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By

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Singularity Free Spacetimes – I : Metric and Fluid Models

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Abstract

We show that the metric for the singularity free family of fluid models [3] can be obtained by a simple and natural inhomogenisation and anisotropisation procedure from Friedman–Robertson–Walker metric with negative curvature. The metric is unique for cylindrically symmetric spacetime with metric potentials being separable functions of radial and time coordinates. It turns out that fluid models separate out into two classes, with $\rho \neq \mu p$ in general but $\rho = 3p$ in particular and $\rho = p$. It is shown that in both the cases radial heat flow can be incorporated without disturbing the singularity free character of the spacetime. Further by introducing massless scalar field it is possible to open out a narrow window for $\mu, 4 \geq \mu \geq 3$.

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I. Introduction

The standard view, the universe had a big-bang singular origin predicted by the standard Friedman–Robertson–Walker (FRW) model, was supported on the formal grounds by the strong singularity theorems [1]. That occurence of singularity is inescapable if we adhere to reasonable energy and causality conditions and to general relativity (GR). To handle the cosmic singularity several attempts have been made by several authors. They include modification of GR, invoking quantum effects and new fields, ascribing unusual properties of matter etc. By sacrificing energy conditions and other physical properties, it is easy to construct a singularity free model, the most well-known example of this class is the deSitter universe. That is there exists a large number of such models where acceptable physical behaviour of matter has been traded off for avoidance of singularity.

Under this background Senovilla's singularity free solution[2] is remarkable that it satisfies the energy conditions and has very acceptable equation of state $\rho = 3p$. This is the first solution of its kind that is free of singularity with physically reasonable behaviour of matter. Further Ruiz and Senovilla [3] have discussed the singularity free character of inhomogeneous cylindrically symmetric spacetime and have separated out a family of singularity free perfect fluid models. It has been shown [4] that these solutions are geodetically complete and are able to escape the singularity theorems because they do not obey the assumption of existence of causal trapped surfaces.

In this paper I we first establish a linkage between the FRW metric and the Ruiz–Senovilla singularity free family. We argue that a natural and simple inhomogenisation and anisotropisation of the FRW open model leads to the singularity free family. Transform FRW metric into the cylindrical coordinates, then inhomogenise and anisotropise it by using the functions of r and t as they occur in the transformed form and the resulting metric is the singularity free family. Further the singularity free family is unique for separation of variables in the cylindrically symmetric metric potentials. It turns out that fluid solutions separate out into two classes, having $\rho \neq \mu p$ in general (but in particular $\rho = 3p$) and $\rho = p$. The stiff fluid solution yields two distinct cosmological vacuum solutions as its matter free ($\rho = 0$) limit [5].

From the point of view of general study of singularity free cosmology it would be of relevance to examine how robust is the singularity free character of the spacetime? That is, can we generate singularity free vacuum solutions or can we introduce heat flow, viscosity etc. retaining the singularity free character? In this context we have [6] considered viscous fluid and found that it is not possible to have viscosity non-negative for all times while heat flow [7] can be incorporated easily. We give

the heat flow generalizations of the two classes of fluid models mentioned above. Further if we introduce massless scalar field with the fluid distribution, then the resulting fluid can have an equation of state $\rho = \mu p$, with $4 \geq \mu \geq 3$.

In §II we describe the inhomogenisation and anisotropisation procedure of FRW metric that leads to the singularity free family of spacetime. We also consider the regularity of Weyl and Ricci curvatures for this metric. §III is devoted to the study of perfect fluid, fluid with radial heat flow and fluid with massless scalar field models. We conclude in §IV with discussion of results obtained in the paper.

In the following paper II a general study of the geodesics of the singularity free metric is carried out and they are shown to be complete indicating the absence of singularity of any kind. We also explicitly exhibit the non-occurrence of compact surfaces that makes the singularity theorems inapplicable to this family of metrics.

II. Singularity Free Metric

Ruiz and Senovilla [3] have separated out a singularity free family of solutions of Einstein's equations for perfect fluid. The spacetime has cylindrical symmetry. With metric potentials as separable functions of r and t the family is unique.

