New Solutions of the Yang-Baxter Equation Without Additivity and Birman-Wenzl Algebra with Colour

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Abstract

New solutions of the Yang-Baxter equation associated with the fundamental representations of $A_n (n > 1), B_n, C_n$ and D_n, without the additivity of spectral parameters, are found in terms of the weight conservation. The Birman-Wenzl algebra with colour is constructed and the explicit examples are given in the cases of B_n, C_n and D_n.

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1 Introduction

The Yang-Baxter equation (YBE) plays a fundamental role in the integrable systems in 1+1 dimensions and the exactly soluble models in statistical mechanics [1-4]. Many solutions of the YBE

$$\check{R}_{12}(x)\check{R}_{23}(x\ y)\ \check{R}_{12}(y) = \check{R}_{23}(y)\check{R}_{12}(xy)\check{R}_{23}(x) \tag{1.1}$$

, $x(=e^u)$ and $y(=e^v)$ are spectral parameters with additivity, have been found. Among them the trigonometric solutions associated with the fundamental representations of A_n, B_n, C_n and D_n were constructed based on the quantum group (QG) [5]. By taking the limit of $\check{R}(x)$ the YBE (1.1) is reduced to the braid relation [6]

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23} \tag{1.2}$$

It is well known that the solutions of the braid relation associated with the representation of the Lie algebra can be constructed from theory of QG [7,8] and calculated in terms of the weight conservation and the extended Kauffman diagrammetic techniques [9,10]. It is shown that there exist two types of the braid group representations (BGRs) associated with the fundamental representations of A_n, B_n, C_n and D_n: standard and exotic [11,12], and the structure of the Birman-Wenzl (BW) algebra [13] exists for the BGRs, not only for the standard ones [8] but also for exotic ones [14], in the cases of B_n, C_n and D_n. The corresponding explicit solutions of the YBE(1.1) can be obtained by the Yang-Baxterization prescription [15,16].

On the other hand, the solutions of the YBE without additivity have been found [17]. As is known, the complicated calculatins were made for deriving the non-additivity of spectral parameters for YBE. It is not clear what the relation is between these solutions and the structure of Lie algebra. Recently, Murakami [18] has found a simple solution of YBE without additivity which can be expressed as a 4x4 matrix:

$$S^{\lambda\mu} = \begin{bmatrix} t_\lambda & & & \\ & 0 & 1 & \\ & t_\lambda t_\mu & t_\mu^{-1}(t_\lambda^2 - 1) & \\ & & & -t_\mu{}^{-1} \end{bmatrix} \tag{1.3}$$

where λ and μ are interpreted in colours, and t_λ, t_μ are coloured parameters. In Ref. [19], we solved directly the YBE

$$\check{R}_{12}(\lambda,\mu)\check{R}_{23}(\lambda,\nu)\check{R}_{12}(\mu,\nu) = \check{R}_{23}(\mu,\nu)\check{R}_{12}(\lambda,\nu)\check{R}_{23}(\lambda,\mu) \tag{1.4}$$

and obtained two solutions:

$$\check{R}_I(\lambda,\mu) = \begin{bmatrix} q & & & \\ & 0 & \eta X(\lambda) & \\ & \eta^{-1}X^{-1}(\mu) & (q-q^{-1})g(\lambda)g^{-1}(\mu) & \\ & & & qX(\lambda)X^{-1}(\mu) \end{bmatrix} \tag{1.5}$$

and

$$\check{R}_{II}(\lambda,\mu) = \begin{bmatrix} q & & & \\ & 0 & \eta X(\lambda) & \\ & \eta^{-1}Y(\mu) & \bar{W}(\lambda,\mu) & \\ & & & -q^{-1}X(\lambda)Y(\mu) \end{bmatrix} \tag{1.6}$$

where $X(\lambda), g(\lambda)$ and $Y(\lambda)$ are arbitrary functions of λ and $\bar{W}(\lambda,\mu)$ satisfies the relation

$$\bar{W}(\lambda,\mu)\bar{W}(\mu,\nu) = (q - q^{-1}X(\mu)Y(\mu))\bar{W}(\lambda,\nu) \tag{1.7}$$

It is shown in Ref. [18] that the solution (1.3) is a special case of solution (1.6). When $\lambda = \mu$, the solutions (1.5) and (1.6) will be reduced to the standard and exotic solutions of the braid group, respectively. In Ref. [20], the new hierarchy of solutions for SU(2), called colored-braid-group representations, have been constructed. As a natural extension, we will seek other solutions of the YBE(1.4) associated with the representations of Lie algebra.

In this paper we will show that there exist the solutions of the YBE(1.4) associated with the fundamental representations of $A_n(n>1), B_n, C_n$ and D_n. When $\lambda = \mu$, these solutions will be reduced to the corresponding BGRs. We construct the BW algebra with colour, which will give the usual BW algebra when all colours are the same. The explicit examples are calculated in the cases of B_n, C_n and D_n. In sec. II, the solutions of the YBE (1.4) associated with the fundamental representations of $A_n(n>1)$ are found in terms of the weight conservation. There exist two types of variable-separation solutions that correspond to the standard and exotic ones. In sec. III, the similar method appearing in sec. II, is used to discuss the cases of B_n, C_n and D_n. In sec. IV, the BW algebra with colour is constructed. It is shown that this structure exist for B_n, C_n and D_n.

2 Solutions of $\check{R}(\lambda,\mu)$ Associated with $A_n(n>1)$.

The structure of the BGR associated with the fundamental representations of A_n has been given in terms of the weight conservation in Ref. [12]. We need to put the spectral parameters without additivity or colours in this structure only. Under the circumstances we assume that $\check{R}(\lambda,\mu)$ has the following form:

$$\check{R}(\lambda,\mu)=\sum_a U_a(\lambda,\mu)e_{aa}\otimes e_{aa}+\sum_{a<b}W^{(a,b)}(\lambda,\mu)e_{aa}\otimes e_{bb}+\sum_{a\neq b}p^{(a,b)}(\lambda,\mu)e_{ab}\otimes e_{ba} \quad (2.1)$$

where $(e_{ab})_{cd}=\delta ac\delta_{bd}, a,b,c,d\epsilon[-\frac{N-1}{2},-\frac{N-1}{2}+1,\ldots,\frac{N-1}{2}], N=n+1, Ua^{(\lambda,\mu)}W^{(a,b)}(\lambda,\mu)$ and $p^{(a,b)}(\lambda,\mu)$ are determined parameters by solving the YBE (1.4).

Substituting (2.1) into (1.4) we obtain the relations :

$$\begin{cases}
p^{(b,c)}(\lambda,\nu)p^{(a,c)}(\mu,\nu) &= p^{(b,c)}(\mu,\nu)p^{(a,c)}(\lambda,\nu) \quad (a<b)\\
p^{(a,b)}(\lambda,\mu)p^{(a,c)}(\lambda,\nu) &= p^{(a,b)}(\lambda,\nu)p^{(a,c)}(\lambda,\mu) \quad (b<c)\\
p^{(a,b)}(\lambda,\mu)U_a(\lambda,\nu) &= p^{(a,b)}(\lambda,\nu)U_a(\lambda,\mu) \quad (a\neq b)\\
U_b(\lambda,\nu)p^{(a,b)}(\mu,\nu) &= U_b(\mu,\nu)p^{(a,b)}(\lambda,\nu) \quad (a\neq b)\\
W^{(a,b)}(\lambda,\mu)U_b(\lambda,\nu)W^{(a,b)}(\mu,\nu) &+ p^{(a,b)}(\lambda,\mu)W^{(a,b)}(\lambda,\nu)p^{(b,a)}(\mu,\nu)\\
&= U_b(\mu,\nu)W^{(a,b)}(\lambda,\nu)U_b(\lambda,\mu) \quad (a<b)\\
U_a(\lambda,\mu)W^{(a,b)}(\lambda,\nu)U_a(\mu,\nu) &= W^{(a,b)}(\mu,\nu)U_a(\lambda,\nu)W^{(a,b)}(\lambda,\mu)\\
&+ p^{(a,b)}(\mu,\nu)W^{(a,b)}(\lambda,\nu)p^{(b,a)}(\lambda,\mu) \quad (a<b)
\end{cases} \quad (2.2)$$

