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Large orders of $1/n$ -expansion
in quantum mechanics

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Asymptotics of large orders of $1/n$ -expansion in quantum mechanics has been found. It has been shown that the coefficients $\epsilon^{(k)}$ grow as $k! a^k$ with $k \rightarrow \infty$, and the dependence of the parameter a on the coupling constant is investigated.

Fig. - 10 , ref. - 28

1. At present $1/n$ -expansion is widely used in various quantum-mechanical problems, see e.g. refs. [1-10] and references therein. We consider below the version of the method proposed in [7], which is applicable not only for discrete spectrum, but also in the case of quasistationary states (resonances). The energy eigenvalues which are complex in the last case ($E_{nl} = E_r - i\Gamma/Z$), can be represented in the form of expansion in powers of the "small parameter" $1/n$,

$$\epsilon = \epsilon' - i\epsilon'' = \epsilon^{(0)} + \frac{\epsilon^{(1)}}{n} + \dots + \frac{\epsilon^{(k)}}{n^k} + \dots, \quad (1)$$

where $n = n_r + l + 1$ is the principal quantum number, l is angular momentum, $\epsilon = 2n^2 E_{nl}$ is the "reduced" energy of nl -state, $\epsilon'' = n^2 \Gamma_{nl}$ and k is the order of $1/n$ -expansion.

The behaviour of the coefficients $\epsilon^{(k)}$ at $k \gg 1$ not only presents some theoretical interest, but is of considerable importance in calculating energy E_{nl} with high accuracy using expansion (1). It is known that divergence of perturbation theory series (PT) in quantum mechanics and field theory is connected with the instability of the vacuum state when coupling constant g changes its sign (the so-called "Dyson phenomenon", established for the first time in QED [11] and later considered for the anharmonic oscillator [12,13], Stark [14-16] and Zeeman [17] effects, and other quantum-mechanical problems).

As a rule, the asymptotics of large orders of PT has the form:

$$E_k \simeq (k\alpha)! a^k k^\beta (c_0 + \frac{c_1}{k} + \frac{c_2}{k^2} + \dots) \equiv$$

$$\equiv (k\alpha + \beta)! a^k (c_0 + \frac{c_1}{k} + \dots), \quad (2)$$

where $E(g) = \sum_k E_k g^k$, $z! \equiv \Gamma(z+1)$ and $\alpha > 0, \beta, a, c_0, c_1$ etc. are calculable constants.

In eq.(1) the expansion parameter is $1/n$ (instead of coupling g), which does not enter the hamiltonian explicitly, and the coefficients $\epsilon^{(k)}$ are rather complicated functions of g , contrary to the case of higher PT orders. So, some modification of Dyson's arguments is needed, which is given below.

2. Asymptotics of large orders of $1/n$ -expansion. Using recurrence relations ¹⁾, we have computed 30 ÷ 50 coefficients $\epsilon^{(k)}$; eq.(2) for them was checked at $k \gg 1$ and parameters of asymptotics α, a, \dots were determined numerically. These calculations have been done for the following problems: the funnel potential

$$V(r) = -r^{-1} + gr, \quad g > 0, \quad (3)$$

the Stark effect in hydrogen and its spherical model (which corresponds to $g \rightarrow -g$ in eq.(3)), the screened Coulomb potential,

$$V(r) = -r^{-1} f(x), \quad x = \mu r, \quad (4)$$

where μ^{-1} is radius of screening and atomic units are used, $\hbar = m = e = 1$. These examples embrace a wide class of potentials used in physics, including the short-range Yukawa and Hulthén

potentials ²), the confining potential (3), frequently used in QCD, and potentials with a barrier.

In all the cases considered it turned out that $\alpha = 1$, e.g. $\epsilon^{(k)} \sim k!$. The dependence of the parameter a in eq.(2) on parameters in the problems is also of some interest, and $\nu = n^2\mu$ is the right parameter for the screened potentials (4), with $\mu = g^{1/2}$ and $f(x) = 1 - x^2$ in the case of potential (3). Finally, $\nu = n^2\epsilon^{1/2} \equiv F^{1/2}$ for the Stark problem, where F is the "reduced" electric field ($F = \epsilon/\epsilon_a$, ϵ is an external electric field and $\epsilon_a \sim \bar{r}^{-2} \sim n^{-4}$ is the atomic field in the electron orbit with principal quantum number n).