We begin with the general form of the metric

$$ds^2 = A^2(dt^2 - dr^2) - B^2 dz^2 - C^2 d\phi^2. \tag{2.1}$$

Here we have $A = A_1(t)A_2(r), B = B_1(t)B_2(r)$, $C = C_1(t)C_2(r)$ and have used the co-ordinate freedom to write $G_{tt} = |g_{rr}|$. The metric admits two space-like Killing vectors $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \phi}$ which are mutually as well as hypersurface orthogonal implying cylindrical symmetry. That means we should write $C_2(r) = rf(r)$ so as to satisfy the regularity conditions on the axis $r = 0$ and the elementary flatness condition in its vicinity.

The prime requirement for a metric to be singularity free is that its coefficients have no zeroes for real values of the co-ordinates and Ricci and Weyl tensors are regular everywhere. Ricci curvatures for the metric (2.1) are given in the Appendix and they will be regular provided $\frac{\dot{A}}{A}, \frac{A'}{A}$ etc. are regular. At the outset $C_2(r) = rf(r)$ would imply $A_1(t) = C_1(t)$ (by demanding : coefficient of $r^{-1} = 0$.) This regularity condition has no reference to matter distribution.

Assuming the matter content of the space-time to be perfect fluid we impose the fluid conditions, $R_{14} = 0$, and $R_{11} = R_{22} = R_{33}$ for the comoving velocity field

$$\mathbf{u} = A dt \tag{2.2}$$

After some calculations it can be shown that $R_{(4)} = 0$ implies that $A_1(t) = C_1(t) = T^{\alpha}(t), B_1(t) = T^{\beta}(t)$ and $A_2(r) = B_2^{q}(r)$. That is the time variation in

the metric stems from one single function. Subsequently $R_{11} = R_{22} = R_{33}$ (from Appendix) will imply

$$\frac{\ddot{T}}{T} + (\alpha + \beta - 1)\frac{\dot{T}^2}{T^2} = k^2 \tag{2.3}$$

$$\frac{C_2''}{C_2} - \frac{B_2''}{B^2} = (\alpha - \beta)k^2 \tag{2.4}$$

$$C_2 = B_2' B_2^{(\alpha-\beta)/\beta}. \tag{2.5}$$

Eq. (2.3) admits the general singularity free solution

$$T(t) = K_1[C(\lambda kt) + t_0)]^{1/\lambda^2} = C^{1/\lambda^2}(\lambda kt), \tag{2.6}$$

where we have chosen the integration constants $K_1 = 1, t_0 = 0$ and written $\alpha + \beta = \lambda^2, C(x) = cosh(x)$. The other solutions for $\lambda^2 \leq 0$ have been ruled out by the singularity free and positivity of density conditions.

In [3] it was assumed that $\alpha + \beta = 1$ and the time dependence steming from the same function $T(t)$ which we have shown above as to follow from $R_{14} = 0$. It turns out that there is no loss of generality in the former assumption either for it would just imply an overall scaling in the expressions for density and pressure (i.e. $C^{\alpha}(kt) \rightarrow C^{\alpha/\lambda^2}(\lambda kt)$). We can hence take $\alpha + \beta = \lambda^2 = 1$. For space-dependance Rûiz and Senovilla employ four arbitrary functions ($g_{44} \neq |g_{11}|$) and obtain the general solution of the system of equations (2.4) and (2.5) with the condition $\alpha + \beta = 1$. The solution in our notation, will read as

$$MX' = C_2/B_2, B_2 = X^{(1-\alpha)/(2\alpha-1)}$$

$$C_2^2 = B_2^{1/2}[(2\alpha - 1)^2 k^2 M^2 X^2 + NX^{(4\alpha-3)/(2\alpha-1)} - L]$$

where $X = X(r)$ and M, N, L are constants of integration. In Ruiz-Senovilla case $X(r)$ is arbitrary which gets determined for us because of the choice $g_{44} = |g_{11}|$. It turns out that we can integrate for X in terms of elementary regular functions, only for $N = 0$ to give the general solution

$$B_2 = C^b(mr), C_2 = m^{-1}S(mr)C^c(mr)$$

where $S(mr) = sinh(mr)$ and $A_2 = B_2^q = C^a(mr)$. The fluid consistency equations (2.4), (2.5) and the equation $R_{14} = 0$ will give relations between the parameters a, b, c, m, k and α.

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Thus we obtain the metric for singularity free space-time as

$$\begin{aligned} ds^2 =& C^{2\alpha}(kt)C^{2a}(mr)(dt^2 - dr^2) - C^{2\beta}(kt)C^{2b}(mr)dz^2 \\ & - m^{-2}S^2(mr)C^{2c}(mr)C^{2\alpha}(kt)d\phi^2 \end{aligned} \tag{2.7}$$

which is unique for perfect fluid distribution with appropriate relations between free parameters.