and

$$\begin{cases}
p^{(a,b)}(\lambda,\mu)W^{(a,c)}(\lambda,\nu)W^{(b,a)}(\mu,\nu) &= W^{(b,c)}(\mu,\nu)p^{(a,b)}(\lambda,\nu)W^{(a,c)}(\lambda,\mu) \quad (b<a<c)\\
W^{(a,b)}(\lambda,\mu)W^{(b,c)}(\lambda,\nu)p^{(a,b)}(\mu,\nu) &= W^{(b,c)}(\mu,\nu)p^{(a,b)}(\lambda,\nu)W^{(a,c)} \quad (a<b<c)\\
W^{(a,b)}(\lambda,\mu)p^{(b,c)}(\lambda,\nu)W^{(a,c)}(\mu,\nu) &= W^{(b,c)}(\mu,\nu)W^{(a,b)}(\lambda,\nu)p^{(b,c)}(\lambda,\mu) \quad (a<b<c)\\
W^{(a,b)}(\lambda,\mu)p^{(b,c)}(\lambda,\nu)W^{(a,c)}(\mu,\nu) &= p^{(b,c)}(\mu,\nu)W^{(a,c)}(\lambda,\nu)W^{(c,b)}(\lambda,\mu) \quad (a<c<b)\\
W^{(a,b)}(\lambda,\mu)W^{(b,c)}(\lambda,\nu)W^{(a,b)}(\mu,\nu) &+ p^{(a,b)}(\lambda,\mu)W^{(a,b)}(\lambda,\nu)p^{(b,a)}(\mu,\nu)\\
=W^{(b,c)}(\mu,\nu)W^{(a,b)}(\lambda,\nu)W^{(b,c)}(\lambda,\mu) &+ p^{(b,c)}(\mu,\nu)W^{(a,c)}(\lambda,\nu)p^{(c,b)}(\lambda,\mu) \quad (a<b<c)
\end{cases} \quad (2.3)$$

The relations (2.3) exist for $n>1$ only. When $n=1$, the solutions of the YBE (1.4) have been discussed in Ref. [19]. So we consider the cases of $n>1$ here.

Let us consider variable-separation solutions of Eqs. (2.2) and (2.3). Suppose

$$\begin{aligned}
U_a(\lambda,\mu) &= g^1{}_a(\lambda)f^1{}_a(\mu)U_a\\
p^{(a,b)}(\lambda,\mu) &= g^2{}_{(a,b)}(\lambda)\ \ f^2{}_{(a,b)}(\mu)
\end{aligned} \quad (2.4)$$

Eq. (2.2) leads to the following constraints:

$$\begin{aligned}
&g_b^1(\lambda)=g_{(a,b)}^2(\lambda)=g_b(\lambda), f_a^1(\mu)=f_{(a,b)}^2(\mu)=f_a(\mu)\\
&(U_a+U_b^{-1})(g_a(\mu)f_a(\mu)-g_b(\mu)f_b(\mu))=0\\
&W^{(a,b)}(\lambda,\mu)W^{(a,b)}(\mu\nu)=(U_a g_a(\mu)f_a(\mu)-U_a^{-1}g_b(\mu)f_b(\mu))W^{(a,b)}(\lambda,\nu)
\end{aligned} \quad (2.5)$$

By solving Eqs. (2.3) and (2.5), up to a common factor $f_a^{-1}(\mu)g_a^{-1}(\mu)$, we obtain the solutions:

$$\begin{aligned}(i) \quad & U_a(\lambda,\mu) = q g_a(\lambda) g_a^{-1}(\mu), p^{(a,b)}(\lambda,\mu) = g_b(\lambda) g_a^{-1}(\mu) \\ & W^{(a,b)}(\lambda,\mu) = (q-q^{-1}) g_b(\lambda) g_a^{-1}(\mu) h_a(\lambda) h_b^{-1}(\lambda) h_a^{-1}(\mu) h_b(\mu)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}(ii) U_a(\lambda,\mu) &= U_a g_a(\lambda) g_a^{-1}(\mu), U_a \epsilon \{q, -q^{-1}\} \\ p^{(a,b)}(\lambda,\mu) &= g_b(\lambda) g_a a^{-1}(\mu) \\ W^{(a,b)}(\lambda,\mu) &= (q-q^{-1}) g_b(\lambda) g_b^{-1}(\mu) h_a(\lambda) h_b^{-1}(\lambda) h_a^{-1}(\mu) h_b(\mu)\end{aligned} \tag{2.7}$$

where q is an arbitrary parameter, $g_a(\lambda)$ and $h_a(\lambda)$ are arbitrary functions of λ. When $\lambda = \mu$, the solutions (2.6) and (2.7) are reduced to the standard and exotic solutions of the braid relation:

$$\begin{aligned}& U_a(\lambda,\lambda) = U_a, \quad U_a \equiv q \quad \text{or} \quad U_a \epsilon \{q, -q^{-1}\} \\ & p^{(a,b)}(\lambda,\lambda) = g_b(\lambda) g_{a(\lambda)}^{-1}, \quad W^{(a,b)}(\lambda,\lambda) = q - q^{-1}, \\ & p^{(a,b)}(\lambda,\lambda) p^{(b,a)}(\lambda,\lambda) = 1\end{aligned} \tag{2.8}$$

3 Solutions of $\check{R}(\lambda,\mu)$ Associated with Bn, Cn and Dn

According to the similar consideration in sec. II, the $\check{R}(\lambda,\mu)$ associated with the fundamental representations of Bn, Cn and Dn has the following structure:

$$\begin{aligned}\check{R}(\lambda,\mu) \quad &= \sum_a U_a(\lambda,\mu) e_{aa} \otimes e_{aa} + \sum_{a<b} W^{(a,b)}(\lambda,\mu) e_{aa} \otimes e_{bb} + \sum_{a \neq b} p^{(a,b)}(\lambda,\mu) e_{ab} \otimes e_{ba} \\ &+ \sum_{a,b} q^{(a,b)} e_{ab} \otimes e_{-a-b}\end{aligned} \tag{3.1}$$

where $a, b \epsilon [-\frac{N-1}{2}, \ldots, \frac{N-1}{2}], N = 2n+1$ and $2n$ for Bn and $Cn(Dn)$, respectively. In (3.1), we assume that

$$q^{(a,b)}(\lambda,\mu)|_{a=\pm b} = q^{(a,b)}_{(\lambda,\mu)}|_{a,b>0} = q^{(a,b)}_{(\lambda,\mu)}|_{a \leq 0, b>|a|} = q^{(a,b)}(\lambda,\mu)|_{b \leq 0, a>|b|} = 0 \tag{3.2}$$

Substituting (3.1) and (3.2) into (1.4) we still obtain Eqs. (2.2) and @2.3) in the cases of $a+b\neq 0, a+c\neq 0$ and $b+c\neq 0$. by solving these equations we determine the parameters $U_a(\lambda,\mu)(a\neq 0), p^{(a,b)}(\lambda,\mu)(a+b\neq 0)$ and $W^{(a,b)}(\lambda,\mu)(a+b\neq 0)$ given by

$$\begin{aligned} U_a(\lambda,\mu) &= qg_a(\lambda)g_a^{-1}(\mu)(a\neq 0) \\ p^{(a,b)}(\lambda,\mu) &= g_a(\lambda)g_a^{-1}(\mu)(a+b\neq 0) \\ W^{(a,b)}(\lambda,\mu) &= (q-q^{-1})g_b(\lambda)g_b^{-1}(\mu)(a+b\neq 0) \end{aligned} \tag{3.3}$$

where, for simplicity, we have taken $U_a = q$ and $h(\lambda) = 1$. The other constraints given by the YBE (1.4) are different for Bn, Cn and Dn. In the following, let us first consider the case of Bn.

