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GRAVITATIONAL COUPLING CONSTANT IN ARBITRARY DIMENSION

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Assuming the equivalence of Friedmann-Robertson-Walker cosmological models and their Newtonian counterparts, we propose, using the Gauss law in arbitrary dimension, a general relation between the Newtonian gravitational constant G and the gravitational coupling constant κ.

1. Introduction

It is now a common belief among cosmologists and relativists that although the spacetime appears smooth, nearly flat and 4-dimensional on large scale, at sufficiently small distances and early times it is highly curved, with all possible topologies and arbitrary dimensions. The initial idea of Kaluza-Klein has been extensively used in unified theories of fundamental interactions [1]. There, the extra dimensions are assumed to be compactified to Planckian size, and therefore do not display themselves in macroscopic processes. The multidimensional cosmologies based on these ideas have been extensively studied in the recent years [2–4].

Usually in cosmological models based on higher dimensions the problem of the dimensionality of the gravitational coupling constant is not tackled on, being tacitly assumed to be

$$\kappa = 8\pi G, \tag{1}$$

where G is the Newtonian gravitational constant. This is, of course, a relation being derived in four dimensions. Even in some textbooks this has not been differentiated [5]. The effective gravitational constant in mutidimensional cosmologies is defined by multiplication of this 4-dimensional value with some volume of the internal space which might even be infinite [6].

There are, however, cosmological models in higher dimensions where this picture does not work and one should be cautious about the value of the coupling constant in dimensions higher than four [7–10]. In fact, Chaterjee [7] and Chaterjee and Bhui [8] calculated a homogeneous model in higher dimensional gravity using the same 4-dimensional relation as in (1).

We intend to show some deficiencies of this misuse and propose a generalization of the gravitational coupling constant for an arbitrary dimension. Any definition of a coupling constant in higher dimension will affect the time dependence of G and therefore may have direct observational consequences [11–13].

In this note we confine ourselves to a mere generalization of κ, using the discrepancies in comparing relativistic and Newtonian cosmologies in arbitrary dimensions. It belongs to the folklore of the theories of gravitation that their weak limit must be the Newtonian theory of gravity. Moreover, on account of the relation (1), there is a Newtonian derivation of the FRW cosmologies. We will show that this derivation using the coupling constant (1) is just valid in (3+1) dimensions and must be changed for arbitrary-dimensional theories. On account of the Gauss theorem we give such a generalization which makes the Newtonian derivation valid in arbitrary dimension.

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2. Friedmann models and Newtonian cosmology in D dimensions

Consider the following Hilbert-Einstein action in $D+1$ dimensional space-time with D its fixed spatial dimension:

$$S_g = -\frac{1}{2\kappa}\int {}^{(D+1)}R\sqrt{-g}\, d^{D+1}x + \frac{1}{2}\int T\sqrt{-g}\, d^{D+1}x, \tag{2}$$

where κ is again the gravitational constant. For simplicity, we consider the $k = 0$ FRW model. In an arbitrary fixed dimension we obtain the following Friedmann equation [9]:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2\kappa}{D(D-1)}\rho, \tag{3}$$

leading to the familiar Friedmann equation in $D = 3$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\rho = \frac{8\pi G}{3}\rho. \tag{4}$$

Now, it is well known that the Friedmann equation (4) can be derived from the Newtonian standpoint. Taking the Newtonian equation of gravitation in D dimension in its usual form

$$\nabla_D^2\,\varphi = 4\pi G\rho, \tag{5}$$

where φ is the gravitational potential and ∇_D is the D-dimensional ∇ operator, the corresponding Friedmann equation can easily be obtained to be

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{D(D-2)}\rho. \tag{6}$$

Now, as expected, for $D = 3$, the familiar relation (4) is obtained, assuming the relation (1). But if $D \neq 3$, one can realize that the agreement between (4) and (6) fails.

3. Modification

Looking for the roots of the factor 8π in (1) we come across the relation

$$R_{00} = \nabla^2\varphi, \tag{7}$$

where φ is the Newtonian gravitational potential.

