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On Singularity Free Spacetimes–II : Geodesic Completeness

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Abstract

We study the geodesics of the singularity free metric considered in the preceding Paper I and show that they are complete. This once again demonstrates the absence of singularity. The geodesic completeness is established in general without reference to any particular matter distribution. The metric is globally hyperbolic and causally stable. The question of inapplicability of the powerful singularity theorems in this case is discussed.

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I. Introduction

In paper I [1], we have deduced a general form of the singularity free spacetime with cylindrical symmetry [2] by effecting a simple and natural inhomogenisation and anisotropisation of the Friedman–Robertson–Walker (FRW) metric with negative curvature. This metric can represent a perfect fluid, a fluid with radial heat flow and massless scalar field as well as two distinct vacuum solutions. The matter field has acceptable physical behaviour and satisfies energy and causality conditions. In here we wish to examine the nature of the geodesics of the general metric, not necessarily representing a fluid model. We shall show that the spacetime is geodesically complete and hence free of singularities. This will however restrain the range of free parameters occurring in the metric.

In §II we reiterate some general features of the metric while §III is devoted to a detailed analysis of the non–spacelike geodesics to establish that they are complete,i.e. they can be extended to the arbitrary values of their affine parameter. That means geodesics never terminate indicating absence of singularity. Since the spacetime is singularity free, how does it avoid the application of general results of the powerful singularity theorems [3]? It turns out that all the energy and causality conditions hold good but the assumption of existence of compact trapped surfaces fails. Finally this as well as global hyperbolicity and causal stability of the metric are discussed in §IV.

The geodesic completeness of the singularity–free radiation spacetime [4] has been established through a detailed analysis [5]. We shall follow the same procedure to study this general case.

II. The Metric

Even at the risk of repeatation we shall briefly reiterate the general features of the metric discussed in detail in Paper I. this is to make the paper self contained enough so that it can be read independently. From Paper I, we take the metric for singularity free spacetime and write

$$ds^2 = C^{2\alpha}(kt)C^{2a}(mr)(dt^2 - dr^2) - C^{2\beta}(kt)C^{2b}(mr)dz^2 - m^{-2}S^2(mr)C^{2\gamma}(kt)$$
$$C^{2c}(mr)d\phi^2 \tag{1}$$

where $C(x) = coshx, S(x) = sinhx$. This will represent a perfect fluid universe with cylindrical symmetry with appropriate constraints on the parameters $\alpha, \beta, \gamma, a, b, c$ and m, k. In fact, it gives rise to a two parameter family (i.e. of the eight parameters only two remain free) of exact solutons of Einstein's equations. In the natural comoving coordinates, the fluid velocity field is given by

$$\mathbf{u} = C^{\alpha}(kt)C^{a}(mr)dt \tag{2}$$

which is orthogonal to the spacelike hypersurface $t =$ constant. The fluid congruence is hence rotation free but it has non-zero expansion and shear which read as follows :

$$\theta = \frac{(2\gamma + \beta)}{A} kT(kt) \tag{3}$$

$$\sigma^2 = \frac{2}{3A^2}(\gamma - \beta)^2 k^2 T^2(kt) \tag{4}$$

where $T(kt) = tanh(kt)$ and $A = C^{\alpha}(kt)C^{a}(mr)$. It is clear that expansion and shear change their sense at $t = 0$, $\theta \gtrless 0$ as $t \gtrless 0$). Both tend to zero as $t \to \pm\infty$ or $r \to \infty$ if $\alpha > 0, a > 0$.

The velocity field (2) is not geodetic and the acceleration vector is

$$\dot{u}_r = -amS(mr)C^{-a-1}(mr)C^{-\alpha}(kt) \tag{5}$$

The non-geodetic character of the congruence plays the crucial role in avoiding formation of singularities. It means that there exists spacelike **pressure gradient** that counteracts gravitational attraction to provide a bounce for the universe when it is in contracting phase for $t < 0$. It converts contraction to expansion at $t = 0$ without formation of a singularity. The presence of shear also helps in this process for not letting the congruence to focus.

The regularity of Weyl and Ricci curvatures and the isotropy of fluid demand $\alpha + \beta = 1$, $\alpha = \gamma$ and,

(i) $b = c, a = -b(1+2b)^{-1}, k = m(1+2b)$

(ii) $b + c = 1, a = -b(1-b), k = 2m$.

These are the only two cases for which the metric (1) yields a family and singularity free models. For the former there exists no equation of state in general, for $b = -1/3$ it gives the Senovilla radiation universe [4] with $\rho = 3p$ while for the latter it is always $\rho = p$ representing the stiff fluid.