(i) The case of Bn. Under the case, besides Eqs. (2.2) and (2.3), the YBE (1.4) gives rise to the constraints:

$$\begin{aligned}
&U_0(\lambda,\mu)U_0(\lambda,\nu) = p^{(o,-a)}(\lambda,\nu)p^{(0,a)}(\lambda,\mu)(a>0) \\
&p^{(a,0)}(\lambda,\nu)p^{(-a,0)}(\mu,\nu) = U_0(\mu,\nu)U_0(\lambda,\nu)(a>0) \\
&p^{(0,a)}(\lambda,\mu)q^{(0,-a)}(\lambda,\nu)p^{(a,-a)}(\mu,\nu) + W^{(0,a)}(\lambda,\mu)p^{(a,0)}(\lambda,\nu)q^{(0,-a)}(\mu,\nu) \\
&\qquad = p^{(a,0)}(\mu,\nu)q^{(0,-a)}(\lambda,\nu)U_a(\lambda,\mu)(a>0) \\
&p^{(0,-a)}(\lambda,\mu)q^{(0,-a)}(\lambda,\mu)U_{-a}(\mu,\nu) = W^{(-a,0)}(\mu,\nu)p^{(0,-a)}(\lambda,\nu)q^{(0,-a)}(\lambda,\mu) \\
&\qquad + p^{(-a,0)}(\mu,\nu)q^{(0,-a)}(\lambda,\mu)p^{(a,-a)}(\lambda,\mu)(a>0) \\
&q^{(-a,0)}(\lambda,\mu)q^{(0,-a)}(\lambda,\nu)p^{(0,-a)}(\lambda,\nu) + W^{(0,a)}(\lambda,\mu)p^{(a,0)}(\lambda,\nu)W^{(-a,0)}(\mu,\nu) \\
&\qquad = p^{(a,0)}(\mu.,\nu)W^{(-a,0)}(\lambda,\nu)W^{(0,a)}(\lambda,\mu)(a>0) \\
&p^{(0,-a)}(\lambda,\mu)W^{(0,a)}(\lambda,\nu)W^{(-a,0)}(\mu,\nu) = q^{(-a,0)}(\mu,\nu)q^{(0,-a)}(\lambda,\nu)p^{(a,0)}(\lambda,\mu) \\
&\qquad + W^{(0,a)}(\mu,\nu)p^{(0,-a)}(\lambda,\nu)W^{(0,a)}(\lambda,\mu)(a>0) \\
&q^{(0,-a)}(\lambda,\mu)q^{(a,-b)}(\lambda,\nu)p^{(-a-b)}(\mu,\nu) + q^{(0,-b)}(\lambda,\mu)p^{(b,-a)}(\lambda,\nu)W^{(-b,-a)}(\mu,\nu) = 0(0<a<b) \\
&q^{(0,-a)}(\mu,\nu)q^{(a,-b)}(\lambda,\nu)p^{(b,a)}(\lambda,\mu) + q^{(0,-b)}(\mu,\nu)W^{(a,b)}(\lambda,\mu) = 0(0<a<b) \\
&q^{(-b,a)}(\lambda,\mu)q^{(-a,0)}(\lambda,\nu)p^{(a,0)}(\mu,\nu) + q^{(-b,0}(\lambda,\mu)p^{(0,a)}(\lambda,\nu)W^{(0,a)}(\mu,\nu) = 0(0<a<b) \\
&q^{(-b,a)}(\mu,\nu)q^{(-a,0)}(\lambda,\nu)p^{(0,-a)}(\lambda,\mu) + q^{(-b,0)}(\mu,\nu)p^{(-a,0)}(\lambda,\nu)W^{(-a,0)}(\lambda,\mu) = 0(0<a<b) \\
&q^{(-a,0)}(\lambda,\mu)q^{(0,-b)}(\lambda,\nu)p^{(0,-b)}(\mu,\nu) + q^{(-a,-b)}(\lambda,\mu)p^{(b,0)}(\lambda,\nu)W^{(-b,0)}(\mu,\nu) = 0(a,b>0) \\
&q^{(-a,0)}(\mu,\nu)q^{(0,-b)}(\lambda,\nu)p^{(b,0)}(\lambda,\mu) + q^{(-a,-b)}(\mu,\nu)p^{(0,-b)}(\lambda,\nu)W^{(0,b)}(\lambda,\mu) = 0(a,b>0) \\
&q^{(0,-a)}(\lambda,\mu)U_a(\lambda,\nu)W^{(-a,a)}(\mu,\nu) + \sum_{b<a} q^{(0,-b)}(\lambda,\mu)W^{(b,a)}(\lambda,\nu)q^{(-b,-a)}(\mu,\nu) \\
&\qquad + U_0(\lambda,\mu)W^{(0,a)}(\lambda,\nu)q^{(0,-a)}(\mu,\nu) = W^{(0,a)}(\mu,\nu)q^{(0.-a)}(\lambda,\nu)U_a(\lambda,\mu)(a>0) \\
&W^{(-a,0)}(\lambda,\mu)q^{(0,-a)}(\lambda,\nu)U_{-a}(\mu,\nu) = q^{(0,-a)}(\mu,\nu)U_{-a}(\lambda,\nu)W^{(-a,a)}(\lambda,\mu)
\end{aligned}$$

$$+\sum_{b<a} q^{(0,-b)}(\mu,\nu)W^{(-a,-b)}(\lambda,\nu)q^{(-b,-a)}(\lambda,\mu) + U_0(\mu,\nu)W^{(-a,0)}(\lambda,\nu)q^{(0,-a)}(\lambda,\mu)(a>0) \tag{3.4}$$