We next argue that the above metric can be deduced through a natural anisotropization and inhomogeneisation of the FRW metric. Consider the FRW metric with negative curvature,

$$ds^2 = dt^2 - T^2(t)\left(\frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right) \tag{2.8}$$

and transform it into cylindrical co-ordinates

$$ds^2 = dt^2 - T^2(t)\left(\frac{d\bar{r}^2}{1+\bar{r}^2} + (1+\bar{r}^2)dz^2 + \bar{r}^2 d\phi^2\right) \tag{2.9}$$

by the transformation

$$r = (S^2(z) + \bar{r}^2 C^2(z))^{1/2}, tan\theta = \frac{\bar{r}}{S(z)\sqrt{1+\bar{r}^2}}. \tag{2.10}$$

Further writing $m\bar{r} = S(m\hat{r})$ and then dropping caps we get

$$ds^2 = dt^2 - T^2(t)(dr^2 + C^2(mr)dz^2 + m^{-2}S^2(mr)d\phi^2). \tag{2.11}$$

Now inhomogenise and anisotropise by taking different powers of $T(t)$ and $C(mr)$ in the metric coefficients. That is the metric (2.7) with $T = C(kt)$. Here $m^{-2}S(mr)$ is just $(C_2(mr) = rf(r))$ to provide 2π periodicity to the angular co-ordinate ϕ and elementary flatness near the axis $r = 0$. Hence it does not participate in the inhomogenisation and anisotropisation procedure. Thus a natural and simple inhomogenisation and anisotropisation of the FRW model leads to the singularity free family given by (2.7).

The metric (2.7) can be written in the spirit of (2.9) as

$$\begin{aligned} ds^2 =& (1+k^2t^2)^{\alpha-1}(1+m^2r^2)^a dt^2 - (1+k^2t^2)^\alpha(1+m^2r^2)^{a-1}dr^2 \\ & - (1+k^2t^2)^\beta(1+m^2r^2)^b dz^2 - r^2(1+k^2t^2)^\alpha(1+m^2r^2)^c d\phi^2 \end{aligned} \tag{2.12}$$

which goes over to the FRW form for $a = 0 = c, b = 1$ and $\alpha = \beta$. If one were to look for simple functions that were regular and free of zeroes, so as to have metric free of singularity, the obvious choice would have been the hyperbolic function $C(mr)$ or $(1 + k^2r^2)$. The amazing thing is that this obvious choice is the right one giving rise to the unique singularity-free cylindrically symmetric metric (2.7).

The kinematic parameters for the metric (2.7) expansion θ, shear σ^2 and acceleration $\dot{u}_r$ are given by

$$\theta = (\alpha + 1)kS(kt)C^{-\alpha-1}(kt)C^{-a}(mr) \tag{2.13}$$

$$\sigma^2 = 2(2\alpha - 1)k^2S^2(kt)C^{-2(\alpha+1)}(kt)C^{-2a}(mr) \tag{2.14}$$

$$\dot{u}_r = -amS(mr)C^{-\alpha}(kt)C^{-a-1}(mr). \tag{2.15}$$

They are all regular and finite throughout the spacetime. The Ricci and Weyl curvatures will also be regular as they involve $\dot{A}/A, A'/A$ etc. which are all regular and finite for the metric (2.7). This would also imply the regular behaviour for the physical parameters ρ and p. There would however be some conditions on the parameters.

It is the non-vanishing of acceleration and shear that is responsible for avoidance of singularity. It is physically conceivable that acceleration and shear donot let the fluid congruence to focus into a small-enough a region to form trapped sufraces leading to singularity. Acceleration of congruence viewed as spatial *pressure gradient* which opposes gravitational attraction and provides the bounce to transform contraction into expansion at $t = 0$. The shear makes goedesics of the congruence to slip through without letting them converge into a small region. It helps in defocussing of the congruence. It is however obvious that their presence alone is not sufficient to avoid singularity as we can easily check by letting $C(kt) \to S(kt)$ in (2.7). Then the spacetime is singular at $t = 0$ with acceleration and shear non-zero. Hence it may be a necessary condition but not sufficient. For sufficiency we should have regularity of the metric and curvatures. Then energy conditions will have to be satisfied.