1/n-expansion is constructed around the classical equilibrium point $x_0(\nu)$ in the effective potential including centrifugal energy. Here we confine ourselves to the $l = n - 1 \gg 1$ states with no radial nodes. The quasiclassical momentum is

$$p(r) = \frac{1}{n} [-\varphi(y, \nu)]^{1/2}, \quad \varphi = y^{-2} - 2y^{-1}f(\nu y) - \epsilon^{(0)}, \quad (5)$$

where $y = n^{-2}r$ and $\epsilon^{(0)}$ is the energy of a classical particle at rest at the equilibrium point, $x_0 = \nu y_0$. The quantities $x_0(\nu)$ and $\epsilon^{(0)}(\nu)$ are determined by the equations [7]

$$\nu = xf - x^2 f', \quad \epsilon^{(0)} = (xf')^2 - f^2 |_{x=x_0} \quad (6)$$

We assume the effective potential to be of the form shown in Fig.1. The width of the highly excited, $n \gg 1$, levels is (within the exponential accuracy)

$$\Gamma_n \approx \text{const.} \exp(-2nQ)n^\sigma, \quad Q(\nu) = \int_{y_0}^{y_1} [\varphi(y, \nu)]^{1/2} dy, \quad (7)$$

where y_0, y_1 are the turning points, see Fig.1, and σ depends on the problem considered. Supposing the analyticity in the variable $\lambda = 1/n$ and using dispersion relations in λ , we obtain

$$\epsilon^{(k)} \approx k! a^k k^{\sigma+1} c_0 [1 + O(1/k)], \quad k \rightarrow \infty \quad (8)$$

$$a = [2Q(\nu)]^{-1}$$

When $y \rightarrow y_0$, then $\varphi(y, \nu) = \omega^2 y_0^{-4} (y - y_0)^2 + \dots$, where ω is dimensionless frequency of vibrations ³ around the equilibrium point $x_0(\nu)$. For sufficiently small values of ν this point is real, as well as all the coefficients $\epsilon^{(k)}$ in (1). With ν increasing the value $\nu = \nu_*$ is achieved, when the collision of two classical orbits occurs, corresponding to the stable (x_0) and unstable equilibrium points in the effective potential. The value ν_* is determined by the first of eqs.(6) with $x = x_*$, while x_* is a root of the equation $f - xf' - x^2 f'' = 0$. It can be shown that at $\nu \rightarrow \nu_*$

$$\omega = C(1 - \nu/\nu_*)^{1/4}, \quad a(\nu) \simeq A(1 - \nu/\nu_*)^{-5/4}, \quad (9)$$

where

$$A = a_1 \{1 + xf''/3f'' |_{x=x_*}\}^{3/4} = \frac{5}{96} C^3, \quad (9a)$$

$$a_1 = 2^{-17/4} \cdot 3^{-1/4} \cdot 5 = 0.19967$$

Note that asymptotics (8) follows from the dispersion relation

$$\epsilon^{(k)} = \frac{1}{\pi} \int_0^\infty \frac{\Gamma_n}{\lambda^{k+3}} d\lambda, \quad \lambda = 1/n$$

and from eq. (7) for the widths of highly excited states with $n \gg 1$. To obtain eq. (9) one should consider the integral (7)

for $Q(\nu)$ in the case when the turning points y_0, y_2 are nearby, and the function $\varphi(y, \nu)$ is considerably simplified. The details will be published elsewhere. Another way to obtain eq. (9) is discussed in Appendix A.

At $\nu > \nu_*$ the coefficients $\epsilon^{(k)}$ and parameter $a(\nu)$ become complex. Evidently, this solution has no direct physical sense in classical mechanics, but proceeding to quantum mechanics it allows one to calculate easily, within $1/n$ -expansion, not only the position E_r , but also the width Γ_n of quasistationary state (see ref.[7-9]).

