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V.V.Fock and A.A.Rosly

Poisson structure
on moduli of flat connections on Riemann surfaces and r-matrix

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V.V.Fock, A.A.Rosly - M., 1992 - 20p.

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Fig. - 8, ref. - 14

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1 Introduction

The moduli space of flat G-bundles a Riemann surface is the classical phase space for Chern-Simons gauge theory and thus it is in a sense the classical limit of WZW conformal field theory. This means that quantizing it one can get a quantization space which turns out to be isomorphic to the space of conformal blocks of the corresponding WZW theory. This statement has been checked by several authors (cf. [4],[13]) by different quantization schemes mainly for the case of Riemann surfaces without boundary. Moduli space of flat connections has many similarities with the space of projective (W-)structures [5] , quantization of which is believed to give quantum (W-)gravity. Apart from this moduli spaces of flat bundles as well as closely related to them moduli spaces of holomorphic bundles (cf. [8]) attracted much attention from the purely mathematical point of view (cf. [3, 7]).*

This paper consists of two parts. In section 3 we discuss in detail Poisson structure on the moduli space of flat bundles on Riemann surfaces with holes. Then we consider by analogous methods the moduli space of projective structures. As a by-product we get a more or less explicit construction for coadjoint orbits for complex Virasoro algebra. In the section 4 we construct a Poisson structure on the space of graph connections in such a way that the action of the graph gauge group is Poissonian w.r.t an appropriate nontrivial Poisson-Lie structure. The considerations of this section are inspired mainly by constructions of refs. [10, 1, 2] where a discrete analog of current algebra was suggested and investigated. Then we prove that the quotient of the space of graph connections by the gauge group coincides with the moduli space of flat connections on a Riemann surface determined by the graph. The quantization of Poisson algebras of functions on these manifolds and related generalization of Turaev's knot algebra [11] will be considered in a forthcoming paper. It will be shown there that the corresponding quantum algebras have the spaces of WZW conformal blocks as their representation spaces.

2 Ciliated fat graphs and Poisson manifolds

The moduli space of flat connections on a compact Riemann surface is by definition a subquotient of a topologically trivial space of all connections. This description is useful also since a nontrivial Poisson manifold (which is the moduli space, or the orbifold to be more precise) represented as a result of a reduction of a trivial symplectic manifold (see sect. 3) for details. The latter has plenty of convenient parameterization unlike the former. The only disadvantage of this description is that the space of connections is infinite dimensional. In this paper (sect. 4) we consider an alternative description of the moduli space in which the role of the space of all connections on a Riemann surface is played by a finite dimensional manifold. The idea is quite familiar both from lattice gauge theory and from Čech cohomology. Namely consider a triangulation of a compact Riemann surface S (with boundary, in general). Then we get a graph formed by the edges and the points of this triangulation. By a graph connection (or lattice gauge field) we mean an assignment of a group element of a gauge group G to each (oriented) edge. The group of lattice gauge transformations G^l acting on the space of graph connections

* See also A.A.Beilinson, V.G.Drinfeld and V.A.Ginzburg. *Differential Operators on Moduli Space of G-bundles.* preprint.

in a natural way is simply a product of several copies of G, one copy for each vertex of the graph. A flat graph connection satisfies the condition that the monodromies around all the faces of the triangulation equals to $1 \in G$. (The monodromy is the product of group elements corresponding to three or more edges of a face, whatever shape of faces is used. One has only to account for the orientation of the edges in an obvious way.) Now, it is a standard assertion that the moduli space of (smooth) flat connections on S is isomorphic to the space of flat graph connections modulo graph gauge transformations. (This is in fact nothing but the statement in Čech cohomology that this space is represented by $H^1(S,G)$.) If we deal with a surface with holes than one can say that some faces of the triangulation are left empty and one does not have to require anything about the corresponding monodromies. It is important to note that if a graph l is obtained from a triangulation of a surface S it can be endowed with an additional structure which (together with the graph itself) contains all the information about the topology of the surface. We suppose that S is oriented. This orientation induces a cyclic order of the ends of edges incident to each vertex. A graph l with a given cyclic order at each vertex is called a fat graph. If S has at least one hole the most economical way is to consider a fat graph with all the faces empty, what is always possible. Conversely, given a fat graph l the corresponding surface can be restored by replacing edges of l by strips glued together at vertices respecting the cyclic order (cf. fig. 1). Summarizing, in order to describe the moduli space $\mathcal{M}$ of flat connections on a surface S with holes we choose a fat graph corresponding to S (this choice is not unique) and consider the quotient of the space of graph connections $\mathcal{A}^l$ by the action of graph gauge transformations, $\mathcal{M} = \mathcal{A}^l/G^l$.

Having described the moduli space as a manifold we are interested now in describing its Poisson structure. Let us forget for a moment that we can defin a Poisson structure on $\mathcal{M}$ by reduction of the space of all (smooth) connections on S and try instead to define a Poisson structure on $\mathcal{A}^l$ in such a way that it can be pulled down on $\mathcal{M}$ [1] We would like to have such a Poisson structure on $\mathcal{A}^l$ that the projection $\mathcal{A}^l \to \mathcal{M}$ will be a Poisson map. This can be achieved if G^l will act on $\mathcal{A}^l$ in a Poisson way (see ref. [10] for the definition of Poisson group actions on Poisson manifolds). For this aim we have to define first a Poisson-Lie structure on G^l itself. The group of graph gauge transformations G^l is the direct product of several copies of G one copy per each vertex of l. The Poisson structure on G^l we define to be the direct product one built from Poisson structures on each copy of G in G^l. The latter can be defined independently at each vertex. (To define a Poisson structure on G one has to choose a classical r-matrix.) Now we look for a Poisson structure on $\mathcal{A}^l$. The requirement that the action of G^l is a Poisson one is almost sufficient to determine the Poisson structure on $\mathcal{A}^l$. The ambiguity amounts in fact to choosing a linear order of ends of edges at each vertex. Therefore instead of fat graphs we have to deal with graphs with linear order. Let us call such graphs ***ciliated fat graphs*** . A ciliated fat graph can be considered as a fat graph with an additional structure (the fat graph underlying a given ciliated fat one is restored uniquely). This additional structure (linear order at each vertex) can be represented by picturing the underlying fat graph on a sheet of paper in such a way that the cyclic order is everywhere, say, counterclockwise and by placing a small cilium at each vertex separating the minimal and the maximal end

[1] As it will be proved in sect. 4 we obtain in this way the same Poisson structure as defined by the reduction procedure from smooth connections.

incident to that vertex. As it was mentioned a fat graph defines a ciliated surface, that is an oriented surface with holes (fig. 1); a ciliated fat graph similarly defines an oriented surface with holes and with some points marked on the boundary (fig. 2). Thus for every ciliated fat graph we have an associated Poisson manifold; namely the space of graph connection endowed with r-matrix Poisson structure. It may happen of course that two different ciliated graphs give isomorphic Poisson manifolds of graph connections. One can show however that the isomorphism class of arising Poisson manifold depends only on the diffeomorphism class of the corresponding ciliated surface.