and

$$\begin{aligned}
&p^{(a,-a)}(\lambda,\nu)U_{-a}(\mu,\nu) = p^{(-b,a)}(\mu,\nu)p^{(b,-a)}(\lambda,\nu)(0<a, b<a) \\
&p^{(a,b)}(\lambda,\mu)p^{(a,a-b)}(\lambda,\nu) = p^{(a,-a)}(\lambda,\nu)U_a(\lambda,\mu)(0<a, b<a) \\
&q^{(-b,-a)}(\lambda,\mu)q^{(a,-b)}(\lambda,\nu)p^{(-a,-b)}(\mu,\nu) + W^{(-b,b)}(\lambda,\mu)p^{(b,-a)}(\lambda,\nu)W^{(-b,-a)}(\mu,\nu) \\
&\qquad = p^{(b,-a)}(\mu,\nu)W^{(-b,-a)}(\lambda,\nu)W^{(-a,b)}(\lambda,\mu)(0<a<b) \\
&p^{(a,-b)}(\lambda,\mu)W^{(-b,a)}(\mu,\nu)W^{(a.b)}(\lambda,\nu) = q^{(-b,a)}(\mu,\nu)q^{(a,-b)}(\lambda,\nu)p^{(b,a)}(\lambda,\mu) \\
&\qquad +W^{(-b,b)}(\mu,\nu)p^{(a.-b)}(\lambda,\nu)W^{(a,b)}(\lambda,\mu)(0<a<b) \\
&q^{(-a,b)}(\lambda,\mu)q^{(-b,-a)}(\lambda,\nu)p^{(b,-a)}(\mu,\nu) + W^{(-a,a)}(\lambda,\mu)p^{(a,b)}(\lambda,\nu)W^{(-a,b)}(\mu,\nu) \\
&p^{(-b,-a)}(\lambda,\mu)W^{(-b,a)}(\lambda,\nu)W^{(-a,-b)}(\mu,\nu) = q^{(-a,b)}(\mu,\nu)q^{(-b,-a)}(\lambda,\nu)p^{(a,-b)}(\lambda,\mu) \\
&\qquad +W^{(-a,a)}(\mu,\nu)p^{(-b,-a)}(\lambda,\nu)W^{(-b,a)}(\lambda,\mu)(0<b<a) \\
&q^{(a,-b)}(\lambda,\mu)U_b(\lambda,\nu)q^{(-b,a)}(\mu,\nu) + \sum_{a<c<b} q^{(a,-c)}(\lambda,\mu)q^{(-c,a)}(\mu,\nu)W^{(c,b)}(\lambda,\nu) \\
&\quad +p^{(a,-a)}(\lambda,\mu)W^{(a,b)}(\lambda,\nu)p^{(-a,a)}(\mu,\nu) = p^{(-a,b)}(\mu,\nu)W^{(a,b)}(\lambda,\nu)p^{(b,-a)}(\lambda,\mu)(0<a<b) \\
p^{(a,-b)}(\mu,\nu)\ &p^{(-b,a)}(\lambda,\mu)W^{(a,b)}(\lambda,\nu) = p^{(a,-b)}(\mu,\nu)p^{(a,-a)}(\mu,\nu)W^{(-b,-a)}(\lambda,\nu)p^{(-a,a)}(\lambda,\mu) \\
&\qquad +q^{(a,-b)}(\mu,\nu)U_{-b}(\lambda,\nu)q^{(-b,a)}(\lambda,\mu) + \sum_{a<c<b} q^{(a,-c)}(\mu,\nu)W^{(-b,-c)}(\lambda,\nu)q^{(-c,a)}(\lambda,\mu)(0<a<b) \\
&q^{(a,-b)}(\lambda,\mu)U_b(\lambda,\nu)W^{(-b,b)}(\mu,\nu) + p^{(a,-a)}(\lambda,\mu)W^{(a,b)}(\lambda,\nu)q^{(-a,-b)}(\mu,\nu) \\
&\quad = W^{(-a,b)}(\mu,\nu)q^{(a,-b)}(\lambda,\nu)U_b(\lambda,\mu)(0<a<b) \\
&W^{(-b,a)}(\lambda,\mu)q^{(a,-b)}(\lambda,\nu)U_{-b}(\mu,\nu) = p^{(a,-a)}(\mu,\nu)W^{(-b,-a)}(\lambda,\nu)q^{(-a,-b)}(\mu,\nu) \\
&\qquad +q^{(a,-b)}(\mu,\nu)U_{-b}(\lambda,\nu)W^{(-b,b)}(\lambda,\mu)(0<a<b)
\end{aligned} \tag{3.5}$$

Substituting (3.3) into (3.4) and (3.5) we obtain the solutions as follows:

$$\begin{aligned}
&U_0(\lambda,\mu) = g_0\ \ (\lambda)g_0^{-1}(\mu), \quad p^{(a,-a)}(\lambda,\mu) = q^{-1}g_{-a}(\lambda)g_a^{-1}(\mu) \\
&W^{(-a,a)}(\lambda,\mu) = (q-q^{-1})(1-q^{-2a+1})g_a(\lambda)g_a^{-1}(\mu) \\
&q^{(0,-a)}(\lambda,\mu) = -(q-q^{-1})\ q^{-a+1/2}g_0(\lambda)g_a^{-1}(\mu)(a>0) \\
&q^{(-a,0)}(\lambda,\mu) = -(q-q^{-1})q^{-a+1/2}g_a(\lambda)g_0^{-1}(\mu)(a>0) \\
&q^{(a,-b)}(\lambda,\mu) = -(q-q^{-1})q^{a-b}g_{-a}(\lambda)g_b^{-1}(\mu)(0<a<b) \\
&q^{(-b,a)}(\lambda,\mu) = -(q-q^{-1})q^{a-b}g_b(\lambda)g_{-a}^{-1}(\mu)(0<a<b) \\
&q^{(-a,-b)}(\lambda,\mu) = -(q-q^{-1})q^{-a-b+1}g_a(\lambda)g_b^{-1}(\mu)(a,b<0)
\end{aligned} \tag{3.6}$$

where $g(\lambda)$ should be satisfied the relation:

$$g_a(\lambda)g_{-a}(\lambda) \text{ is independent of index a} \tag{3.7}$$

(ii) The cases of c_n and D_n. Under the case the YBE (1.4) leads to the algebraic equations (3.3) and (3.5). The distinction between C_n and D_n is that the element $W^{(-1/2,1/2)}(\lambda,\mu)$ of $\check{R}(\lambda,\mu)$ - matrix is equal to nonzero and zero for C_n and D_n, respectively. By solving Eqs. (3.5) directly we obtain the solutions:

$$\begin{aligned}
U_a(\lambda,\mu) &= qg_a(\lambda)g_a^{-1}(\mu), \quad p^{(a,b)}(\lambda,\mu) = g_b(\lambda)g_a^{-1}(\mu) \qquad (a+b\neq 0) \\
W^{(a,b)}(\lambda,\mu) &= (q-q^{-1})g_b(\lambda)g_b^{-1}(\mu)(a+b\neq 0), \quad p^{(a,-a)}(\lambda,\mu) = q^{-1}g_{-a}(\lambda)g_a^{-1}(\mu) \\
q^{(a,-b)}(\lambda,\mu) &= -(q-q^{-1})q^{a-b}g_{-a}(\lambda)g_b^{-1}(\mu) \qquad (0<a<b) \\
q^{(-b,a)}(\lambda,\mu) &= -(q-q^{-1})\, q^{a-b}g_b(\lambda)g_{-a}^{-1}(\mu) \quad (0<a<b) \\
W^{(-a,a)}(\lambda,\mu) &= \begin{cases} (q-q^{-1})(1+q^{-2a-1})g_a(\lambda)g_a^{-1}(\mu) & \text{for } C_n \\ (q-q^{-1})(1-q^{-2a+1})g_a(\lambda)g_a^{-1}(\mu) & \text{for } D_n \end{cases} \\
q^{(-a,-b)}(\lambda,\mu) &= \begin{cases} (q-q^{-1})q^{-a-b-1}g_a(\lambda)g_b^{-1}(\mu) & \text{for } C_n \\ -(q-q^{-1})q^{-a-b+1}g_a(\lambda)g_b^{-1}(\mu) & \text{for } D_n \end{cases}
\end{aligned} \tag{3.8}$$

where $a,b\epsilon(-n+1/2,-n+3/2,\ldots,n-1/2)$ and $g_a(\lambda)$ are also constrained by the relation (3.7).

We emphasize that the solutions (3.3), (3.6) and (3.8) are one type of the solutions associated with the fundamental representations of B_n, C_n and D_n only, which will be reduced to the standard BGRs when $\lambda=\mu$. It is worth mentioning that there also exists another type of the solutions corresponding to the exotic BGRs, which can be obtained by changing the coefficients depending on λ and μ of the elements of $\check{R}(\lambda,\mu)$-matrix. Considering that the calculation is lengthy and complicated we will not write it here.