In [8], the geodesics of the metric (2.7) have been extensively studied and it has been shown that the above space-time is geodesically complete for $\alpha \geq 0, \alpha + \beta \geq 0, \alpha \geq \beta, a \geq 0, a + b \geq 0, a \geq b$ and $b \leq 0$. The geodesic completeness is established without reference to any matter distribution. It is obvious that the space-time is causally stable and satisfies stronger causality and energy conditions. Thus there occurs no singularity in the space-time.

For the metric (2.7) we give below the Ricci and Weyl curvatures to show explicitly that they are regular. The expressions read as follows:

$$\begin{aligned}2(\psi_0+\psi_4)A^2 =& m^2(2a-3c+b-1)-k^2(\alpha-\beta)+2\alpha(\alpha-\beta)k^2T^2(kt)\\ &+m^2[2a(c-b)+(b-c)(b+c-1)]T^2(mr)\end{aligned} \tag{2.16}$$

$$\begin{aligned}2(\psi_0-\psi_4)A^2 =& mk(\alpha+\beta-1)T(kt)T^{-1}(mr)+\{2\alpha(c-b)\\ &+\alpha(2a-b-c)+\beta(b+c-2a)+(b-c)\}T(mr)\end{aligned} \tag{2.17}$$

$$\begin{aligned}2\psi_2A^2 =& k^2(1-2\alpha)+m^2(2a+b-3c-1)\\ &+(b+c-2a+2bc-b^2-c^2)m^2T^2(mr)\\ &+(2\alpha+\alpha^2+\beta^2-2\alpha\beta-1)k^2T^2(kt)\end{aligned} \tag{2.18}$$

$$\begin{aligned}A^2R_{11} =& m^2(1+b+3c)-\alpha k^2+\{b(b-1)+c(c-1)-a(b+c+1)\}m^2T^2(mr)\\ &-\{\alpha(\alpha+\beta-1)\}k^2T^2(kt)\end{aligned} \tag{2.19}$$

$$A^2R_{22} = 2bm^2-\beta k^2-\beta(\alpha+\beta-1)k^2T^2(kt)+\{b(b-1)+bc\}m^2T^2(mr) \tag{2.20}$$

$$\begin{aligned}A^2R_{33} =& m^2(b+3c+1)-\alpha k^2-\alpha(\alpha+\beta-1)k^2T^2(kt)\\ &+\{c(c-1)+bc\}m^2T^2(mr)\end{aligned} \tag{2.21}$$

$$\begin{aligned}A^2R_{44} =& (2\alpha+\beta)k^2-2am^2+a(1-b-c)m^2T^2(mr)\\ &+\{\beta(\beta-1)+\alpha(\alpha-1)-\alpha(\alpha+\beta+1)\}k^2T^2(kt)\end{aligned} \tag{2.22}$$

$$A^2R_{14} = mk[\beta(b-a)-\alpha(a+b)]T(kt)T(mr) \tag{2.23}$$

where

The above quantities will be singularity free if the coefficients of $T^{-1}(mr)$ vanish, so we obtain the condition

$$\alpha + \beta = 1. \tag{2.25}$$

With α, β as restrained above, Ricci and Weyl curvatures are regular and finite everywhere indicating that the metric is free of any kind of singularity. Now we can introduce matter to determine other parameters occurring in R_{ik} consistent with the energy conditions.

The conditions for perfect fluid $R_{14} = 0$ and $R_{(11)} = R_{(22)} = R_{(33)}$ would imply

$$(\alpha - \beta)(\alpha + \beta - 1) = 0 \tag{2.26}$$

$$(\beta - \alpha)k^2 = (b - 1 - 3c)m^2 \tag{2.27}$$

$$(b - c)(b + c - 1) = 0 \tag{.2.28}$$

$$a(b + c + 1) = b(b - 1 - c) \tag{2.29}$$

$$\alpha(a + b) = \beta(b - a). \tag{2.30}$$

The first condition is satisfied in view of (2.25) which has to be obeyed always as it is dictated by the regularity of Weyl tensor.

The remaining conditions give rise to the following two cases

(i) $b = c$

(ii) $b + c = 1$.