It may be worth mentioning some distinguished examples of graphs and corresporeding manifolds. The Poisson manifold corresponding to a graph consisting only of two vertices and one edge (fig. 3a) coincides with the Poisson-Lie group G provided the r-matrices chosen at the vertices are related by the operation of permutation of tensor factors ($r_{12} \mapsto r_{21}$). With the same condition on r-matrices, a graph consisting of two vertices and two edges connecting them (fig. 3b) yields the manifold $G \times G$ endowed with a Poisson-Lie structure coinciding with that of the double $D_- \simeq G \times G$. If we take the same r-matrices at two vertices we get D_+ as our Poisson manifold (see ref. [10] for definitions of doubles). Finally, the graph consisting of one vertex and one edge (fig. 3c) correspon!s to the Poisson manifold G^*, the dual Poisson-Lie group.

The following operations with graphs are important to discuss: *i*) erasure of an edge (fig. 4), *ii*) contraction of an edge (fig. 5), and *iii*) gluing graph(s) (fig. 6). The linear orders at the vertices touched by such an operation descend from those of the original graph in a more or less obvious way (cf. figs. 4,5,6). We have to mention only that there are in fact two ways to contract an edge which differ in what happens to the cilia. The operation of gluing deserves some explanation. Given two vertices on a graph with the same number N of ends of edges incident to them we can form a new graph by erasing both vertices and gluing together this liberated edges. (The k-th end liberated at one ver tex is to be glued to the $(N-k)$-th end at the other vertex.) Note that with help of this operation one can glue together two different graphs obtaining a single new one.

For the operations on graphs just described there exist natural maps between the corresponding spaces of graph connections. These maps are in fact projections in directions shown by the arrows in figs. 4,5,6. A pleasant feature is that these maps turn out to be Poisson maps. More precisely, in case of gluing one has to require that the r-matrices at two vertices to be glued are related by permutation of tensor factors. Consider for instance a map corresponding to gluing together two simplest graphs (fig. 7a) each of which represents the Poisson-Lie group G (an edge with two vertices). The result of gluing is again the graph of the same shape while the corresponding map of graph connections, $G \times G \to G$ is simply the group product which is known to be a Poisson one. Similarly, gluing together the graphs representing D_- gives the Poisson map $D_- \times D_- \to D_-$ (fig. 7b) corresponding to the group multiplication. Contracting one of two edges of the D_- graph (fig 7c) one obtains the Poisson map $D_- \to G^*$. As a Poisson manifold the dual group G^* can be identified with the coset D_-/G_Δ where G_Δ is the diagonal subgroup in $D_- \simeq G \times G$ (cf. ref. [1]). The isomorphism of G^* with the coset D_-/G_Δ shows that there is a Poisson action of D_- on G^*. i.e. a Poisson map $D_- \times G^* \to G^*$ which again can be described by gluing graphs (as shown in fig 7d). Looking at the pictures above suggests the following generalization of the notion of a

double. Namely, we can define a Poisson-Lie group, called in general a ***polyuble***[2], by the ciliated fat graph consisting of two vertices and several edges connecting them (analogously to the case of the double the r-matrices at two vertices should be related by the operation of permuta tion of tensor factors, while the order of ends should be opposite; fig. 7e). An immediate observation is that on the space of graph connections $\mathcal{A}^l$ for an arbitrary ciliated fat graph l there is a Poisson action of a polyuble $P(n)$ adjusted to each vertex n (see, fig. 7f). Thus the space $\mathcal{A}^l$ is a homogeneous space for the group P^l which is the direct product (in the sense of Poisson groups) of $P(n)$'s. Note also that the group of graph gauge transformations G^l which gives us the moduli space $\mathcal{M} = \mathcal{A}^l/G^l$ is a Poisson subgroup in P^l. (Any individual polyuble P, disregarding for the moment the Poisson structure, is a product $G \times \ldots \times G$ and contains the diagonal subgroup which turns out to be a Poisson subgroup.)

Finally, it is worth mentioning thatsome particular cases of Poisson manifolds defined by graphs have been considered in literature. Namely the Poisson manifold of graph connections on a graph corresponding to the boundary of a polygon was suggested in ref. [10] as a discrete approximation of current algebra coadjoint space. (See also refs. [1, 2] where this discrete approximation was used to investigate WZW conformal model.)

3 Poisson structure of moduli spaces

In this section we shall describe a Poisson structure on the space of flat connections modulo gauge transformations on Riemann surfaces with holesby means of a reduction of the space of all smooth connectionson them. Then we shall give an analogous description for the space of projective structures on such surfaces.

Let S be a compact Riemann surface with holes. Let $\mathcal{A}$ be a space of all G-connections on it (where G is a complex Lie group with Lie algebra $\mathfrak{g}$ possessing a nondegenerate invariant quadratic form which we denote by tr). The space $\mathcal{A}$ is in a natural way a symplectic manifold with the symplectic structure

$$\Omega = tr \int \delta A \wedge \delta A, \tag{1}$$

where $A \in \mathcal{A}$ is a $\mathfrak{g}$-valued 1-form on S, δ is an external differential on $\mathcal{A}$, tr is the Killing form on $\mathfrak{g}$ and $\wedge$ is a shorthand way to denote the wedge product both on $\mathcal{A}$ and on S. This symplectic structure is well known to be invariant with respect to the gauge transformations

$$A \mapsto g^{-1}Ag + g^{-1}dg, \tag{2}$$

where g is a G-valued function on S.

Now let us try to define the momentum mapping for this action. One can easily check that infinitesimal gauge transformation ϵ is generated by the Hamiltonian function

$$H_\epsilon = tr \int_S \epsilon(dA + A \wedge A) + tr \int_{\partial S} \epsilon A. \tag{3}$$

[2] We dedicate the Poisson groups of this type to I.V.Polyubin

The Hamiltonian generating a given transformation is defined only up to an additive constant and therefore the Poisson brackets between them in general reproduce the commutation relations between the elements of the gauge algebra only up to a cocycle:

$$\{H_{\epsilon_1}, H_{\epsilon_2}\} = H_{[\epsilon_1,\epsilon_2]} + c(\epsilon_1, \epsilon_2). \tag{4}$$

In our case

$$c(\epsilon_1, \epsilon_2) = tr \int_{\partial S} \epsilon_1 d\epsilon_2. \tag{5}$$

One can prove that this cocycle is nontrivial and therefore we can define the momentum mapping not for the algebra of gauge transformations itself, but only for its central extension by the Maurer-Cartan cocycle (5).