Now, the coefficient in the Poisson equation, i.e. 4π has been obtained using the Gauss law for 3-dimensional space. Thus we should first derive the correct coefficient for D-dimensional space. Applying Guass's law for a D-dimensional volume, we find the Poisson equation for arbitrary fixed dimension:

$$\nabla^2\varphi = \frac{2\pi^{D/2}G}{(D/2-1)!}\rho. \tag{8}$$

On the other hand, we get for arbitrary $D \neq 2$

$$R_{00} = \left(\frac{D-2}{D-1}\right)\kappa\rho. \tag{9}$$

A comparison of (7), (8), and (9) gives us the following modified Einstein gravitational constant for any D except $D = 2$:

$$\kappa_D = \frac{2(D-1)\pi^{D/2}G}{(D-2)(D/2-1)!}. \tag{10}$$

The correct form of the Friedmann equation in an arbitrary fixed dimension, except $D = 2$, is now derived to be

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi^{D/2}G}{D(D-2)(D/2-1)!}\rho. \tag{11}$$

As is easily seen, the above relation is in complete agreement with its Newtonian counterpart in all dimensions except $D = 2$.

For the case of $2+1$ dimension, we have to revisit equation (9) which is

$$\begin{aligned} R_{00} &= \frac{1}{2}g_{00}R + \kappa T_{00} \\ &= \frac{\kappa}{D-1}\left[(D-2)T_{00} + T^i{}_i\right] \\ &= \frac{\kappa}{D-1}\left[(D-2)\rho + Dp\right] \end{aligned} \tag{12}$$

for a perfect fluid universe. It is clear from Eq. (12) that the first term is identically equal to zero for $D = 2$. Therefore, in this case we get

$$R_{00} \cong \kappa(0\times\rho + 2p) = 2\pi G\rho, \qquad \text{for } D = 2. \tag{13}$$

In a matter-dominated universe — in which we usually compare the weak field limit of the relativistic equations with the Newtonian ones — there is ignorable radiation as compared to matter. For this special case, however, we can no more ignore the presence of radiation as compared to the vanishing first term in (13). Thus, by setting $p = (\rho/D) = (\rho/2)$, κ can be found to be

$$\kappa_2 = 2\pi G, \tag{14}$$

hence the Friedmann equation (3) becomes simply

$$\left(\frac{\dot{a}}{a}\right)^2 = 2\pi G\rho. \tag{15}$$

So the Einstein gravitational coupling constant would be in general

$$\kappa_D = \begin{cases} 2\pi G, & \text{for } D = 2, \\ \dfrac{2(D-1)\pi^{D/2}G}{(D-2)(D/2-1)!}, & \text{for } D \neq 2, \end{cases} \tag{16}$$

and thus

$$\left(\frac{\dot{a}}{a}\right)^2 = \begin{cases} 2\pi G\rho, & \text{for } D = 2, \\ \dfrac{4\pi^{D/2}G}{D(D-2)(D/2-1)!}\rho, & \text{for } D \neq 2. \end{cases} \tag{17}$$

However, ρ scales as $a^{-D(1+w)}$ where w is a constant number depending on the equation of state: it is either zero or $(1/D)$. Substituting the ρ into Eq. (17), we get the following law for the expansion factor in different dimensions:

$$a(t) \propto \begin{cases} t^{2/(D_R+1)}, & \text{radiation dominated}, \\ t^{2/D_M}, & \text{matter dominated}. \end{cases} \tag{18}$$

For instance, for $D = 1$, $D = 2$ and $D = 3$, $a(t)$ scales as

$$a(t) \propto \begin{cases} -t^2, & \text{for } D = 1, \\ t, & \text{for } D = 2, \\ t^{2/3}, & \text{for } D = 3. \end{cases} \tag{19}$$

in a matter-dominated universe and as

$$a(t) \propto \begin{cases} -t, & \text{for } D = 1, \\ t^{2/3}, & \text{for } D = 2, \\ t^{1/2}, & \text{for } D = 3. \end{cases} \tag{20}$$

in a radiation-dominated universe. By comparison one can find $D_M - 1 = D_R$, i.e., the time behaviour of $a(t)$ in each radiation-dominated dimension is same as in a matter dominated phase but one dimension less. For example, the behaviour of $a(t)$ in a $D = 4$ radiation-dominated phase is identical to that of a $D = 3$ matter-dominated one, etc.

It is worth mentioning that the strange behaviour of $a(t)$ for $D = 1$, with a minus sign, is due to applying Eq. (12). This can be explained by recalling the fact that Eq. (12) cannot be derived from the Einstein equations

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{21}$$

in the case $D = 1$. The result $a(t) < 0$ in (19) and (20) for $D = 1$, which is a consequence of adopting (12) as the dynamical equation of gravity, leads to a negative density. This is, of course, physically problematic.

4. Conclusion

The dimensional dependence of the gravitational constant κ may have various and serious field-theoretic and astrophysical consequences hitherto unnoticed.

It would be interesting, e.g., to see which changes are to be expected if the results of this note are combined with the Kaluza-Klein paradigm. The consequences on time variation of G within various models is another very interesting cosmological issue which is under investigation.

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