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Inhomogeneity and non-singularity of cosmological models

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Abstract

We establish that spherical symmetry cannot sustain inhomogeneity and for its inclusion one has to use cylindrical fluid models which can be free of the big-bang singularity.

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1 Introduction

Relativistic cosmology is characterised by the two main features: expansion of the Universe and its birth in a hot big-bang singular event. These are the predictions of the standard Friedman-Robertson-Walker (FRW) model which is represented by a solution of Einstein's field equations for gravitation (general relativity, GR). Even though these predictions are strongly supported by the present day observations, it is pertinent to ask: Is the big-bang singular beginning unique, i.e. does GR not admit a solution without the big-bang? For it is one thing to have the unique possibility and quite the other that observations have to pick up one of the two possibilities admitted by the theory. In the case of the big-bang, this question is even more in order for the FRW that imbibes it has very special properties of homogeneity, isotropy and spherical symmetry. One of the main points of this paper is to show that the big-bang can be avoided by considering inhomogeneous fluid models. It turns out that inhomogeneity cannot be accommodated in spherical symmetry. For its inclusion we will have to consider cylindrical symmetry which admits inhomogeneous fluid models with or without big-bang singularity. It may be noted that the non-singular models satisfy all physical conditions and have very acceptable physical behaviour.

The general belief that the big-bang singularity is inevitable was based on the singularity theorems [1] which predicted the inevitability of singularity in GR under physically reasonable assumptions. Hence all the effort was focussed on how to "manage" the big-bang singularity. Quantum effects and new fields as well as modification of GR were invoked to deal with the very dense state of the early Universe. A new field of quantum cosmology and early Universe cosmology has come into being with very intense activity.

In 1990 Senovilla [2] produced a counter example to the general belief by obtaining an exact solution of Einstein's equations free of singularity as well as satisfying all physical conditions. It represented a radiation Universe with $\rho = 3p > 0$ which was inhomogeneous, anisotropic and cylindrically symmetric. Thus the big-bang is not the unique prediction of the theory. Hence it is in order to address the question, what causes or what can avoid the big-bang?

In Section 2 we set up the general equations for an orthogonal metric separable in space and time variables for a perfect fluid distribution. In Section 3 we consider inclusion of inhomogeneity which leads to cylindrical models with or without singularity. In Section 4 we conclude with a discussion.

2 Field equations

In cosmology we always assume separability of the spacetime metric in space and time variables. This is a simplifying assumption that helps in solving the equations as well as in understanding the overall behaviour of the Universe. We consider the general orthogonal metric,

$$ds^2 = Ddt^2 - Adx_1^2 - Bdx_2^2 - Cdx_3^2 \tag{2.1}$$

where $A = A(t)A(x_\alpha)$ and so on. Einstein's field equations for the fluid model are given by

$$R_{ik} = -8\pi(T_{ik} - \frac{1}{2}Tg_{ik}) \tag{2.2}$$

where

$$T_{ik} = (\rho + p)u_i u_k - pg_{ik}. \tag{2.3}$$

We take the co-moving fluid congruence, $u_i = \sqrt{D}\delta_i^0$.
Now the fluid conditions will require

$$T_{ab} = 0 \text{ for } a \neq b \tag{2.4}$$

and

$$T_1^1 = T_2^2 = T_3^3 \tag{2.5}$$

which are in all 8 conditions.

To go any further we have to consider the explicit expressions for T_{ik}, which look very formidable [3]. We have however, discovered a good bit of order in them. Given one each of diagonal and off diagonal components, we can write all the others by prescribing proper permutation rules. Let us begin with

$$\begin{aligned} -32\pi AT_0^1 &= -2\left(\frac{B_0}{B} + \frac{C_0}{C}\right)_1 + \frac{A_0}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right) + \frac{B_0}{B}\left(-\frac{B_1}{B} + \frac{D_1}{D}\right) \\ &\quad + \frac{C_0}{C}\left(-\frac{C_1}{C} + \frac{D_1}{D}\right) \qquad (2.6) \\ -32\pi T_1^1 &= \frac{1}{A}\left[\frac{B_1C_1}{BC} + \frac{D_1}{D}\left(\frac{B_1}{B} + \frac{C_1}{C}\right)\right] \\ &\quad + \frac{1}{B}\left[2\left(\frac{C_2}{C} + \frac{D_2}{D}\right)_2 + \frac{C_2}{C}\left(-\frac{B_2}{B} + \frac{C_2}{C}\right) + \frac{D_2}{D}\left(-\frac{B_2}{B} + \frac{C_2}{C} + \frac{D_2}{D}\right)\right] \end{aligned}$$

$$+\frac{1}{C}\left[2\left(\frac{B_3}{B}+\frac{D_3}{D}\right)_3+\frac{B_3}{B}\left(\frac{B_3}{B}-\frac{C_3}{C}\right)+\frac{D_3}{D}\left(\frac{B_3}{B}-\frac{C_3}{C}+\frac{D_3}{D}\right)\right]$$
$$+\frac{1}{D}\left[-2\left(\frac{B_0}{B}+\frac{C_0}{C}\right)_0-\frac{B_0}{B}\left(\frac{B_0}{B}+\frac{C_0}{C}-\frac{D_0}{D}\right)-\frac{C_0}{C}\left(\frac{C_0}{C}-\frac{D_0}{D}\right)\right] \tag{2.7}$$

where subscripts represent partial derivatives, i.e. $A_0=\frac{\partial A}{\partial t}, A_1=\frac{\partial A}{\partial x_1}$ etc.