Let $\hat{g}$ denote the algebra of gauge transformations centrally extended by (5) and let $\hat{G}$ be the corresponding group. The space $\hat{g}$ is the space of pairs (ϵ, z), where ϵ is an element of the gauge algebra and z is a complex number. Let us consider the space $\hat{g}^*$ consisting of triples (R, B, z) where R is a g-valued two form on S, B is a g-valued 1-form on the boundary of S and z is a complex number. There is a nondegenerate pairing $<,>$ between $\hat{g}$ and $\hat{g}^*$

$$<(R, B, z), (\epsilon, z)> = tr \int_S \epsilon R + tr \int_{\partial S} \epsilon B + zz. \tag{6}$$

The momentum mapping for the action of $\hat{g}$ can be defined now as a mapping $\mathcal{A} \to \hat{g}^*$, given by curvature and the restriction of connection form to the boundary.

$$A \mapsto (dA + A \wedge A, A|_{\partial S}, 1) \tag{7}$$

Now consider the Hamiltonian reduction of $\mathcal{A}$ with respect to $\hat{G}_0$, the group of gauge transformations equal to the identity on the boundary, over the zero value of the corresponding momentum mapping. The reduced space $\mathcal{M}_0$ is the space of flat connections on S modulo gauge transformations from $\hat{G}_0$. This space can also be considered as the space of boundary restrictions of flat connections. It is well known that the space of G-connections on a circle can be identified with the coadjoint space of the affine Kac-Moody algebra with the standard Kirillov Poisson structure. The following proposition shows that these two Poisson structures are related:

Proposition 1 *The mapping from the space $\mathcal{M}_0$ to the Kac-Moody coadjoint representation space sending a flat connection on the Riemann surface S to its restriction to a component of the boundary is a Poisson mapping.*

Proof. This mapping is essentially the momentum mapping for the action of gauge transformations. □

Now let us consider the quotient of the space $\mathcal{M}_0$ by the whole group $\hat{G}$ (the group $\hat{G}$ acts on $\mathcal{M}_0$ because the group $\hat{G}_0$ of gauge transformations equal to the identity on the boundary is normal in $\hat{G}$). The quotient space $\mathcal{M}$ is a finite dimensional Poisson manifold. Its symplectic leaves are in one-to-one correspondence with the coadjoint orbits of the centrally extended group of gauge transformations which in turn are parameterized by the conjugacy classes of monodromies around the holes. Thus we have

Proposition 2 *The space of all flat G-connections modulo gauge transformations on a Riemann surface with holes inherits a Poisson structure from the space of all G-connections. The symplectic leaves of this structure are parameterized by the conjugacy classes of monodromies around holes.*

As an example let us apply this construction in case when the Riemann surface S is an annulus. Its boundary consists of two connected components and therefore we have two Poisson mappings $\mu_{1,2} : \mathcal{M}_0 \to \mathbf{k}^*$ where $\mathbf{k}^*$ is the dual space to the Kac-Moody algebra $\mathbf{k}$.

Proposition 3 *Two Poisson mappings μ_1 and μ_2 are dual.*

Here the duality means that the space of functions on $\mathcal{M}_0$ commuting with all the functions pulled back by one mapping is just the space of functions pulled back by the other one. Dual pairs are useful to describe symplectic leaves of Poisson manifolds by blowing up the points (see [12] for details). In our case it means that every symplectic leaf of $\mathbf{k}^*$ can be represented as $\mu_2\mu_1^{-1}(x)$ for some $x \in \mathbf{k}^*$ i.e. they are the sets of all connections on one boundary component of the annulus which can be extended in a flat way to the whole annulus giving a fixed connection on the other boundary component. This gives, of course, a well known answer that symplectic leaves of $\mathbf{k}^*$ are just the gauge orbits. We have considered this trivial example in order to illustrate an analogous construction for the Virasoro algebra in which case the answer is less trivial.

Now let us proceed to the generalization of the above constructions to the case of moduli space of projective structures and discuss possible generalizations to the spaces of W_n-projective structures. This was done in [5] for the case of closed Riemann surfaces (i.e. without holes). Briefly, the relation between the space of $SL(2, \mathbf{C})$ connections and the space of projective structures stated in [5] is the following. Let us call an $SL(2, \mathbf{C})$-connection A to be *nondegenerate* if for any real tangent vector v the element in the upper right corner of the matrix $i_v A$ is nonzero. Then the space of projective structures is isomorphic as a symplectic manifold to the space of all nondegenerate flat connections modulo non strictly lower triangular gauge transformations and diffeomorphisms. This statement can be easily generalized to the case of a Riemann surface S with boundary. Consider the space $\mathcal{P}_0$ of all nondegenerate flat connections on S modulo diffeomorphisms equal to the identity on the boundary and lower triangular gauge transformations. boundary. One can check that it is a symplectic manifold with symplectic structure inherited from that of the space of all $SL(2, \mathbf{C})$-connections. Note that a nondegenerate connection 1-form modulo gauge transformations restricted to the boundary can be transformed in a unique way by a lower triangular gauge transformation to the form

$$A = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}. \tag{8}$$

The group of all diffeomorphisms acts on $\mathcal{P}_0$ and analogously to the situation considered in the first part of this section we have to extend the diffeomorphism group in order to define the momentum mapping. On the Lie algebraic level this extension is given by the Gelfand-Fuchs cocycle:

$$\{H_{v_1}, H_{v_2}\} = H_{[v_1,v_2]} + \int_{\partial S} v_1 \partial_x^3 v_2 \tag{9}$$

where v_1 and v_2 are two vector fields on S tangent to the boundary, x is a coordinate on the boundary and the Hamiltonian functions are given by

$$H_v = \int_{\partial S} vT. \tag{10}$$

The momentum mapping gives us a set of mappings $\mathcal{P}_0 \to Vir^*$ where Vir is the Virasoro algebra. For the case of S being an annulus we get a dual pair of Poisson mappings allowing us to construct Virasoro symplectic leaves. The answer is that a symplectic leave is a set of all connections of the form (8) such that they can be extended in a flat nondegenerate way from one boundary component of the annulus to the whole annulus giving a fixed connection on the other boundary. Note, that this construction gives the answer for complex Virasoro coadjoint orbits, where the standard methods do not work because the complex Virasoro group does not exist (cf. [9]).

Note that this construction could be generalized for the W_n algebra case. The only problem is to formulate an appropriate notion of nondegeneracy.

4 Graph connections

In this section we shall construct a Poisson structure on the space of graph connections $\mathcal{A}^l$, in such a way that the lattice gauge group endowed with nontrivial r-matrix Lie-Poisson structure acts on $\mathcal{A}^l$ in a Poisson way.