These are the only two possibilities for singularity free fluid models satisfying the energy conditions. The former in general represents a fluid without an equation of state. Only when $b = -1/3$, it represents Senovilla's [2] radiation model with $\rho = 3p$. The latter will always have $\rho = p$, representing stiff fluid model [5]. In this case when we put $\rho = 0$, we get two distinct vacuum solutions as its matter free limit [5]. The point to be noted is that these are all general unique solutions for the singularity free form (2.7). It is also interesting that the two cases $\rho \neq \mu p$ in general (but $\rho = \frac{1}{3}p$ in particular) and $\rho = p$ separate out nicely.

III Fluid Models

We shall consider the perfect fluid, fluid with radial heat flux and fluid with massless scalar field models. It is possible to introduce heat flux in both the cases

considered above. Inclusion of massless scalar field opens out a narrow window $4 \geq \mu \geq 3$ for $\rho = \mu p$ which otherwise had only the discrete value $\mu = 3$ allowed in the case (i).

3.1 Perfect fluid

Einstein's equations for non-empty spacetime are

$$R_{ik} = -8\pi(T_{ik} - \frac{1}{2}Tg_{ik}) \tag{3.1}$$

where for perfect fluid

$$T_{ik} = (p+\rho)u_i u_k - pg_{ik}, u_i u^i = 1. \tag{3.2}$$

Case (i) $b = c$.
We write from eqs. (2.27)–(2.30)

$$\alpha = (1+b)(1+2b)^{-1}, a = -b(1+2b)^{-1}, \frac{m}{k} = n = (1+2b)^{-1}. \tag{3.3}$$

This determines the metric (2.7) in terms of the two free parameters b and k, say.

The density and pressure have the expressions :

$$8\pi p A^2 = \frac{b^2(1-2b)}{(1+2b)^3}k^2C^{-2}(nkr) - \frac{(3b+1)(b+1)}{(1+2b)^2}k^2C^{-2}(kt) \tag{3.4}$$

$$8\pi p A^2 = \frac{b(2b-1)(2+3b)}{(1+2b)^3}k^2C^{-2}(nkr) - \frac{(3b+1)(b+1)}{(1+2b)^2}k^2C^{-2}(kt) \tag{3.5}$$

where $A = C^{\alpha}(kt)C^{a}(nkr)$. This is the Ruiz-Senovilla [3] model. For satisfaction of the energy conditions we shall require $-\frac{1}{2} < b \leq -\frac{1}{3}$ and the Senovilla radiation model results for $b = -\frac{1}{3}$. With b so restricted, ρ and p are positive for entire spacetime. For $b \neq -\frac{1}{3}$, there is no equation of state of the type $\rho = \mu p$. The maximum density would occur at $t = 0$ and $r = 0$ which decreases to zero as $t \to \pm\infty$ and $r \to \infty$. The parameter k can be identified with ρ_{max} which can be freely chosen. At a given t, ρ is largest at the origin $r = 0$ and at given r, it is largest at $t = 0$.

Case (ii) $b + c = 1$.

In this case we have

$$\alpha = \frac{1}{2}(2-b), a = b(b-1), c = 1-b, n^2 = \frac{m^2}{k^2} = \frac{1}{4} \tag{3.6}$$

and we get the stiff fluid [5] model with

$$8\pi\rho = 8\pi p = \left(\frac{b^2-4}{4A^2}\right)k^2C^{-2}(kt). \tag{3.7}$$

Clearly the energy condition will be obeyed for $b^2 \geq 4$, i.e. b lying outside the interval $-2 \leq b \leq 2$, the end points of which give the two distinct vacuum cosmological solutions [5]. These solutions are given as follows:
For $b = 2$

$$ds^2 = C^4(mr)(dt^2 - dr^2 - C^2(2mt)dz^2) - m^{-2}S^2(mr)C^{-2}(mr)d\phi^2. \tag{3.8}$$

For $b = -2$

$$\begin{aligned} ds^2 =& C^4(2mt)[C^{12}(mr)(dt^2 - dr^2) - C^6(mr)m^{-2}S^2(mr)d\phi^2] \\ & - C^{-2}(2mt)C^{-4}(mr)dz^2. \end{aligned} \tag{3.9}$$

It can be easily verified from eqs. (2.16)–(2.18) that both the solutions are Petrov type I. The former (3.6) is static as Weyl scalars have no time dependence, while the latter can asymptotially represent a plane gravitational wave. Since there are no obvious localized sources for the solutions, hence the only possible source can be gravitational radiation. However, one knows that in realistic situation the metric for gravitational radiation is of type I almost everywhere [7].