4 BW Algebra With Colour

In the $\check{R}(\lambda.\mu)$ satisfied the YBE (1.4), the parameters λ and μ can be understood as the colours. According to this consideration, we try to construct BW algeba with colour. As is known, the usual BW algebra is generated by the unit I, the braid operators G_i and the monoid operators E_i and depends on two independent parameters m and ℓ [13]. In order to colour the usual BW algebra, we introduce the operators $I_i(\lambda,\mu), RG_i(\lambda,\mu)$ and $E_i(\lambda,\mu)$ instead of I, G_i and E_i . $G_i(\lambda,\mu)$ and $I_i(\lambda,\mu)$ satisfy the relations:

$$\begin{aligned} G_i(\lambda,\mu)G_{i+1}(\lambda,\nu)G_i(\mu,\nu) &= G_{i+1}(\mu,\nu)G_i(\lambda,\nu)G_{i+1}(\lambda,\mu) \\ G_i(\lambda,\mu)G_j(\nu,\rho) &= G_j(\nu,\rho)G_i(\lambda,\mu), \quad |i-j|\geq 2 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} I_i(\lambda,\mu)I_{i+1}(\lambda,\nu)I_i(\mu,\nu) &= I_{i+1}(\mu,\nu)I_i(\lambda,\nu)I_{i+1}(\lambda,\mu) \\ I_i(\lambda,\mu)I_j(\nu,\rho) &= I_i(\lambda,\mu),\ |i-j|\geq 2 \\ I_i(\lambda,\mu)I_i(\mu,\lambda) &= I_i(\lambda,\lambda)=I_i(\mu,\mu)=I \end{aligned} \tag{4.2}$$

where I is unit. We define the following algebraic relations:

$$\begin{aligned} E_i(\lambda,\mu) &= m^{-1}(G_i(\lambda,\mu)+G_i^{-1}(\mu,\lambda))-I_i(\lambda,\mu) \\ E_i(\lambda,\mu)G_i(\mu,\lambda) &= G_i(\lambda,\mu)E_i(\mu,\lambda)=\ell^{-1}E_i(\lambda,\mu)I_i(\mu,\lambda)=\ell^{-1}I_i(\lambda,\mu)E_i(\mu,\lambda) \\ E_i(\lambda,\mu)E_{i+1}(\lambda,\nu)E_i(\mu,\nu) &= I_i(\lambda,\mu)I_{i+1}(\lambda,\nu)E_i(\mu,\nu) \\ &= E_i(\lambda,\mu)I_{i+1}(\lambda,\nu)I_i(\mu,\nu)=I_{i+1}(\mu,\nu)E_i(\lambda,\nu)I_{i+1}(\lambda,\mu) \\ E_{i+1}(\mu,\nu)E_i(\lambda,\nu)E_{i+1}(\lambda,\mu) &= I_{i+1}(\mu,\nu)I_i(\lambda,\nu)E_{i+1}(\lambda,\mu) \\ &= E_{i+1}(\mu,\nu)I_i(\lambda,\nu)I_{i+1}=I_i(\lambda,\mu)E_{i+1}(\lambda,\nu)I_i(\mu,\nu) \\ G_i(\lambda,\mu)G_{i+1}(\lambda,\nu)E_i(\mu,\nu) &= E_{i+1}(\mu,\nu)G_i(\lambda,\nu)G_{i+1}(\lambda,\mu) \\ &= E_{i+1}(\mu,\nu)E_i(\lambda,\nu)I_{i+1}(\lambda,\mu)=I_i(\lambda,\mu)E_{i+1}(\lambda,\nu)E_i(\mu,\nu) \\ E_i(\lambda,\mu)G_{i+1}(\lambda,\nu)G_i(\mu,\nu) &= G_{i+1}(\mu,\nu)G_i(\lambda,\nu)E_{i+1}(\lambda,\mu) \\ &= E_i(\lambda,\mu)E_{i+1}(\lambda,\nu)I_i(\mu,\nu)=I_{i+1}(\mu,\nu)E_i(\lambda,\nu)E_{i+1}(\lambda,\mu) \\ G_i(\lambda,\mu)I_{i+1}(\lambda,\nu)I_i(\mu,\nu) &= I_{i+1}(\mu,\nu)G_i(\lambda,\nu)I_{i+1}(\lambda,\mu)=I_i(\lambda,\mu)I_{i+1}(\lambda,\nu)G_i(\mu,\nu) \\ G_{i+1}(\mu,\nu)I_i(\lambda,\nu)I_{i+1}(\lambda,\mu) &= I_i(\lambda,\mu)G_{i+1}I_i(\mu,\nu)=I_{i+1}(\mu,\nu)I_i(\lambda,\nu)G_{i+1}(\lambda,\mu) \end{aligned} \tag{4.3}$$

where m,ℓ are arbitrary parameters independing on the colours, $G_i^{-1}(\mu,\lambda)$ is the inverse of $G_i(\mu,\lambda)$. In terms of the algebraic relations (4.1)-(4.3), it is easy to derive other algebraic relations given by

$$\begin{aligned} G_i(\lambda,\mu)I_i(\mu,\lambda) &= I_i(\lambda,\mu)G_i(\mu,\lambda) \\ E_i(\lambda,\mu)I_i(\mu,\lambda) &= I_i(\lambda,\mu)E_i(\mu,\lambda) \\ E_i(\lambda,\mu)E_i(\mu,\lambda) &= (m^{-1}(\ell+\ell^{-1})-1)E_i(\lambda,\mu)I_i(\mu,\lambda) \\ G_i(\lambda,\mu)G_i(\mu,\lambda) &= m(G_i(\lambda,\mu)+\ell^{-1}E_i(\lambda,\mu)I_i(\mu,\lambda)+I \\ G_i(\lambda,\mu)G_i(\mu,\lambda)G_i(\lambda,\mu) &= (m+\ell^{-1})G_i(\lambda,\mu)G_i(\mu,\lambda)I_i(\lambda,\mu) \end{aligned}$$