Let l be a ciliated fat graph homotopically equivalent to a Riemann surface S with holes. Denote by $E(l)$ the set of ends of edges of l and by $N(l)$ the set of its vertices. Each element of $N(l)$ corresponds to with the subset of $E(l)$ of ends of edges incident to the given vertex. In what follows we shall identify elements of $N(l)$ with the corresponding subsets. A mapping which sends an end of an edge α to the opposite end of the same edge $\alpha^\vee$ is an involution of the set $E(l)$. The ciliated fat graph structure of l defines an ording incide each $n \in N(l)$. One can easily see that such data – a set divided into ordered subsets and an involution of it without fixed points – unambiguously define a ciliated fat graph. Let $[\alpha]$ be the vertex containing α and $[\alpha, \alpha^\vee]$ be the edge linking α and $\alpha^\vee$.

Call a ***graph connection*** on a graph l an assignment of an element $\mathbf{A}_\alpha$ of a group G to each $\alpha \in E(l)$ such that

$$\mathbf{A}_{\alpha^\vee} = \mathbf{A}_\alpha^{-1}. \tag{11}$$

The lattice gauge group G^l is a product of finite dimensional groups G — one for each vertex of the graph. The group G^l acts on $\mathcal{A}^l$ in a natural way:

$$\mathbf{A}_\alpha \mapsto \mathbf{g}_{\alpha^\vee}^{-1} \mathbf{A}_\alpha \mathbf{g}_\alpha \tag{12}$$

The space of graph connections can be considered as a quotient space of the space of flat connections on a surface S. Indeed let us blow up the fat graph in order to obtain a surface S with the graph drawn on it. Then for a (smooth) connection A on S we can construct a graph connection on l assigning to $\alpha \in E(l)$ the parallel transport operator along the edge linking $\alpha^\vee$ and α. This graph connection does not change if we transform the connection A by a gauge transformation equal to the identity at the vertices. It

is clear that every graph connection can be continuously extended to the surface and therefore the space of graph connections $\mathcal{A}^l$ can be represented as a quotient

$$\mathcal{A}^l = \{A \in \mathcal{A} | dA + A \wedge A = 0\}/\hat{G}_1. \tag{13}$$

where $\hat{G}_1$ is the group of gauge transformations equal to the identity on vertices. Of course this representation is defined unambiguously up to the action of the graph gauge group and therefore the isomorphism between the spaces $\mathcal{M}$ and $\mathcal{A}^l/G^l$ is canonical.

A priori the space $\mathcal{A}^l$ has no Poisson structure, while the space $\mathcal{A}^l/G^l$ does. Our aim is to introduce a Poisson structure on $\mathcal{A}^l$ compatible with that on $\mathcal{A}^l/G^l$ and with graph gauge group action.

Let us fix for each vertex n of the graph a solution $r(n) \in \mathrm{g} \otimes \mathrm{g}$ of the Yang-Baxter equation:

$$[r_{12}(n), r_{13}(n)] + [r_{12}(n), r_{23}(n)] + [r_{13}(n), r_{23}(n)] = 0 \tag{14}$$

such that

$$r_{12}(n) + r_{21}(n) = t, \tag{15}$$

where $t \in \mathrm{g} \otimes \mathrm{g}$ is the quadratic Casimir:

$$t = \sum e_i \otimes e_i, \tag{16}$$

where $\{e_i\}$ is an orthonormal basis in g.

Let us define a bivector field B on $\mathcal{A}^l$:

$$B = \sum_{n \in N(l)} \Big(\sum_{\alpha, \beta \in n; \alpha < \beta} r^{ij}(n) X_i^\alpha \wedge X_j^\beta + \frac{1}{2} \sum_{\alpha \in n} r^{ij}(n) X_i^\alpha \wedge X_j^\alpha \Big) \tag{17}$$

where $X_i^\alpha = L_i^\alpha - R_i^{\alpha^\vee}$, L_i^α and R_i^α are respectively the left- and right-invariant vector fields corresponding to the element $e_i \in g$ on the group assigned to $\alpha \in E(l)$ and $r^{ij}(n)$ is an r-matrix chosen for the vertex n written in the basis $\{e_i\}$. Note that the vector fields X_i^α are chosen to be consistent with the eq.(11).

Proposition 4 *a) The bivector P defines a Poisson structure on $\mathcal{A}^l$. b) The group G^l endowed with the direct product Poisson-Lie structure acts on $\mathcal{A}^l$ in a Poisson way.*

The proof can be obtained by a straightforward check.

Proposition 5 *Let $[\alpha, \alpha^\vee]$ be an edge and n be a vertex containing α. Suppose that $[\alpha^\vee] \notin n$. Then the quotient $\mathcal{A}^l/G(n)$ is isomorphic to the Poisson manifold $\mathcal{A}^{l'}$, where l' is a graph obtained from l by contracting the edge $[\alpha, \alpha^\vee]$ and such that the maximal remaining end at the vertex $[\alpha^\vee]$ become the maximal one for the new vertex.*

The isomorphism being evident from representation of graph connections as quotient of all connections, the proof can be given by a direct coordinate check of coincidence of two Poisson strictures.

Let us proceed now to the correspondence between spaces of graph connections and spaces of ordinary connections.

Proposition 6 *The quotient of the space of graph connections by the graph gauge group is isomorphic as a Poisson manifold to the quotient of the space of all flat connections on the corresponding Riemann surface by the gauge group.*

Remark. Before giving the proof let us note that this statement shows that all ambiguities in the construction of the space $\mathcal{A}^l$ – such as choices of ordering and of r-matrices – do not influence the Poisson structure of its quotient by the gauge group, though it is impossible to introduce a Poisson structure on $\mathcal{A}^l$ compatible with that on the gauge quotient without fixing nontrivial r-matrices. Note also that as topologicallys these moduli spacess are always isomorphic to a product of several copies of the group G modulo the overall G-conjugation, though they are not isomorphic to each other as Poisson manifolds. For example a sphere with three holes and a torus with one hole give topologically the same spaces, $(G \times G)/AdG$, while the Poisson structure is trivial for the former case and nontrivial for the latter one.

Proof. First of all let us describe a linear basis in the space of all functions on $\mathcal{A}^l$. Let us assign an irrep π_α of G in a space V_α to each $\alpha \in E(l)$ in such a way that $\pi_{\alpha^\vee} = \pi_\alpha^*$ and assign an intertwiner $C_n \in \mathrm{Inv}(\otimes_{\alpha\in n} V_\alpha^*)$ to each vertex n. We can consider matrices from $\mathrm{End}V_\alpha$ as belonging to $V = \otimes_{\alpha\in E(l)} V_\alpha$ and the intertwiners C_n as belonging to its dual V^*.

Let us call the data $\{C_n, \pi_\alpha\}$ the *coloring* of the graph. Define for further needs an operation of contracting an edge of a colored graph. The coloring of a new vertex of the graph with the edge $[\alpha, \alpha^\vee]$ contracted will be $< C_{[\alpha]} \otimes, C_{[\alpha^\vee]} >_\alpha$, where $<,>_\alpha$ is a natural pairing between V_α and V_α^*.

