover, similar results can be obtained if the energy momentum tensor of the perfect fluid is replaced by the energy momentum tensor of imperfect (i.e., viscous and heat conducting) and anistropic fluids.

Acknowledgement: The author acknowledges kind and warm hospitality and highly stimulating atomosphere of IUCAA where this work was completed.

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