The density and pressure of the fluid in general read as follows :

$$\begin{aligned} 8\pi\rho A^2 =& (a-3b)m^2 + \frac{a}{b}k^2 + \frac{(b-a)(3b+a)}{4b^2}k^2T^2(kt) \\ &+ \frac{1}{2}(a-3b)(b+c-1)m^2T^2(mr) \end{aligned} \tag{6}$$

$$\begin{aligned} 8\pi p A^2 =& (a+b)m^2 - k^2 + \frac{(b-a)(3b+a)}{4b^2}k^2T^2(kt) \\ &+ \frac{1}{2}(a+b)(b+c-1)T^2(mr) \end{aligned} \tag{7}$$

where $A = C^\alpha(kt)C^a(mr)$. It can be easily verified from the above expressions that $\rho = 3p$ for $b = -1/3 = c$ and $\rho = p$ for $b + c = 1$. At a given t, ρ and p are maximum at $r = 0$ and the overall maximum occurs at $t = 0$ and $r = 0$. The parameter m or k can be identified with the largest value of energy density, that can be chosen as large (or small) as one wishes. ρ and p can be made positive everywhere satisfying the strong energy condition. Since ρ and p are regular and finite everywhere, it indicates absence of physical (or Ricci) singularity. We have verified that the metric (1) is free of Weyl singularity as all Weyl curvatures are regular and finite and so are the kinematic parameters.

The metric admits two spacelike killing vectors $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \phi}$ which are mutually as well as hypersurface orthogonal. The spacetime is cylindrically symmetric with 2π period for the angular coordinate ϕ. The singularity at $r = 0$ is the coordinate

singularity of cylindrical coordinates. The t coordinate decreases or increases along every past or future directed non-spacelike curve, and hence it defines a cosmic time for the spacetime. The gradient of t always remains timelike which implies the causal stability of the metric. It is the stronger condition that includes weaker chronology and causality conditions.

Having seen that the metric (1) is free of the curvature singularities and also satisfies proper causality conditions, we shall in the next Section show that the spacetime is geodesically complete. That means it cannot be extended, ruling out the possibility of hidden singularities. This is though quite transparent from the form of the metric, we shall however show it explicitly by studying the geodesics of the metric in detail.

III. Geodesics

To demonstrate the geodesic completeness of the metric, we should show that all non-spacelike geodesics can be extended to arbitrary values of the affine parameter. For this let us write the geodesics of the metric (1) following the standard procedure and they would read as follows :

$$C^{2\alpha}(kt)C^{2a}(mr)(\dot{t}^2 - \dot{r}^2) - L^2C^{-2\beta}(kt)C^{-2b}(mr) - m^2M^2S^{-2}(mr)C^{-2\gamma}(kt)C^{-2c}(mr) = \delta, \tag{8}$$

$$C^{2\beta}(kt)C^{2b}(mr)\dot{z} = L, \tag{9}$$

$$m^{-2}S^2(mr)C^{2\gamma}(kt)C^{2c}(mr)\dot{\phi} = M, \tag{10}$$

$$\ddot{r} + amT(mr)(\dot{t}^2 + \dot{r}^2) + 2\alpha kT(kt)\dot{t}\dot{r} - M^2m^3S^{-3}(mr)C^{-2(\gamma+\alpha)}(kt)C^{-2(c+a)+1}(mr)(1 + cT^2(mr)) - L^2bmS(mr)C^{-2(\beta+\alpha)}(kt)C^{-2(b+a)-1}(mr) = 0, \tag{11}$$

$$\ddot{t} + \alpha k T(kt)(\dot{t}^2 + \dot{r}^2) + 2amT(mr)\dot{t}\dot{r}$$
$$+ \gamma k m^2 M^2 S^{-2}(mr) S(kt) C^{-2(\gamma+\alpha)-1}(kt) C^{-2(c+a)}(mr)$$
$$+ \beta k L^2 S(kt) C^{-2(\beta+\alpha)-1}(kt) C^{-2(b+a)}(mr) = 0. \qquad (12)$$

A dot denotes derivative with respect to the affine parameter, and L and M are the two constants of motion corresponding to the two killing vectors representing the conserved z and ϕ momenta. The another constant of motion is δ due to the rest mass which is one for timelike and zero for null geodesics. Since all the terms in equations (8–12) are non-singular hence the solutions to the equations will exist and they will be unique.

We shall now examine the behaviour of first and second derivatives of the coordinates relative to the affine parameter. We should put finite bounds on the first derivatives [6] which will imply that the geodesics are complete. However, the second derivatives should not be singular to ensure that the field is overall non-singular. Without any loss of generality we can restrict to the future pointing geodesics. In this discussion, we donot restrict to fluid distributions, i.e. the parameters $\alpha, \beta, \gamma, a, b, c, m, k$ are treated as arbitrary. Following the ref. [5], we shall first consider the particular geodesics and shall then come to the general case.