For each coloring we can define a function ψ on $\mathcal{A}^l$

$$\psi(\{\mathbf{A}_\alpha\}) = < \otimes_n C_n, \otimes_{\alpha\in E_1(l)} \pi_\alpha(\mathbf{A}_\alpha) > \tag{18}$$

where $E_1(l) \subset E(l)$ is a set of ends of edges containing exactly one end of each edge. The ambiguity in choice of this set is unessential because of the condition $\pi_{\alpha^\vee}(\mathbf{A}_{\alpha^\vee}) = \pi_\alpha(\mathbf{A}_\alpha)$.

The set of such functions determined by all possible colorings evidently forms a linear basis in the space of functions on $\mathcal{A}^l$.

Note that given a graph drawn on a surface S each coloring defines also a function on the space $\mathcal{A}$ of G-connections on S and on the space of flat connections modulo gauge transformations $\mathcal{M}$ in particular. For the latter case the set of functions determined by colored graphs forms in fact a basis in the space of all functions.

Now we shall calculate Poisson brackets of such functions on $\mathcal{A}^l$ and on $\mathcal{A}$ and show that the results coincide.

Let us start with the space $\mathcal{A}^l$. Rewrite the bivector (17) in the form

$$B = \sum_n (r^{ij}(n) X_i^\Delta(n) \otimes X_j^\Delta(n) + \sum_{\alpha,\beta\in n} (n,\alpha,\beta) t^{ij} X_i^\alpha \otimes X_i^\beta), \tag{19}$$

where $X_i^\Delta(n) = \sum_{\alpha\in n} X_i^\alpha$ and $(n,\alpha,\beta) = \begin{cases} 1 & \alpha > \beta \\ 0 & \alpha = \beta \\ -1 & \alpha < \beta \end{cases}$.

The vector fields $X_i^\Delta(n)$ just generate the gauge transformations. Thus in calculating the brackets between gauge invariant functions we can drop the first term in (19) The

final answer for brackets will be

$$\{\psi, \psi'\} = < C \otimes C', T(\Pi \otimes \Pi') > \tag{20}$$

where $C = \otimes_n C_n, C' = \otimes_n C'_n$, $\Pi = \otimes_{\alpha\in E_1(l)} \pi_\alpha(\mathbf{A}_\alpha)$, $\Pi' = \otimes_{\alpha\in E_1(l)} \pi'_\alpha(\mathbf{A}_\alpha)$, ψ and ψ' are the functions determined by colorings $\{C_n, \pi_\alpha\}$ and $\{C'_n, \pi'_\alpha\}$ respectively, $T = \sum_{\alpha,\beta\in n; \alpha>\beta} t_n^{\alpha\beta}$ and $t_n^{\alpha\beta}$ is the Casimir element represented in the space $V_\alpha \otimes V_\beta$ and hence, naturally, in $V \otimes V$. One can check that the value of this expression is independent of the choice of the set $E_1(l)$.

This expression for Poisson brackets can be represented graphically as follows. Let $(l, \{C_n, \pi_\alpha\})$ and $(l', \{C'_n, \pi'_\alpha\})$ be two colored graphs and let $\alpha \in E(l)$ and $\alpha' \in E(l')$ be some ends of edges of the corresponding graphs. Define a colored graph $(l, \{C_n, \pi_\alpha\}) \cup_{\alpha\alpha'} (l', \{C'_n, \pi'_\alpha\})$ to be a graph obtained from l and l' by gluing together vertices $[\alpha]$ and $[\alpha']$. [3] The new vertex is to be colored by $(C_{[\alpha]} \otimes C'_{[\alpha']}) t^{\alpha,\alpha'}$ and the colors of edges and of other vertices remain unchanged. Thus we have

$$\{\psi, \psi'\} = \sum_n \sum_{\alpha,\beta\in n} (n,\alpha,\beta)(l, \{C_n, \pi_\alpha\}) \cup_{\alpha\beta} (l, \{C'_n, \pi'_\beta\}) \tag{21}$$

Now let us proceed to the calculation of Poisson brackets in $\mathcal{M}$ resting on the definition of this space given in sect. 3. For this aim we shall calculate the brackets in the space $\mathcal{A}$ of all connections w.r.t. its Poisson structure,

$$\{\psi, \psi'\} = \int_S \frac{\delta\psi}{\delta A} \wedge \frac{\delta\psi'}{\delta A} \tag{22}$$

and then restrict the result to the space of flat connections. The following proposition gives an expression for Poisson brackets of functions on $\mathcal{A}$ given by a colored graph drawn on the surface S.

Proposition 7 *Let $(l, \{C_n, \pi_\alpha\})$ and $(l', \{C'_n, \pi'_\alpha\})$ be two transversal colored graphs drawn on a surface S. Than the Poisson brackets of the corresponding functions ψ and ψ' are given by the expression*

$$\{\psi, \psi'\} = \sum_{x\in l\cap l'} (-)^{sign(x)} (l, \{C_n, \pi_\alpha\}) \cup_{\alpha(x)\alpha'(x)} (l', \{C'_n, \pi'_\alpha\}) \tag{23}$$

where $\alpha(x) \in E(l)$ and $\alpha'(x) \in E(l')$ are such ends of edges that $x = [\alpha(x), \alpha(x)^\vee] \cap [\alpha'(x), \alpha'(x)^\vee]$ and $sign(x) = 1$ if the oriented edge $[\alpha(x), \alpha(x)^\vee]$ intersect the oriented edge $[\alpha(x), \alpha(x)^\vee]$ from right to left and -1 otherwise.

One can check that the r.h.s. of (23) is well defined i.e. it is independent of the choice of $\alpha(x)$ and $\alpha'(x)$.

This formula is inconsistent when applied to functions which are determined by the same graph. However as far as we are interested only in values of functions on flat connections we can deform one graph in the surface to make them transversal and apply the formula (23).

[3] We hope there will be no confusion with the operation of gluing discussed in sect. 2. Here we simply identify two vertices.