3. Some examples.

a) We begin with the potential (4), where

$$f(x) = \exp(-x) \cdot x^{\lambda-1}, \quad \lambda > 0 \quad (10)$$

($\lambda = 1$ corresponds to the Yukawa potential, $\lambda = 2$ - to the exponential potential). The equations for x_0 and ω take the following form:

$$[x_0^{\lambda+1} + (2-\lambda)x_0^\lambda]e^{-x_0} = \nu, \quad \omega^2 = \lambda - \frac{x_0(1-\lambda+x_0)}{2-\lambda+x_0},$$

hence

$$x_* = \lambda + \frac{1}{2}[(1+4\lambda)^{1/2} - 1],$$

$$\nu_* = \frac{1}{2\lambda}[1 + (1+4\lambda)^{1/2}]x_*^{\lambda+1}e^{-x_*}, \quad (10a)$$

$$A = \frac{5}{96}[1+4\lambda - (1+4\lambda)^{1/2}]^{3/4},$$

see also eqs. (A.15), (A.16) and Table 1.

b) Another example is

$$f(x) = \exp(-x^\beta/\beta), \quad \beta > 0 \quad (11)$$

In this case

$$(x_0 + x_0^{\beta+1})f(x_0) = \nu, \quad \omega = \left[\frac{1 + \beta x_0^\beta - x_0^{2\beta}}{1 + x_0^\beta} \right]^{1/2},$$

$$x_* = \left[\left(1 + \frac{\beta^2}{4}\right)^{1/2} + \frac{\beta}{2} \right]^{1/2},$$

$$A = \frac{5}{96}[\beta^2 + 4 + (\beta - 2)(\beta^2 + 4)^{1/2}]^{3/4} \quad (11a)$$

Here $\beta = 1$ corresponds to the Yukawa potential, $\beta = 2$ - to the Gaussian screening (No.3 in Table 1).

c) For the generalized funnel potential

$$V(r) = -\frac{1}{r} + \frac{g}{N}r^N, \quad N > -1 \quad (12)$$

we get ($g < 0$)

$$\nu_* = (N+1)(N+2)^{-\frac{N+2}{N+1}}, \quad A = \frac{5}{3} \cdot 2^{-17/4}(N+2)^{3/4} = (13)$$

$$= a_1[(N+2)/3]^{3/4},$$

where a_1 is the coefficient in eq. (9a).

d) Consider the Stark effect in a hydrogen atom for the (n_1, n_2, m) state, where n_1, n_2, m are the parabolic quantum numbers and $n = n_1 + n_2 + |m| + 1$. Using the results of papers [8,9], we obtain at $n \gg 1$:

$$a = \frac{3}{2} F [-\epsilon^{(0)}(F)]^{-3/2} / J(F), \quad (14)$$

where $\epsilon^{(0)}$ is the first term in expansion (1), $F = u^4 \epsilon$,

$$J(F) = \frac{3}{2} \int_{u_1}^{u_2} \frac{du}{u} (A' - B'u + u^2 - u^3)^{1/2}, \quad (14a)$$

$A' = m^2 F^2 [-\epsilon^{(0)}]^{-3}$, $B' = 4\beta_1 F [\epsilon^{(0)}]^{-2}$ and $\beta_2 = \beta_2(F)$ is the separation constant corresponding to the parabolic coordinate $\eta = \tau - z$.

Hence, in the region of weak fields

$$a = \frac{3}{2} F - \frac{9}{4h} (2n_2 + |m| + 1) F^2 \ln F + O(F^2) \quad (15)$$

The $(0, 0, n-1)$ states correspond to circular electron orbits perpendicular to the direction of electric field ϵ . In this case the integral $J(F)$ can be expressed through elementary functions and

$$a = \frac{1}{2} \left[z + \frac{z^3}{3(1-z^2)} - \text{Arth}z \right]^{-1}, \quad (16)$$

where

$$z = (1 - 3\tau)^{1/2} (1 - \tau)^{-1}, \quad \tau(1 - \tau^2)^4 = F, \quad 0 < \tau < 1/3 \quad (17)$$

The collision of two classical solutions occurs at $\tau = 1/3$, or $F = F_c = 2^{12} \cdot 3^{-9} = 0.2081$, where the parameter of the asymptotics behaves similarly to eq.(9):

$$a(F) = A f^{-5/4} (1 + b f^{1/2} + b_1 f + \dots), \quad (18)$$

$A = 2^{-3/4} \cdot 3^{-3/2} \cdot 5 = 0.5722$ and $f = 1 - F/F_c \rightarrow 0$ (for calculation details see Appendix B).