3.2 Fluid with heat flow

The energy momentum tensor for fluid with heat flow is given by

$$T_{ik} = (p+\rho)u_iu_k - pg_{ik} + (q_iu_k + q_ku_i), u_iu^i = 1, u_iq^i = 0. \tag{3.10}$$

Here q_i is the heat flow vector. Here we are relaxing the condition (2.30) by taking radial heat flow. We take the tetrad components of q_i as $q_{(a)} = (q, 0, 0, 0)$. The parameters ρ, p and q are given by

$$\begin{aligned} 8\pi\rho A^2 =& (a-3b)m^2 + (\beta-\alpha)k^2 + (\beta^2 + 2\alpha\beta + \alpha - \beta)k^2T^2(kt) \\ & + \frac{1}{2}(a-3b)(b+c-1)m^2T^2(mr) \end{aligned} \tag{3.11}$$

$$8\pi pA^2 = (a+b)m^2 - (\alpha+\beta)k^2 + (\alpha+\beta-\beta^2)k^2T^2(kt) + \frac{1}{2}(a+b)(b+c-1)m^2T^2(mr) \tag{3.12}$$

$$8\pi qA^2 = mk[\alpha(a+b) - \beta(b-a)]T(kt)T(mr) \tag{3.13}$$

where $A = C^{\alpha}(kt)C^{a}(mr)$.

Case (i) $b = c$.

In this case we have

$$b = c, \alpha = 1-\beta, a = -\frac{b}{1+2b}, n^2 = \frac{m^2}{k^2} = \frac{1-2\beta}{1+2b} \tag{3.14}$$

We treat b and n as arbitrary parameters. The physical parameters are then given by

$$8\pi qA^2 = \frac{k^2nb}{(1+2b)}\{(n^2-1) + 4b(b+1)n^2\}T(kt)T(nkr) \tag{3.15}$$

$$8\pi pA^2 = \frac{k^2}{4}(n^2-1)\{1-n^2(1+2b)^2\} + \frac{b^2(1-2b)}{(1+2b)}n^2k^2C^{-2}(nkr) + \frac{1}{4}\{n^2(1+2b)-3\}\{1+n^2(1+2b)\}C^{-2}(kt) \tag{3.16}$$

$$8\pi\rho A^2 = \frac{k^2}{4}(3+n^2)\{1-n^2(1+2b)^2\} - \frac{b(1-2b)(2+3b)}{(1+2b)}n^2k^2C^{-2}(nkr) + \frac{1}{4}\{n^2(1+2b)-3\}\{1+n^2(1+2b)\}C^{-2}(kt). \tag{3.17}$$

From the above equations it can be easily seen that the physical requirements $p \geq 0, \rho \geq 0$ are satisfied for

$$-\frac{1}{2} < b \leq -\frac{1}{3}, \qquad \frac{3}{1+2b} < n^2 \leq \frac{1}{(1+2b)^2}. \tag{3.18}$$

From (3.15) it is clear that $q = 0$ implies $n^2 = (1+2b)^{-2}$ and the model reduces to the perfect fluid (3.3). For $n = 3$ we have the heat flow generalization of the Senovilla radiation model [7].

Case (ii) $b + c = 1$

Here we write

$$a = b(b-1), c = 1 - b, \alpha = 1 - \beta, n^2 = \frac{m^2}{k^2} = \frac{2\beta - 1}{4(b-1)} \tag{3.19}$$

and obtain

$$8\pi q A^2 = b(b-1)(4n^2 - 1)nk^2 T(kt)T(nkr) \tag{3.20}$$

$$\begin{aligned} 8\pi p A^2 =& \frac{k^2}{4}(4n^2 - 1)\{1 - 4n^2(b-1)^2\} \\ &+ \frac{k^2}{4}[4n^2(b-1) - 1][4n^2(b-1) + 3]C^{-2}(kt) \end{aligned} \tag{3.21}$$

$$\begin{aligned} 8\pi \rho A^2 =& -\frac{k^2}{4}[4n^2 - 1][3 + 4n^2(b-1)^2] \\ &+ \frac{k^2}{4}[4n^2(b-1) - 1][4n^2(b-1) + 3]C^{-2}(kt). \end{aligned} \tag{3.22}$$

The physical requirements $\rho > 0$ and $p \geq 0$ restrict n and b to the ranges

$$0 < n^2 \leq \frac{1}{4}, \qquad 1 < b < 2. \tag{3.23}$$

For $n^2 = \frac{1}{4}$, it reduces to the stiff fluid model (3.6).