Let us describe now a particular deformation which will give an expression for brackets in a form which can be compared to eq.(20). We rotate each edge slightly around its middle in counterclockwise direction. Then we place vertices of the deformed graph between the greatest end and the smallest end at the corresponding vertex of the original graph and connect each end of the rotated edge with the corresponding shifted vertex in clockwise direction as shown on fig. 8. For this particular deformation we have one intersection point $x_{[\alpha,\alpha^\vee]}$ at the middle of each edge with $\mathrm{sign}(x_{[\alpha,\alpha^\vee]}) = 1$ and one intersection point $x_{\alpha,\beta}$ for each couple of edges incident to a vertex n and such that $\alpha > \beta$ with $\mathrm{sign}(x_{\alpha,\beta}) = -1$. Thus the Poisson brackets for coincident graphs can be expressed in the form:

$$\begin{aligned}\{\psi(l,\{C_n,\pi_\gamma\}),\psi(l,\{C'_n,\pi'_\gamma\})\} &= \sum_{\alpha\in E_1(l)} \psi((l,\{C_n,\pi_\gamma\}) \cup_{x_{[\alpha,\alpha^\vee]},x_{[\alpha,\alpha^\vee]}} (l,\{C'_n,\pi'_\gamma\})) - \\ &- \sum_n \sum_{\alpha,\beta\in n;\alpha>\beta} \psi((l,\{C_n,\pi_\gamma\}) \cup_{\alpha,\beta} (l,\{C'_n,\pi'_\gamma\}))\end{aligned} \tag{24}$$

Then using the identities

$$\psi((l,\{C_n,\pi_\gamma\}) \cup_{\alpha,\alpha} (l,\{C'_n,\pi'_\gamma\})) = \psi((l,\{C_n,\pi_\gamma\}) \cup_{\alpha^\vee,\alpha^\vee} (l,\{C'_n,\pi'_\gamma\})) \tag{25}$$

and

$$\psi((l,\{C_n,\pi_\gamma\}) \cup_{\alpha,\alpha} (l,\{C'_n,\pi'_\gamma\})) = \sum_{\beta\in[\alpha];\beta\neq\alpha} \psi((l,\{C_n,\pi_\gamma\}) \cup_{\alpha,\beta} (l,\{C'_n,\pi'_\gamma\})) \tag{26}$$

we come just to eq.(20) □.

5 Concluding remarks

In this paper we have considered only the case of a complex group G. What's is about its real forms? If one wishes to get a real valued Poisson structure on the space of graph connections [4] corresponding to a real group then one has to find a real solution of the classical Yang-Baxter equation (14). For some real forms, e.g. for $SL(n,\mathbf{R})$, this is certainly possible, while for others, like $SU(n)$, for instance, it seems to be not the case. One possible way to deal with such real forms is to consent to considering quasi Poisson manifolds with a definite violation of Jacobi identity, which under quantization must lead to quasiassociative algebras.

All the above considerations have been performed to prepare to define and investigate properties of the quantum moduli space of flat bundles. It is expected to be an algebra, irreducible representations of which are the spaces of conformal blocks. Some evidence for believing that it will be really so one can extract just from classical consideration. For example conformal blocks of WZW theory are labelled by irreducible representations of the gauge group G, which in turn are in correspondence with conjugacy classes of G. Here we have observed that symplectic leaves of moduli spaces of flat bundles are labelled just by conjugacy classes.

[4] Though it might be only the moduli space for which the reality condition for Poisson structure is essential

The space of graph connections was introduced mainly to simplify the consideration of the moduli space. Topologically it is simply a product of groups, instead of the moduli space which *a priori* has no preferred parameterization. The next step postponed to a forthcoming paper should be to quantize the graph connections spaces by some modification of quantum group technique and then obtain quantum moduli spaces by taking quotients on the quantum level. Note that the consideration of the space of graph connections gives the moduli space of smooth flat connections on Riemann surfaces without taking any continuous limit. Nevertheless it still would be interesting to consider a continuous limit (cf. [1, 2]). In the classical case this limit is believed to be a space of all connections on a surface modulo gauge transformations and its quantization would give a universal algebra connected with the space of conformal blocks with any number of punctures. It is to be mentioned here one more interplay between Kac-Moody algebras and the space of connections modulo gauge transformations. A generic (0,1)-connection on the complex plane can be gauge transformed to zero by an unambiguously defined gauge transformation equal to the unity at the infinity. Thus we can parameterize the space of all g connections modulo gauge transformations equal to the unity at the infinity by (1,0)-connections $A(z)$. The Poisson bracket induced by the symplectic structure (1) is

$$\{F,G\}(A) = tr \int_{S\times S} \frac{A(z)-A(w)}{z-w} \left[\frac{\delta F}{\delta A(z)}; \frac{\delta G}{\delta A(w)}\right]. \tag{27}$$

This is a linear bracket and thus it defines a Lie algebra structure on the space of linear functionals on A:

$$\{A^a(z), A^b(w)\} = f^{ab}_c \frac{A^c(z)-A^c(w)}{z-w}, \tag{28}$$

where f^{ab}_c are structure constants of g. This Lie algebra is quite similar to the loop algebra with r-matrix bracket (cf. [6]) and contains it as a subalgebra generated by holomorphic A's.

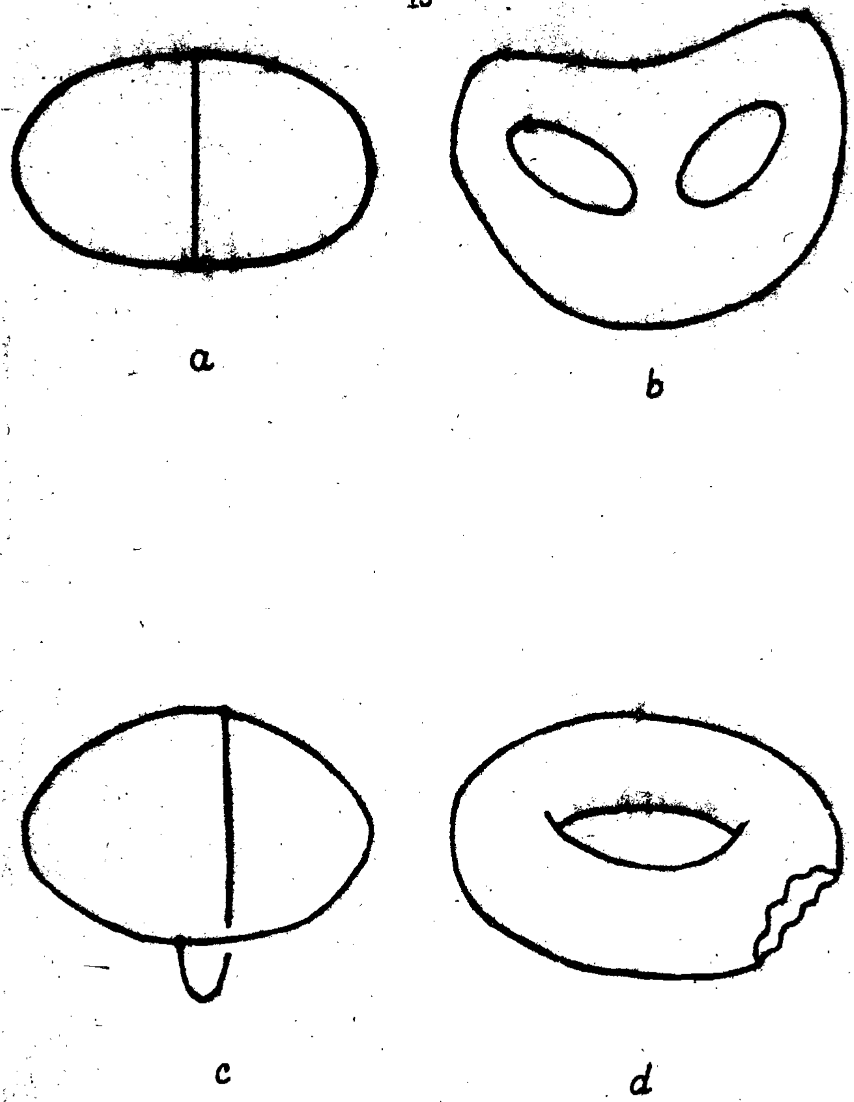

Fig. 1 Example of fat graphs and surfaces corresponding to them.
The cyclic orders at vertices are understood to be counterclockwise.
The graph (a) gives the disk with two holes (b).
The graph (b) gives a torus with one hole (d).