e) The formulae for the funnel potential (3) can be obtained from the preceding ones when substituting $g \rightarrow -g$,

$$\nu = x_0 + x_0^3, \quad z = \omega = [(1 + 3x_0^2)/(1 + x_0^2)]^{1/2} \quad (19)$$

So, $z > 1$ and $\text{Arth}z = \text{Arth}z^{-1} \pm \frac{1}{2}\pi i$. Therefore, the asymptotical parameter a becomes complex, which corresponds to oscillations of the coefficients $\epsilon^{(k)}$ with $k \rightarrow \infty$. In particular, at $g \rightarrow \infty$ we have: $z \rightarrow 3^{1/2}$,

$$a(\infty) = \frac{1}{2} [3^{1/2} - \ln(2 + 3^{1/2}) + i\pi]^{-1} \quad (20)$$

and $|a(\infty)| = 0.1578$ (compare with Figs.2 and 3).

The parameters a for the Stark effect, spherical model (12) with $N = 1$ and the funnel potential (3) are shown in Fig.2 - curves 1,2 and 3, correspondingly. Note that $a(\nu) \rightarrow \infty$, when $\nu \rightarrow \nu_c$. Thus, the coefficients $\epsilon^{(k)}(\nu)$ sharply grow, and $1/n$ -expansion itself is no longer applicable in this region. It was observed already in the first attempts to sum series (1) for $\nu \approx \nu_c$ [7], and the underlying reason becomes clear from Fig.2. However, at $\nu > \nu_c$, the parameter $a(\nu)$ decreases with ν increasing, and applicability of $1/n$ -expansion is restored. In this region the coefficients $\epsilon^{(k)}$ in eq.(1) are complex, thus the first few terms of expansion (1) determine the width of a quasistationary state with a reasonably high accuracy.

Similar results were obtained also for the Yukawa and Hulthén potentials [18], see Fig.4 and 5.

4. $1/n$ -expansion and the problem of two centres.

The nonrelativistic problem of two Coulomb centres,

$$V(\vec{r}) = -\frac{Z_1}{r_1} - \frac{Z_2}{r_2},$$

$$r_{1,2} = [\rho^2 + (z \pm R/2)^2]^{1/2}, \quad \rho = (x^2 + y^2)^{1/2}, \quad (21)$$

is encountered frequently in different branches of physics, including the theory of molecules, μ -catalysis, etc. In this case the coefficients $\epsilon^{(k)}$ depend on the internuclear distance R , where the first term $\epsilon^{(0)}$ corresponds to the electron energy on the classical orbit which is determined by the equilibrium condition of the forces acting on the electron in its rest frame. Here we confine ourselves to the case $Z_1 = Z_2 = 1$ (the molecular ion H_2^+). For the states with $m = n - 1, n \rightarrow \infty$ (or, equivalently, for $n = 1$ and $D \rightarrow \infty$, where D is space dimensionality) the equations can be written in a parametric form,

$$\begin{aligned} \epsilon^{(0)} &= -2(1-\tau)^2(1+\tau), \\ \epsilon^{(1)} &= 2(1-\tau)^3\{(2n_1+1)[1+(1+3\tau)^{1/2}] - \\ &\quad -(2n_2+1)[1-(1-3\tau)^{1/2}]\}, \\ \bar{R} &\equiv n^{-2}R = \tau^{1/2}(1-\tau)^{-2}, \end{aligned} \quad (22)$$

where $0 < \tau < 1/3, \bar{R} < R_* = 3^{3/2} \cdot 2^{-2} = 1.299$ and $E = n^{-2}\epsilon$ is the electronic term energy, while $\tau = \cos^2\alpha$ and α is the angle at Z vertex in a triangle (Z, Z, e) . The numerical analysis

shows [10,19,20] that $\epsilon^{(k)}$ grow as factorials at $k \rightarrow \infty$, while the parameter $a = a(\bar{R})$ increases at $\bar{R} \rightarrow R_*$, see Fig.6 taken from ref.[19]. Here we present a few analytical results.