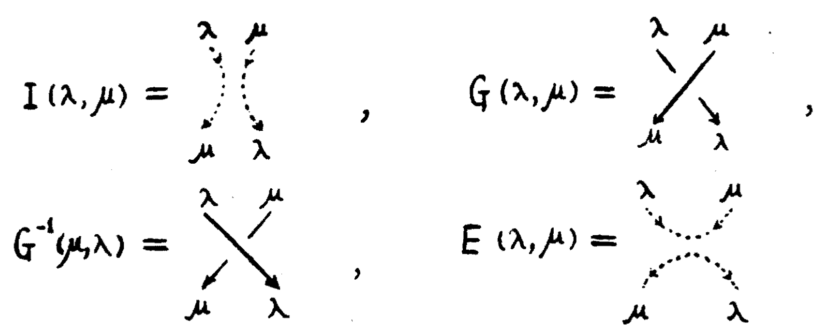
$$
\begin{aligned}
-(\ell^{-1}m+1)G_i(\lambda,\mu) &+ \ell^{-1}I_i(\lambda,\mu) \\
G_i(\lambda,\mu)E_{i+1}(\lambda,\nu)E_i(\mu,\nu) &= G_{i+1}(\nu,\mu)E_i(\lambda,\nu)I_{i+1}(\lambda,\mu) = I_i(\lambda,\mu)G_{i+1}^{-1}(\nu,\lambda)E_i(\mu,\nu) \\
G_{i+1}(\mu,\nu)E_i(\lambda,\nu)E_{i+1}(\lambda,\mu) &= G_i^{-1}(\mu,\nu)E_{i+1}(\lambda,\nu)I_i(\mu,\nu) = I_{i+1}(\mu,\nu)G_i^{-1}(\nu,\lambda)E_{i+1}(\lambda,\mu) \\
E_i(\lambda,\mu)E_{i+1}(\lambda,\nu)G_i(\mu,\nu) &= E_i(\lambda,\mu)G_{i+1}^{-1}(\nu,\lambda)I_i(\mu,\nu) = I_{i+1}(\mu,\nu)E_i(\lambda,\nu)G_{i+1}^{-1}(\mu,\nu) \\
E_{i+1}(\mu,\nu)E_i(\lambda,\nu)G_{i+1}(\lambda,\mu) &= E_{i+1}(\mu,\nu)G_i^{-1}(\nu,\mu) = I_i(\lambda,\mu)E_{i+1}(\lambda,\nu)G_i^{-1}(\nu,\mu) \\
E_i(\lambda,\mu)G_{i+1}(\lambda,\nu)E_i(\mu,\nu) &= \ell I_{i+1}(\mu,\nu)E_i(\lambda,\nu)I_{i+1}(\lambda,\mu) \\
E_{i+1}(\mu,\nu)G_i(\lambda,\nu)E_{i+1}(\lambda,\mu) &= \ell I_i(\lambda,\mu)E_{i+1}(\lambda,\nu)I_i(\mu,\nu) \\
G_i(\lambda,\mu)E_{i+1}(\lambda,\nu)G_i(\mu,\nu) &= G_{i+1}^{-1}(\nu,\mu)E_i(\lambda,\nu)G_{i+1}^{-1}(\mu,\lambda) \\
G_{i+1}(\mu,\nu)E_i(\lambda,\nu)G_{i+1}(\lambda,\mu) &= G_i^{-1}(\mu,\lambda)E_{i+1}(\lambda,\nu)G_i^{-1}(\nu,\mu) \\
E_i(\lambda,\mu)I_{i+1}(\lambda,\nu)E_i(\mu,\nu) &= (m^{-1}(\ell+\ell^{-1})-1)E_i(\lambda,\mu)I_{i+1}(\lambda,\nu)I_i(\mu,\nu) \\
E_{i+1}(\mu,\nu)I_i(\lambda,\nu)E_{i+1}(\lambda,\mu) &= (m^{-1}(\ell+\ell^{-1})-1)I_{i+1}(\mu,\nu)I_i(\lambda,\nu)E_{i+1}(\lambda,\mu) \\
G_i(\lambda,\mu)I_{i+1}(\lambda,\nu)G_i(\mu,\nu) &= (m(G_i(\lambda,\mu)+\ell^{-1}E_i(\lambda,\mu))+I_i(\lambda,\mu))I_{i+1}(\lambda,\nu)I_i(\mu,\nu) \\
G_{i+1}(\mu,\nu)I_i(\lambda,\nu)G_{i+1}(\lambda,\mu) &= (m(G_{i+1}(\mu,\nu)+\ell^{-1}E_{i+1}(\mu,\nu))+I_{i+1}(\mu,\nu))I_i(\lambda,\nu)I_{i+1}(\lambda,\mu)
\end{aligned}
\tag{4.4}
$$

For reason that the algebraic relations (4.1)-(4.4) will be reduced to the BW algebra when $\lambda = \mu = \nu$ [13], we shall call the algebra generated by the operators $I_i(\lambda,\mu), G_i(\lambda,\mu)$ and $E_i(\lambda,\mu)$ the BW algebra with colour. In the following we will show that the $\check{R}(\lambda,\mu)$ associated with the fundamental representations of B_n, C_n and D_n admits this structure.

Before doing so, we express these generators as the tensor product:

$$A_i(\lambda,\mu) = I^{(1)}(\lambda_1)\otimes\cdots\otimes I^{(i-1)}(\lambda_{i-1})\otimes A(\lambda,\mu)\otimes I^{(i+2)}(\lambda_{i+1})\otimes\cdots\otimes I^{(n)}(\lambda_n) \quad (4.5)$$

where $A(\lambda,\mu)$ stands for $I(\lambda,\mu), G(\lambda,\mu)$ and $E(\lambda,\mu), I^{(j)}(\lambda j)$ is the unit denoting by color λj. In terms of the method appearing in Ref. [6.9], we introduce the diagrammatic symbols defined by



(4.6)

the algebraic relations (4.1)-(4.4) can be represented by Figs. 1-4 (the coefficients are omitted), respectively.

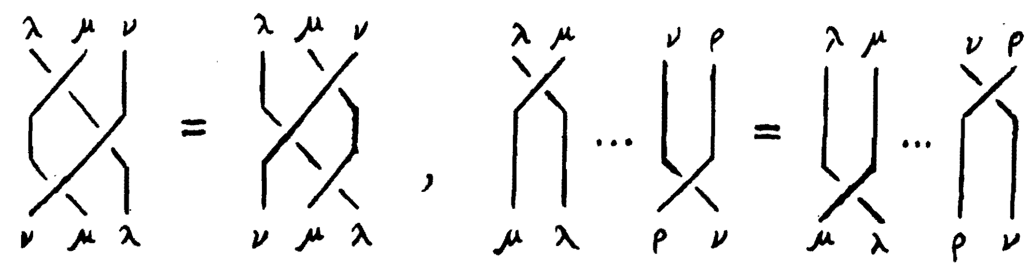


FIG. 1 Yang-Baxter algebra with colour.

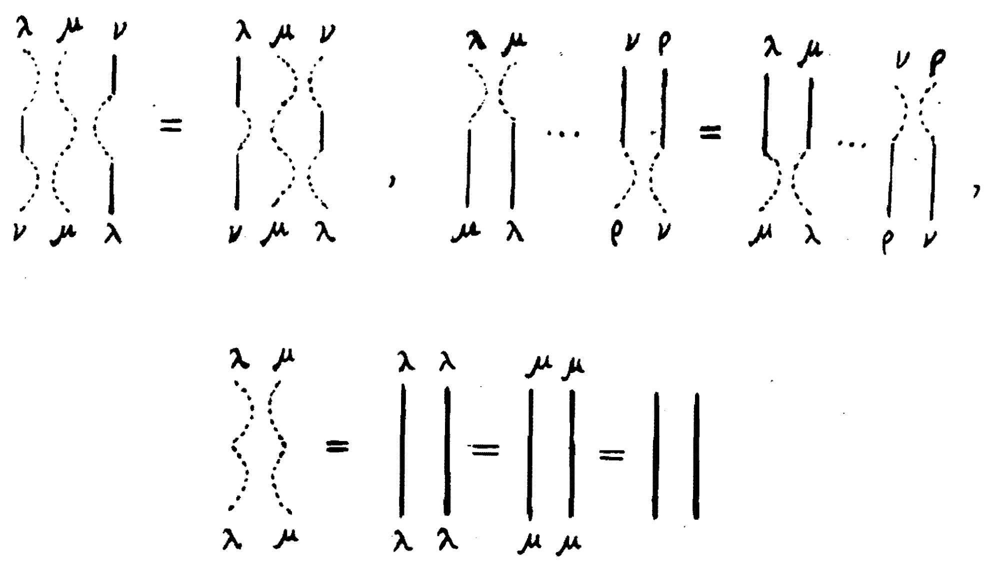


FIG. 2 The operator $I(\lambda, \mu)$ satisfies the relations.