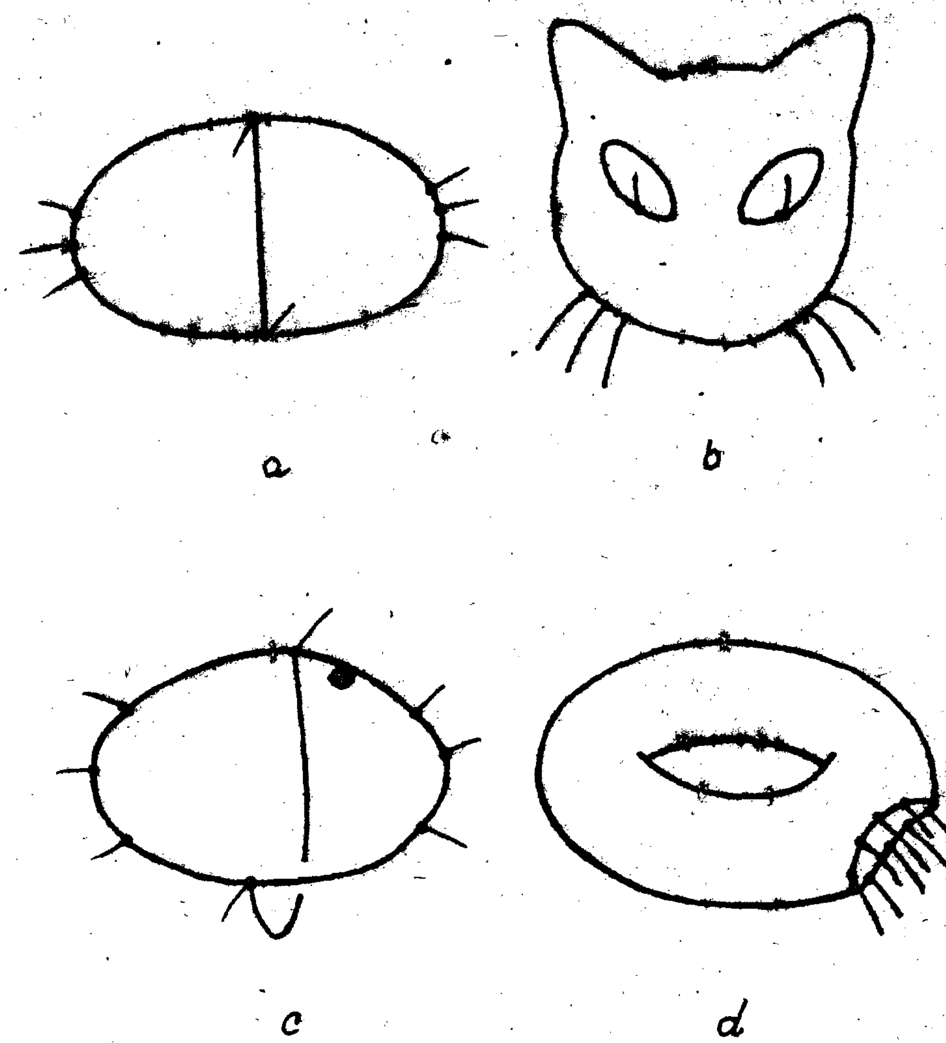

Fig. 2 Examples of ciliated fat graphs and corresponding ciliated surfaces.
Cilia are indicated by small strokes at the vertices.
The graph (a) gives the disk with two holes (b).
The graph (c) gives the torus with one hole (d).

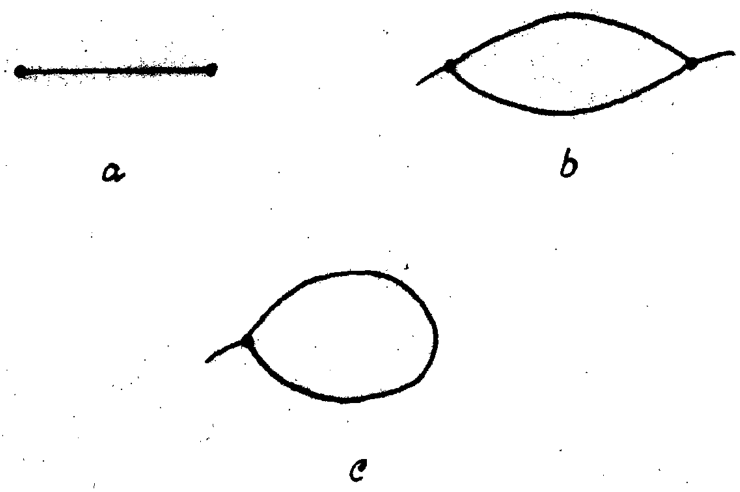

Fig. 3 The graphs corresponding to
(a) the Poisson-Lie group G,
(b) its double $D \simeq G \times G$,
(c) its dual Poisson-Lie group G^*.

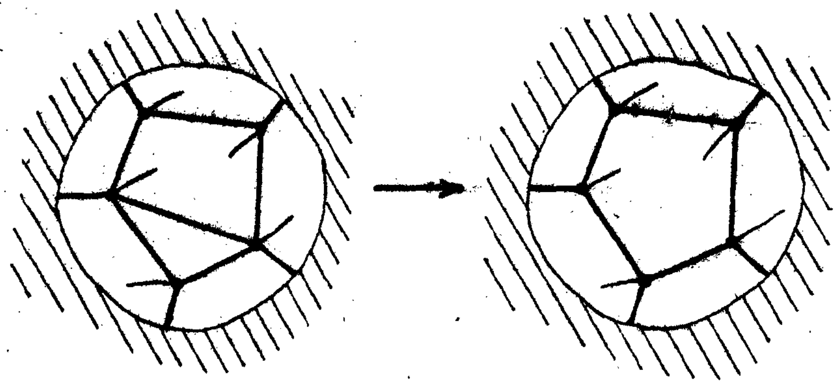

Fig. 4 Operation of erasing an edge. The shaded region represents the remainder of the graph.