(a) Fluuid congruence : The simplest geodesics are the ones associated with the fluid congruence. For these we have $\dot{r} = \dot{\phi} = \dot{z} = 0$. It follows from eq. (2) that the only geodesic possible in this case has to lie on the axis $r = 0$ and it is given by

$$\dot{t} = C^{-\alpha}(kt) \leq 1 \quad \text{for} \quad \alpha \geq 0.$$

That means α must be non-negative and then the geodesics are obviously complete.

(b) Axial geodesics : For the geodesics lying on the axis we have $\dot{r} = 0 = \dot{\phi}, r = 0$ and so we write

$$\dot{z} = \dot{L} C^{-2\beta}(kt),$$

$$\dot{t} = C^{-\alpha}(kt)[L^2C^{-2\beta}(kt) + \delta]^{1/2} \leq (L^2 + \delta)^{1/2},$$

provided $\alpha + \beta \geq 0$. Then the derivatives will be bounded everywhere and the geodesics will be complete. Note that the coordinate t attains infinite value only when so does the affine parameter. This property will be shared by all the geodesics and will not be referred to henceforth.

(c) Radial null geodesics : Here $\dot{z} = 0 = \dot{\phi}, \delta = 0$, the first integrals give

$$\dot{t} = |\dot{r}|, \quad \dot{t} = hC^{-2\alpha}(kt)C^{-2a}(mr), \quad h = \text{const}.$$

It is clear that $|\dot{r}| = \dot{t} \leq h$ provided $\alpha \geq 0, a \geq 0$ and then the geodesics will be complete. The geodesics with fixed ϕ comfortably continue along $\pi + \phi$ after crossing the axis.

(d) Radial timelike geodesics: Here $\delta = 1$, $\dot{z} = 0 = \dot{\phi}$. Let us parametrise $\dot{t}$ and $\dot{r}$ by writing

$$\dot{t} = C(v)A(r,t), \dot{r} = S(v)A(r,t), \dot{v} = B(r,t,v)A(r,t)$$

where $A = C^{-\alpha}(kt)C^{-a}(mr)$. Then

$$B = -[amT(mr)C(v) + \alpha kT(kt)S(v)].$$

We have $a \geq 0, \alpha \geq 0$ and take $k, m \geq 0$ which will imply $\dot{v} \leq 0$. We shall consider the role of the second derivatives in the end. The same reasoning will apply here to show that geodesics are complete.

(e) Null geodesics with zero angular momentum : In this case we have $\dot{\phi} = 0, \delta = 0$ and as before we write

$$\dot{t} = C(v)E(r,t), \dot{r} = S(v)E(r,t),$$

and

$$\dot{v} = E(r,t)F(r,t,v).$$

Then we obtain

$$E = |L|C^{-(\alpha+\beta)}(kt)C^{-(a+b)}(mr),$$

$$F = -[k(\alpha-\beta)T(kt)S(v) + m(a-b)T(mr)C(v)]$$

We shall require $\alpha+\beta \geq 0$, $a+b \geq 0$, $\alpha \geq \beta$ and $a \geq b$ for $\dot{v} \leq 0$.

(f) Null geodesics on the hypersurfaces $z = const.$: These are defined by $\dot{z} = 0, \delta = 0$, and as before we have

$$\dot{t} = C(v)P(r,t), \dot{r} = S(v)P(r,t)$$

and

$$\dot{v} = P(r,t)D(r,v)$$

where

$$P = |M|mS^{-1}(mr)C^{-(\gamma+\alpha)}(kt)C^{-(a+c)}(mr),$$

$$D = k(\gamma-\alpha)T(kt)S(v) + mC(v)[(c-a)T(mr) + T^{-1}(mr)].$$

As has been noted earlier that the regularity of the Ricci tensor requires $\alpha = \gamma$. Hence $D = D(r,v)$ as assumed.

In this case we can obtain one of the equations of the orbit that on integration yields

$$C(v) = N^{-1}S(mr)C^{c-a}(mr), N = \text{const.} > 0.$$

Since $C(v) \geq 1$, the coordinate r is bounded between the roots of $S(mr)C^{c-a}(mr) = N$. Clearly $|v|$ is bounded and the geodesics are complete. Among them there could be a circular geodesic, corresponding to the double root, which will go around the axis at the fixed radius.