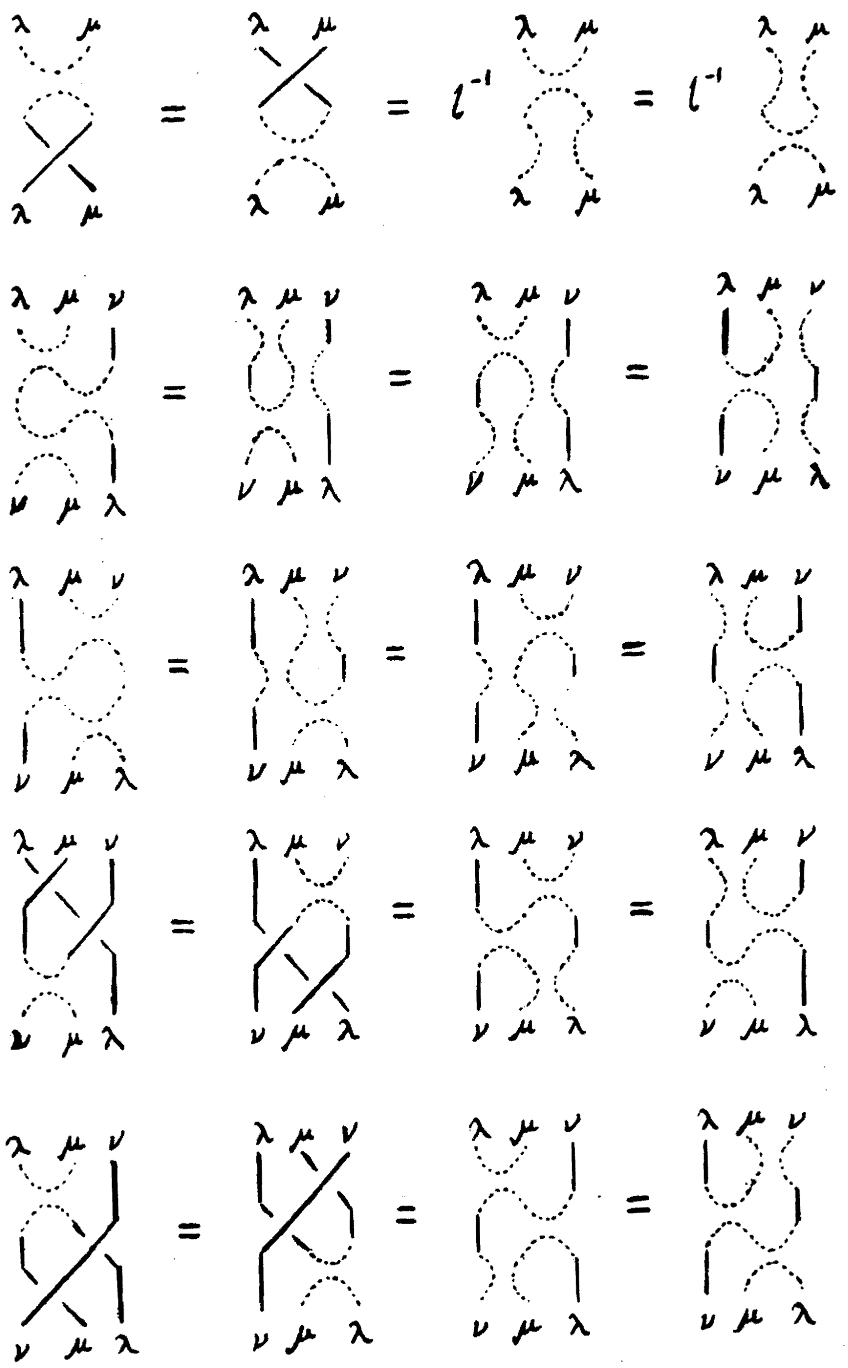


FIG. 3 Some algebraic relations given by (4.3).

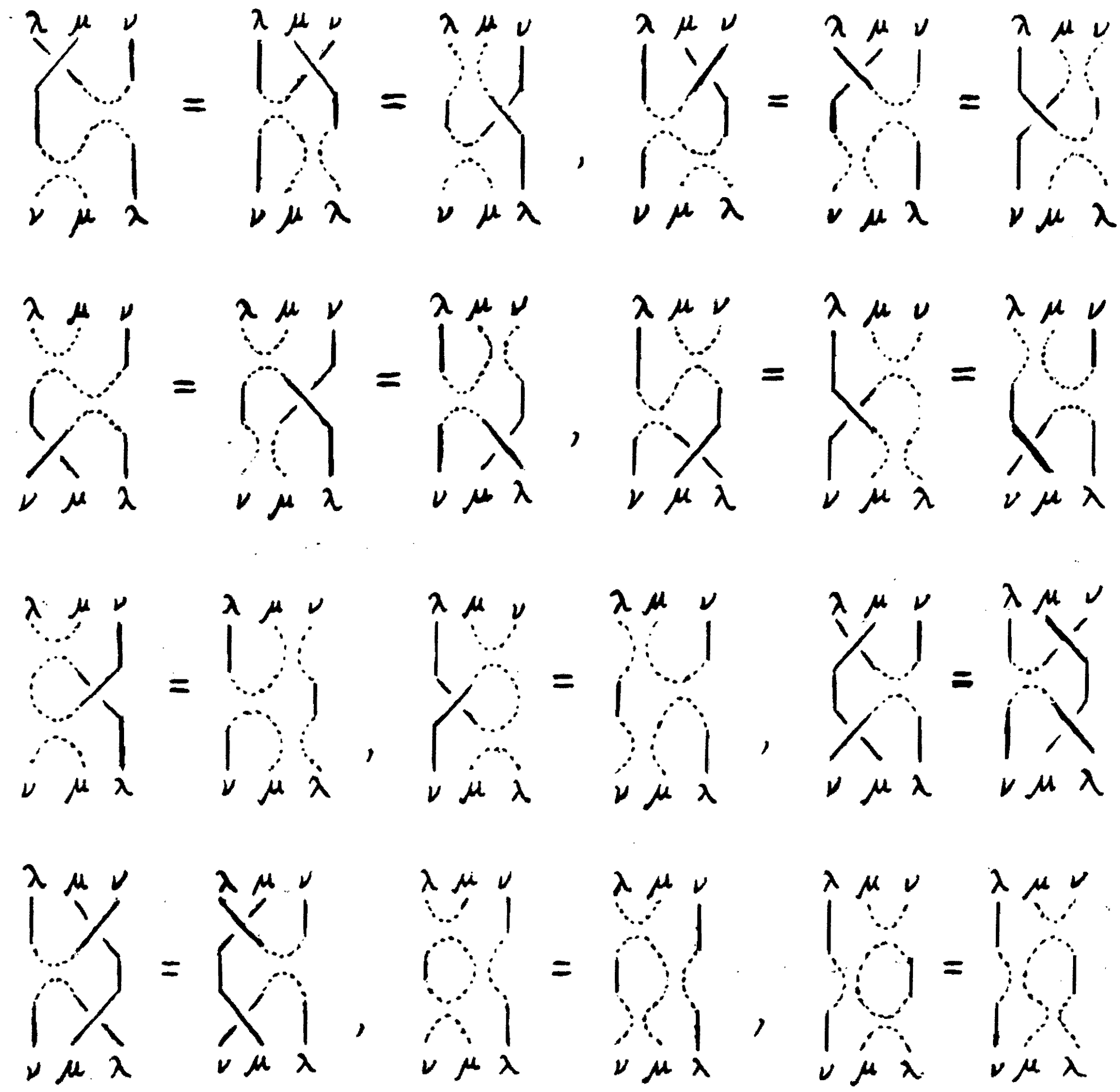


FIG. 4 Some algebraic relations given by (4.4).