The successive cyclic permutations $1\to 2\to 3\to 1$, $A\to B\to C\to A$ yield T_0^2, T_0^3 from $T_0^1; T_3^2, T_1^3$ from T_2^1; and T_2^2, T_3^3 from T_1^1. To get T_2^1 from T_0^1 we write $0\to i2$ (i.e. $A_0\to iA_2, T_0^1\to iT_2^1$) and $B\to C\to D\to B$ while T_0^0 follows fromT_1^1 when $2\to 3\to 1\to i0\to -2$ $(T_1^1\to T_0^0)$ and $A\to D\to B\to C\to A$. Thus we can write all 10 T_{ik}, given the two (one each of diagonal and off diagonal).

For the metric (2.1), the kinematic parameters; expansion, shear and acceleration are given by

$$\theta = \frac{1}{2\sqrt{D}}\left(\frac{A_0}{A}+\frac{B_0}{B}+\frac{C_0}{C}\right) \tag{2.8}$$

$$\sigma^2 = \frac{1}{36D}\left[\left(\frac{B_0}{B}+\frac{C_0}{C}-\frac{2A_0}{A}\right)^2+\left(\frac{C_0}{C}+\frac{A_0}{A}-\frac{2B_0}{B}\right)^2+\left(\frac{A_0}{A}+\frac{B_0}{B}-\frac{2C_0}{C}\right)^2\right] \tag{2.9}$$

$$\dot{u}_\alpha = -\frac{D_\alpha}{2D} \tag{2.10}$$

and the vorticity $w_{ik}=0$. Further we also note, in general,

$$\theta,\alpha = \frac{3}{2}\left[(\sigma^i_\alpha+w^i_\alpha);i-(\sigma_{\alpha i}+w_{\alpha i})\dot{u}^i\right] \tag{2.11}$$

$$= \theta\dot{u}_\alpha+\frac{1}{\sqrt{g_{00}}}\left(\ln\sqrt{|g/g_{00}|}\right)_{0^\alpha} \tag{2.12}$$

where we have taken $u_i=\sqrt{g_{00}}\delta^0_i$.

This means whenever the spacetime is free of shear and vorticity, its expansion is constant over the hypersurface orthogonal to the velocity field u_i. Further, if the quantity g/g_{00} is separable in space and time, then acceleration will also vanish. That is for the separable metric (2.1), acceleration can be non-zero only if shear is non-zero [4].

According to the Raychaudhuri equation [5], singularity can be avoided for the metric (2.1) only if acceleration is non-zero which requires non-zero shear. Hence non-singular spacetimes represented by the metric (2.1) must of necessity be both inhomogeneous and anisotropic. This is a general statement without reference to the matter content of the spacetime.

3 Inhomogeneity

First of all let us write the fluid conditions (2.4)and (2.5) explicitly. They will read as:

$$\frac{A_0}{A}\left(\frac{B_1}{B}+\frac{C_1}{C}\right)+\frac{B_0}{B}\left(-\frac{B_1}{B}+\frac{D_1}{D}\right)+\frac{C_0}{C}\left(-\frac{C_1}{C}+\frac{D_1}{D}\right)=0 \tag{3.1}$$

$$-\ 2\left(\frac{C_1}{C}+\frac{D_1}{D}\right)_2+\frac{B_1}{B}\left(\frac{C_2}{C}+\frac{D_2}{D}\right)+\frac{C_1}{C}\left(-\frac{C_2}{C}+\frac{A_2}{A}\right)+\frac{D_1}{D}\left(-\frac{D_2}{D}+\frac{A_2}{A}\right)=0 \tag{3.2}$$