(g) General non-spacelike geodesics : Now we come to the general case. From eq. (11) it follows that $\ddot{r}$ will be negative for positive t and increasing r provided $b \leq 0$ (for large r the term with L^2 will dominate over that with M^2). With $\ddot{r} < 0$, the r-coordinate cannot diverge to infinity in a finite proper time. As r decreases the $\dot{t}\dot{r}$ term will dominate over the $(\dot{t}^2 + \dot{r}^2)$ term while the L^2 term will tend to zero as $r \rightarrow 0$. In the vicinity of the axis $r = 0$, $\ddot{r} > 0$ for decreasing r and $t > 0$. Thus the geodesics cannot collapse quickly enough into the axis to become singular. It is this feature that really provides the *bounce off* to the universe, turning contraction to expansion at $t = 0$.

The above arguments will apply to the cases (d) and (e) above and hence the geodesics will be complete for them. As regards the t-coordinate, $\ddot{t}$ should be negative for large values of t, so that it keeps its growth in check, not letting it diverge to infinity for finite values of the affine parameter. From eq. (12), it can be seen that this will be so even if $\beta \leq 0$ because the $(\dot{t}^2 + \dot{r}^2)$ will always be the dominant term (for decreasing r the $\dot{t}\dot{r}$ term will not be relevant). $\dot{z}$ is always regular so long as t and r are regular. $\dot{\phi}$ diverges as $r \rightarrow 0$ but with $\dot{\phi} \neq 0$, the geodesics can never reach $r = 0$ as has been demonstrated in (f) above. In the neighbourhood of the axis eq. (10) simply represents the centrifugal effect.

It is straightforward to carry out the similar calculations as done in the previous cases but they are very combersome and not very enlightening and hence we will not report them here. It can be verified that the general geodesics are also complete. Let us now collect together all the restrictions prescribed by the above considerations on the parameters and they are :

$$\alpha \geq 0, \alpha + \beta \geq 0, \alpha \geq \beta$$

$$a \geq 0, a+b \geq 0, a \geq b, b \leq 0, k \geq 0, m \geq 0$$

and $\alpha = \gamma$ implied by the regularity of the Ricci tensor. It is interesting to note that α, β and a, b satisfy the similar conditions. All this follows purely by requiring the geodesics to be complete without any reference to the matter distribution. That is in this framework one can *a priori* ensure the geodesic completeness by adhering to the above conditions on the parameters. The singularity free perfect fluid models discussed in Paper I satisfy all these restrictions.

IV. Discussion

It follows that all the conclusions drawn in Ref. [5] for the radiation model [4] remain true for this general case as well. We summarise some of them in the following:

The surfaces $t =$ const. are global cauchy surfaces and hence the spacetime is globally hyperbolic. This will mean that it is also causally simple. This is quite obvious from this analysis of the previous section. Since every $t =$ const. hypersurface is a global cauchy surface and t defines a time function in the spacetime, it follows that every non-spacelike curve (need not be geodesic) can be extended to arbitrary values of its generalised affine parameter as it has to cross all the cauchy surfaces. That means the spacetime is *singularity free.*

It is transparent from the expression for ρ and p that the strong energy as well as the generic conditions are satisfied. This indicates that our spacetime satisfies all the energy and causality conditions. However, it fails to obey the condition of the existence of compact trapped surfaces, which has been assumed in the singularity theorems. And this lets it escape the application of the theorems. We shall now demonstrate that the spacetime does not admit trapped surfaces.

For this we should compute trace of the two null second fundamental forms [3] and that is given by

$$(g^{ab}X_{ab})_{\mp} = \frac{1}{\sqrt{2}}C^{-\alpha}(kt)C^{-a+1}(mr)S^{-1}(mr)$$
$$[\mp m \mp m(b+c)T^2(mr) - k(\gamma+\beta)T(kt)T(mr)]$$

It is clear that $(X^a_a)_- \leq 0$ provided $b+c+1 \geq 0$ and $(X^a_a)_+ \geq 0$ for $m(1+b+c) \geq k(\alpha+\beta)$. Since the traces have opposite signs, hence there are no closed trapped surfaces.

This means that the outgoing and incoming radial null geodesics are respectively expanding and contracting everywhere. For existence of a closed trapped surface, in some region they should all be contracting, i.e. the trace should be > 0 for the both in some region. It can be verified that the singularity free models of Paper I satisfy the conditions just deduced. Thus there occur no closed trapped surfaces in the singularity free family of metrics.

In Paper I (and Ref. [2]), we have deduced a general singularity free form for the cylindrically symmetric metric. In this paper, we have demonstrated by analysing its geodesics that the general metric is *really free* of singularities with proper restrictions on the parameters. This is a general analysis without reference to any matter distribution. That is the constraints on the parameters deduced above have always to be respected may what be the energy content of the universe.

The non-existence of closed trapped surfaces can be ensured in advance so as to escape the application of the singularity theorems [3]. This however does not conflict with the requirements of the energy as well as the fluid consistency conditions. That is dropping of this assumption has no adverse physical implications, makes the theorems inapplicable and provides a singularity free cosmological model.

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