As is known, the linear representation of the generator $\check{R}(\lambda,\mu)(I(\lambda,\mu)$ and $E(\lambda,\mu))$ is the $\check{R}(\lambda,\mu)$-matrix on the space $V(\lambda)\otimes V(\mu)(\check{R}(\lambda,\mu): V(\lambda)\otimes V(\mu)\rightarrow V(\mu)\otimes V(\lambda))$. The $\check{R}(\lambda,\mu)$-matrix associated with the fundamental representations of B_n, C_n and D_n has been given in Sec. III. The corresponding $\check{R}^{-1}(\mu,\lambda)$ has the form:

$$
\begin{aligned}
\check{R}(\mu,\lambda) &= \sum_{a} U'_a(\mu,\lambda)e_{aa}\otimes e_{aa} + \sum_{a>b} W'^{(a,b)}(\mu,\lambda)\otimes e_{bb} + \sum_{a\neq b} p'^{(a,b)}(\mu,\lambda)e_{ab}\otimes e_{ba} \\
&\quad + \sum_{a,b} q'^{(a,b)}(\mu,\lambda)e_{ab}\otimes e_{-a-b)} \qquad (4.7)
\end{aligned}
$$

and

$$q'^{(a,b)}(\mu,\lambda)|_{a=\pm b} = q'^{(a,b)}(\mu,\lambda)|_{a,b<0} = q'^{(a,b)}|_{\substack{a\le 0\\ 0<(a)}} = q'^{(a,b)}|_{\substack{b\le 0\\ a<(b)}} = 0 \tag{4.8}$$

The nonzero elements of the $\check{R}^{-1}(\mu,\lambda)$ are given by

$$\begin{gathered}
U'_a(\mu,\lambda) = q^{-1}g_a(\lambda)g_a^{-1}(\mu)(a\neq 0), U'_0(\mu,\lambda) = g_0(\lambda)g_0^{-1}(\mu)\\
W'^{(a,b)}(\mu,\lambda) = -(q-q^{-1})g_b(\lambda)g_b^{-1}(\mu)(a+b\neq 0).\\
W'^{(a,-a)}(\mu,\lambda) = -(q-q^{-1})(1-q^{2a-1}, g_{-a}(\lambda)g_{-a}^{-1}(\mu)(a>0)\\
p'^{(a,b)}(\mu,\lambda) = g_b(\lambda)g_a^{-1}(\mu)(a+b\neq 0), p'^{(-a,a)}(\mu,\lambda) = qg_a(\lambda)g_{-a}^{-1}(\mu)\\
q'^{(a,0)}(\mu,\lambda) = (q-q^{-1})q^{a-1/2}g_{-a}(\lambda)g_0^{-1}(\mu)(a>0)\\
q'^{(0,a)}(\mu,\lambda) = (q-q^{-1})q^{a-1/2}g_b(\lambda)g_{-a}{}^{-1}(\mu)(a>0)\\
q'^{(-b,a)}(\mu,\lambda) = (q-q^{-1})q^{-b+1}g_b(\lambda)g_{-a}{}^{-1}(0<b<a)\\
q'^{(a,-b)}(\mu,\lambda) = (q-q^{-1})q^{-b+2}g_{-a}(\lambda)g_b{}^{-1}(\mu)(0<b<a)\\
q'^{(a,b)}(\mu,\lambda) = (q-q^{-1})q^{a+b-1}g_{-a}(\lambda)g_b{}^{-1}(\ m)(a,b>0)
\end{gathered}\tag{4.9}$$

for B_n and

$$\begin{aligned}
&U'_a(\mu,\lambda) = q^{-1}g_a(\lambda)g_a{}^{-1}(\mu), \qquad W'^{(a,b)}(\mu,\lambda) = -(q-q^{-1}), g_b(\lambda)g_b^{-1}(\mu)(a+b\neq 0)\\
&p'^{(a,b)}(\mu,\lambda) = g_b(\lambda)g_a^{-1}(\mu)(a+b\neq 0), p'^{(-a,a)} = qg_a(\lambda)g_{-a}{}^{-1}(\ \mu\ )\\
&q'^{(a,b)}(\mu,\lambda) = (q-q^{-1})g_{-a}(\lambda)g_{-b}{}^{-1}(\mu)\begin{cases} -q^{a+b+1} \text{ for } C_n\\ q^{a+b-1} \text{ for } D_n\end{cases} (a,b>0)\\
&W'^{(a,-a)}(\mu,\lambda) = -(q-q^{-1})g_{-a}(\lambda)g_{-a}{}^{-1}(\mu)\begin{cases} (1+q^{2a+1}) \text{ for } C_n\\ (1-q^{2a-1}) \text{ for } D_n\end{cases}
\end{aligned}\tag{4.10}$$

for C_n and D_n.

Setting

$$\begin{aligned}
G(\lambda,\mu) = (-1)^{1/2}\check{R}(\lambda,\mu), \qquad & m = (-1)^{1/2}(q\ \ -q^{-1})\\
\ell = (-1)^{-1/2}\lambda^{-1}, \qquad & \lambda = \begin{cases} q^{-2n} \text{ for } B_n\\ -q^{-2n-1} \text{ for } C_n\\ q^{-2n+1} \text{ for } D_n\end{cases}
\end{aligned}\tag{4.11}$$

from 5th equation in the relation (4.4) we obtain

$$I(\lambda,\mu) = \sum_{a,b} g_b(\lambda)g_b^{-1}(\mu)e_{aa}\otimes e_{bb} \tag{4.12}$$

Substituting (4.11) and (4.12) into the first equation in (4.3) we derive to $E(\lambda,\mu)$ in the following form:

$$E(\lambda,\mu) = -\sum_{a,b}\epsilon_a\epsilon_{-b}\gamma_a\gamma_b g_{-a}(\lambda)g_{-b}{}^{-1}(\mu) \qquad e_{ab}\otimes e_{-a-b} \tag{4.13}$$

where $\epsilon_a = 1$ for B_n and D_n, $\epsilon_a = 1(a < 0)$ or $-1(a > 0)$ for C_n, and

$$\gamma_a = q^{\tilde{a}} \tag{4.14}$$

$$\tilde{a} = \begin{cases} a + 1/2, (a < 0) \\ a(a = 0) \\ a - 1/2(a > 0) \end{cases} \text{for } B_n \text{ and } D_n, \tilde{a} = \begin{cases} a - 1/2, (a < 0) \\ a + 1/2, (a > 0) \text{ for } C_n \end{cases} \tag{4.15}$$

By the direct calculation we can prove that the matrices $I(\lambda, \mu), G(\lambda, \mu)$ and $E(\lambda, \mu)$ given by (4.11)-(4.15) satisfy the algebraic relations (4.1)-(4.3). It is shown that the $\check{R}(\lambda, \mu)$-matrix associated with the fundamental representations of B_n, C_n and D_n admits the structure of the BW algebra with colour.

Discussion In this paper, we have derived new solutions of the YBE (1.4) associated with the fundamental representations of $A_n(n > 1), B_n, C_n$ and D_n, and constructed the BW algebra with colour. It has been shown that the $\check{R}(\lambda, \mu)$ for B_n, C_n and D_n admits this algebraic structure.

It is worth notice that there also exist solutions of the YBE (1.4) associated with other representations of Lie algebra, but which can not give rise to the BW algebra with colour. This situation will appear in the high dimensions.

We would like to point out that starting from the $\check{R}(\lambda, \mu)$-matrix we can construct "second" Yang-Baxterization prescription to generate the solutions of YBE (1.1), if we regard the parameters λ and μ as the colours. In this prescription, the solutions of YBE (1.1) will satisfy the initial copndition, the unitarity condition, etc. The details will be discussed in a following paper.

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