The phenomenological expression for the heat conduction is given by

$$q_k = \psi\left(\frac{\partial F}{\partial x^i} + F\dot{u}_i\right)(\delta^i_k - k^i u_k) \tag{3.24}$$

where ψ is the thermal conductivity and F is the temperature. For the cases under consideration only radial heat flow is retained. Hence eq. (3.24) can be integrated if ψ is a function of t alone to give

$$\begin{aligned} F =& l(t)C^a(nkr) \\ &- \frac{k}{16\pi a\psi}\{\alpha(a+b) - \beta(b-a)\}C^{-a}(nkr)C^{-\alpha}(kt)T(kt) \end{aligned} \tag{3.25}$$

where $l(t)$ is an arbitrary function of t. Thus temperature distributions for the cases (i) and (ii) can be obtained.

The heat flux has been considered in the evolution of the cosmologial models by several authors (see [7]).

3.3. Fluid with massless scalar field

The energy momentum tensor for a perfect fluid with massless scalar fields V is given by

$$T_{ik} = (\rho + p)u_i u_k - pg_{ik} + V_i V_k - \frac{1}{2} g_{ik} V^a V_a \tag{3.26}$$

where $u^i u_i = 1$ and $V_i = \frac{\partial V}{\partial x^i}$. We take the scalar field V to be function of t only. V has to satisfy the equation

$$[\sqrt{-g} V^i]_{,i} = 0 \tag{3.27}$$

the first integral of which is given by

$$\dot{V} = lC^{-(\beta+\gamma)}(kt) \tag{3.28}$$

where l is a constant of integration. The introduction of the scalar field does not affect the fluid conditions and hence the free parameters continue to be governed by the conditions (2.26)–(2.30). In this case also the solution separate into two classes, (i) $b = c$ and (ii) $b + c = 1$. In the latter case the solution represents a stiff fluid in the presence of scalar field. Since any stiff fluid solution also has a scalar field interpretation this case does not lead to any new possibility. However, in the former case $b = c$, we obtain

$$\alpha = \frac{1+b}{1+2b}, \qquad a = \frac{-b}{1+2b}, \qquad n^2 = \frac{m^2}{k^2} = (1+2b)^{-2}, \qquad \beta = 1 - \alpha. \tag{3.29}$$

The density and pressure have the expressions :

$$\begin{aligned} 8\pi\rho A^2 =& \frac{b(2b-1)(2+3b)}{(1+2b)^3} k^2 C^{-2}(nkr) \\ &- \left[\frac{(b+1)(3b+1)}{(1+2b)^2} k^2 + l^2\right] C^{-2}(kt) \end{aligned} \tag{3.30}$$

$$\begin{aligned} 8\pi p A^2 =& \frac{b^2(1-2b)}{(1+2b)^3} k^2 C^{-2}(nkr) \\ &- \left[\frac{(b+1)(3b+1)}{(1+2b)^2} k^2 + l^2\right] C^{-2}(kt) \end{aligned} \tag{3.31}$$

where $A = C^{\alpha}(kt)C^{a}(nkr)$.
For $b = -\frac{1}{3}$, the solution represents a scalar-field generalization of Senovilla radiation model. For $l^2 = -\frac{(b+1)(3b+1)}{(1+2b)^2}k^2$ the fluid distribution of the model has an equation of state $\rho = \mu p$ where $\mu = -\frac{1}{b}(2+3b)$. The positivity of ρ and p restricts b to the range $-\frac{1}{2} < b \leq -\frac{1}{3}$ so that $3 \leq \mu \leq 4$.

IV. Discussion

We have argued that the singularity free family of perfect fluid spacetimes obtained by Ruiz and Senovilla [3] can be inferred through a simple and natural inhomogenisation and anisotropisation of FRW metric for the open universe. This establishes a kind of linkage between the two. If one were to write a singularity free metric, without reference to anything else, the natural choice for metric functions would have been hyperbolic or quadratic functions without zeros. The amazing thing is that this obvious choice is the right and the only choice. However, one has to assume cylindrical symmetry in place of spherical symmetry.