$$\begin{aligned}
&\frac{1}{A}\left[-2\left(\frac{C_1}{C}+\frac{D_1}{D}\right)_1+\frac{C_1}{C}\left(\frac{A_1}{A}+\frac{B_1}{B}-\frac{C_1}{C}\right)+\frac{D_1}{D}\left(\frac{A_1}{A}+\frac{B_1}{B}-\frac{D_1}{D}\right)\right]\\
+\ &\frac{1}{B}\left[2\left(\frac{C_2}{C}+\frac{D_2}{D}\right)_2-\frac{C_2}{C}\left(\frac{A_2}{A}+\frac{B_2}{B}-\frac{C_2}{C}\right)+\frac{D_2}{D}\left(\frac{A_2}{A}+\frac{B_2}{B}-\frac{D_2}{D}\right)\right]\\
+\ &\frac{1}{C}\left[2\left(\frac{B_3}{B}-\frac{A_3}{A}\right)_3-\frac{A_3}{A}\left(\frac{A_3}{A}-\frac{C_3}{C}+\frac{D_3}{D}\right)+\frac{B_3}{B}\left(\frac{B_3}{B}-\frac{C_3}{C}+\frac{D_3}{D}\right)\right]\\
+\ &\frac{1}{D}\left[2\left(\frac{A_0}{A}-\frac{B_0}{B}\right)_0+\frac{A_0}{A}\left(\frac{A_0}{A}+\frac{C_0}{C}-\frac{D_0}{D}\right)-\frac{B_0}{B}\left(\frac{B_0}{B}+\frac{C_0}{C}-\frac{D_0}{D}\right)\right]=0.
\end{aligned} \tag{3.3}$$

Equations (3.1), (3,2) represent 3 equations each corresponding to $T_{\alpha 0}=0$ and $T_{\alpha\beta}=0$ for $\alpha\neq\beta$ while (3.3) represents two equations $T^1_1=T^2_2=T^3_3$. The cyclic permutation $A\to B\to B\to C\to A$, $1\to 2\to 3\to 1$ will yield the other members.

Now if we let the metric (2.1) have radial symmetry, i.e. $A=A(r)A(t)$ etc., $r^2=x_1^2+x_2^2+x_3^3$, then equation (3.1) implies

$$\left(\frac{A_0}{A}-\frac{B_0}{B}\right)N=0,\ \left(\frac{B_0}{B}-\frac{C_0}{C}\right)N=0 \tag{3.4}$$

where

$$N=\frac{A'}{A}+\frac{B'}{B}+\frac{C'}{C}-\frac{D'}{D} \tag{3.5}$$

and a dash denotes derivative w.r.t.r. That means either

(i) $\frac{A_0}{A} = \frac{B_0}{B} = \frac{C_0}{C}$

or

(ii) $N = 0$.

In the former case (i) $\sigma = 0$ (equation 2.9)) which will lead to vanishing of acceleration implying, according to the Raychaudhuri equation [4], inavoidability of singularity. The other equations will determine the FRW homogeneous and isotropic model uniquely.

In the other case (ii), $D = ABC$,. then equation (3.2) will imply the three relations of the type

$$AB' - A'B = k_1 r \tag{3.6}$$

where k_1, is a constant. Substituting them in (3.3) and some simple manipulations will lead to A, B, C being independent of r. This is the homogeneous Bianchi I model.

Thus radial symmetry (i.e. spherical symmetry) can only admit FRW and Bianchi I models, which are both homogeneous. Hence inhomogeneous fluid models cannot be radially symmetric. For inclusion of inhomogeneity it is necessary to give up spherical symmetry for a less restrictive symmetry like cylindrical symmetry [6].

In cylindrical symmetry, the above case (i) will imply $\frac{A_0}{A} = \frac{B_0}{B} \neq \frac{C_0}{C}$ which means $\sigma \neq 0 \Rightarrow \dot{u}_\alpha \neq 0$. That means singularity can be avoided.

It turns out that cylindrical symmetry admits inhomogeneous fluid models with or without singularity [7]. The singularity free family has been identified and it is shown to be unique, i.e. it constitutes the complete set of non-singular cylindrical models [8]. Bianchi I homogeneous model is, of course, admitted. That means cylindrical symmetry does not rule out homogeneity like spherical symmetry does for inhomogeneity.

4 Discussion

The big-bang was a remarkable prediction of the theory which was strongly supported by the observation. However the question that does the theory provide big-bang as the only possibility is well in order even if observations confirm the big-bang firmly

and solidly. The investigation in this direction was not considered seriously because of the universal application of the singularity theorems [1]. We now know that this was a wrong impression that got universal acceptance, perhaps because the observations strongly supported big-bang.

The assumption that restricts the applicability of the theorems is the existence of compact trapped surfaces, meaning gravity to become strong enough to trap photons in a compact surface. The field equations are supposed to determine the behaviour of the field entirely and any additional prescription will amount to a special circumstance. What really happens is that singularity free models do not admit compact trapped surfaces and hence the theorems become inapplicable.

It is not desirable that the Universe be homogeneous and isotropic at all scales. Inhomogeneity must be incorporated in early times. We have shown that inhomogeneity can be incorporated as well as the big-bang singularity can be avoided if we trade off spherical symmetry to cylindrical symmetry. We are presently attempting to establish that cylindrical symmetry is the unique setting for non-singular models [9].

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