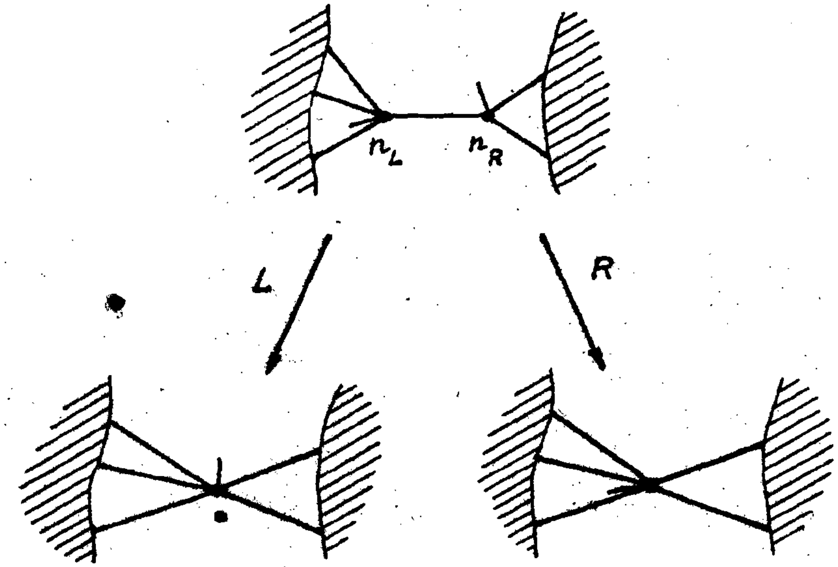

Fig. 5 Operations of contractions of an edge.
L and R are the two different ways of contraction.
L corresponds to factoring by gauge transformation at the vertex n_L
R corresponds to factoring by gauge transformation at the vertex n_R

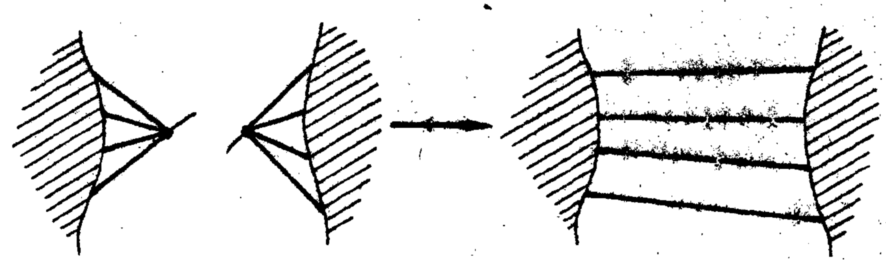

Fig. 6 Operation of gluing graphs.

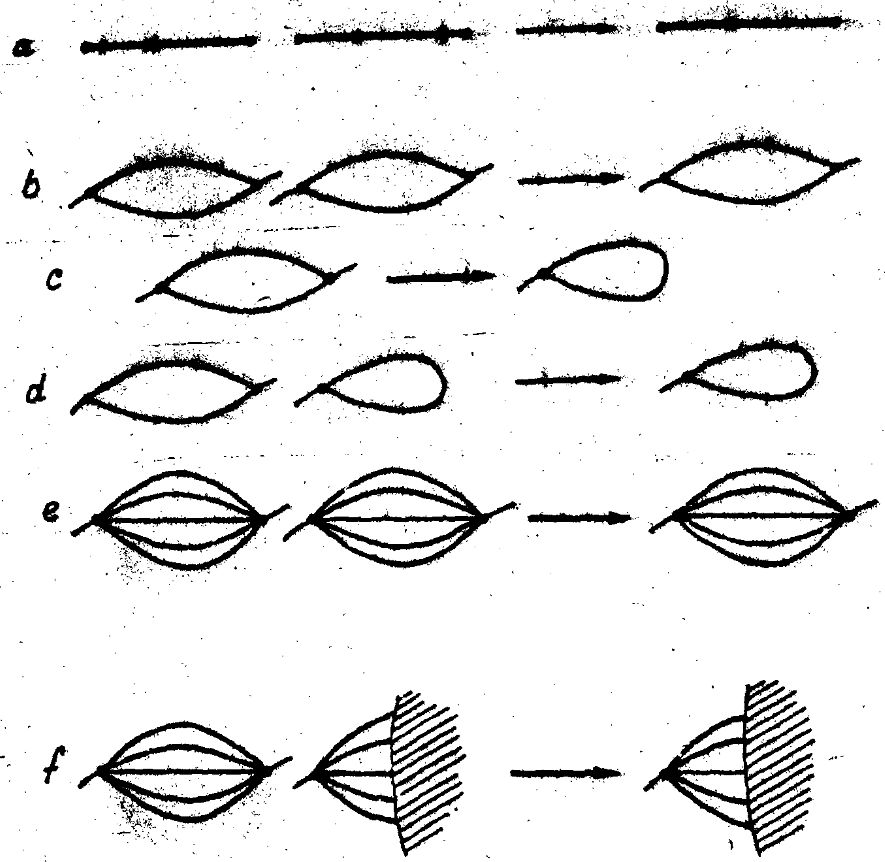

Fig. 7 Some particular cases of gluing graphs which correspond to natural operations in Poisson-Lie groups:
(a) multiplication in G,
(b) multiplication in D,
(c) projection $D \to G^*$,
(d) action of D on G^*,
(e) multiplication in the 5-uble,
(f) action of the 5-uble on a space of graph connections.

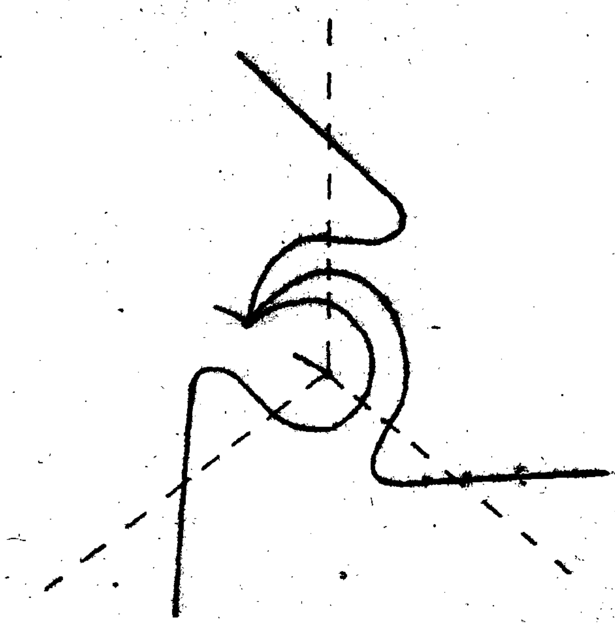

Fig. 8 The particular way of deforming a graph drawn on a surface which gives two transversal graphs; the original graph is shown by the broken line.

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