It turns out that the metric (2.7) is the unique singularity free spacetime for cylindrically symmetric spacetime with separable variables. In the context of cosmology separability is an appropriate assumption for the overall behaviour of universe should not depend upon the interactions arising out of interaction between variables. We are already taking a big step deeper by giving up homogeneity and isotropy of the today's universe as described by FRW metric. This was motivated by the consideration that universe has to have inhomogeneity and anisotropy at early times so as to have generic initial conditions as well as to facilitate formation of structures. It is obvious that in order to consider anisotropy and inhomogeneity effectively one has to consider less restrictive symmetry than spherical. That leads naturally to cylindrical symmetry. We would hence argue that choice of cylindrical symmetry is almost determined by inhomogeneity and anisotropy of the early universe. There can, however, be more general metric without symmetry but cylindrical symmetry is the simplest choice.

For the avoidance of singularity acceleration and shear play the crucial role. The former provides the bounce to the universe at $t = 0$ where contraction turns into expansion while the latter makes the fluid geodesic congruence to slip through without letting them to focus in a small enough region to form compact surfaces leading to singularity. The overall scenario is : universe has very low density tending to zero at $t \to -\infty$, wherefrom it starts contracting and attains the dense state at $t = 0$ where contraction changes to expansion and it expands to low density again at $t \to \infty$. The momentum gained during the contraction phase takes the universe through $t = 0$ to the expanding phase. The maximum density at $t = 0$ and $r = 0$ can be made as large or small as one pleases by choosing the parameter k. All

through all the physical and kinematic parameters remain regular and finite. The presence of acceleration and shear may be necessary but by no means sufficient to avoid singularity. In addition the spacetime should be regular.

Let us recall that $\alpha = \gamma$ was dictated by the cylindrical symmetry while $\alpha + \beta = 1$ for the metric (2.7) is demanded by the Weyl regularity. These two are general conditions without reference to any matter field. That is of the three α, β, γ only one is free for the singularity free spacetime. It may be noted that all the fluid models (which are the only ones for the metric (2.7)) discussed in §III obey the conditions, $\alpha \geq 0, \alpha + \beta \geq 0, \alpha \geq \beta, a \geq 0, a + b \geq 0, a \geq b$ and $b \leq 0$, obtained in the following paper II for completteness of geodesics.

The most important question for the singularity free models is how to evolve them into the standard FRW model which successfully describes the present day universe. The affirmative answer to this question will bring these models into the active arena of cosmology and would perhaps have very significant role to play in the early universe cosmology. The main difficulty here is that the anisotropy measure (σ/θ) of the metric (2.7) is constant which means it remains anisotropic for all times. It would be interesting to find a singularity free solution with σ/θ decreasing with t so that it can isotropise at late times to go over to FRW model. This brings us to another important question : do there exist other singularity free solutions with lesser or no symmetry, or is cylindrical symmetry singled out? These are the questions that are currently engaging us.

A brief report of important results discussed here has been submitted for publication elsewhere [10].

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Appendix

For the metric (2.1) we introduce the tetrad $\theta^1 = Adr, \theta^2 = Bdz, \theta^3 = Cd\phi, \theta^4 = Adt$. The non-vanishing tetrad componentes R_{ab} of the Ricci tensor for (2.1) are given by

$$R_{14} = \frac{1}{A^2}\left[\frac{\dot{B}'}{B} + \frac{\dot{C}'}{C} - \frac{\dot{A}}{A}\left(\frac{B'}{B} + \frac{C'}{C}\right) - \frac{A'}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right]$$

$$\begin{aligned} R_{44} =& \frac{1}{A^2}\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{A}}{A}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right. \\ & \left. - \frac{A''}{A} + \frac{A'}{A}\left(\frac{A'}{A} - \frac{B'}{B} - \frac{C'}{C}\right)\right], \end{aligned}$$

$$\begin{aligned} R_{11} =& \frac{1}{A^2}\left[\frac{A''}{A} + \frac{B''}{B} + \frac{C''}{C} - \frac{A'}{A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{C'}{C}\right)\right. \\ & \left. - \frac{\ddot{A}}{A} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right)\right], \end{aligned}$$

$$R_{22} = \frac{1}{A^2}\left[\frac{B''}{B} + \frac{B'C'}{BC} - \frac{\ddot{B}}{B} - \frac{\dot{B}\dot{C}}{BC}\right]$$

$$R_{33} = \frac{1}{A^2}\left[\frac{C''}{C} + \frac{B'C'}{BC} - \frac{\ddot{C}}{C} - \frac{\dot{B}\dot{C}}{BC}\right]$$

A prime and a dot indicate differentiation with respect to r and t respectively.