If $0 < \bar{R} < R_*$, then (ref.[21] and Appendix C):

$$a(\bar{R}) = -\frac{1}{2}(\text{Arth } \zeta - \zeta)^{-1}, \quad \zeta = (1-3\tau)^{1/2}(1-\tau)^{-1}, \quad (23)$$

and $\tau = \tau(\bar{R})$ is determined in the preceding equations. So, the singularity of the Borel transform [20] closest to the origin is at $\delta_0 = 1/2a < 0$. The series (1) is alternating in sign and can be summed up with the help of Padé approximants. In a recent paper [20] the dependence of δ_0 on \bar{R} was established numerically with a high accuracy. The values of the Borel parameter δ_0 , given in [20], are in a very good agreement with the analytical formula (23), as can be seen from Table 2.

When $\bar{R} = R_*$, the three classical orbits (stable and unstable) coincide, so the rearrangement of $1/n$ -expansion occurs at this point. If $\bar{R} > R_*$, we have

$$a(\bar{R}) = \frac{1}{4}[\zeta(1-\zeta^2)^{-1} - \text{Arth } \zeta]^{-1} \quad (24)$$

and

$$\zeta = \left(\frac{3\tau-1}{3\tau-\tau^2}\right)^{1/2}, \quad \bar{R} = 8\tau^{3/2}(1-\tau)^{-1}(1+\tau)^{-2}, \quad 1/3 < \tau < 1.$$

In this case $a = 1/2\delta_0 > 0$, so the terms of the $1/n$ -expansions are of the same sign. The derivation of eqs.(23), (24) follows the same lines as in eq.(16), and is given in Appendix C. It is notable that the singularity of $a(\bar{R})$ differs from eq.(9) and is no longer symmetrical in this case:

$$a(\bar{R}) \simeq A_{\pm} |1 - \bar{R}/R_*|^{-3/2}, \quad \bar{R} \rightarrow R_*, \quad (25)$$

where $A_+ = 3^{-1/2}$ at $\bar{R} > R_*$ and $A_- = -(2/3)^{1/2}$ at $\bar{R} < R_*$.

5. Thus, large orders of $1/n$ -expansion increase as factorials⁴⁾. This explains why in many quantum mechanical problems⁵⁾ it is necessary to calculate $\sim 30 \div 50$ coefficients $\epsilon^{(k)}$ and to use one of the summation methods to obtain the energy ϵ_{nl} with the accuracy required for experiments. At present the summation of divergent series occurring in quantum mechanics is developed fairly well and, in principle, presents no insuperable difficulties.

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Appendix A

Higher Orders of the $1/n$ -expansion

Let the potential be

$$V(r) = -\frac{1}{r} f(x) \equiv -\mu^{-1} v(x), \quad (A.1)$$

where $\hbar = m = 1$, $x = \mu r$, $v(x) = f(x)/x$, $\mu^{-1} \equiv R$ is the screening radius, energy $E_{nl} = \epsilon/2n$, $n = p + l + 1$ is principal and p -radial quantum numbers. The Schrödinger equation takes the form

$$\frac{1}{n} \frac{d^2 \chi}{d\xi^2} + \left(\frac{x_0}{\nu}\right)^2 \{\epsilon + 2\nu v(x_0(1 + \xi n^{-1/2}))\} - \left(1 - \frac{2p+1}{n} + \frac{p(p+1)}{n^2}\right) (1 + \xi n^{-1/2})^{-2} \chi = 0, \quad (A.2)$$

which is convenient when $n \rightarrow \infty$ and $p = 0, 1, 2, \dots$ is fixed. Here $\nu = n^2 \mu$, $x = x_0(1 + \xi n^{-1/2})$ and $x_0 = x_0(\nu)$ is the classical equilibrium point determined by the first of eqs. (6). The variable ξ , unlike r , is bounded as $n \rightarrow \infty$. Expanding the quantities in (A.2) near $x = x_0$, one obtains:

$$x_0^3 v'(x_0) = -\nu, \\ \epsilon^{(0)} = x^3 v'(xv' + 2v) |_{x=x_0} \equiv (xf')^2 - f^2 |_{x=x_0} \quad (A.3)$$

and the equation for an anharmonic oscillator,

$$\frac{d^2 \chi}{d\xi^2} + \left\{ \left(\frac{x_0}{\nu}\right)^2 [\epsilon^{(1)} + \frac{\epsilon^{(2)}}{n} + \dots] + (2p+1) - \omega^2 \xi^2 + \dots \right\} \chi = 0$$

$$\begin{aligned}
& +n^{-1/2}[\sigma\xi^3 - 2(2p+1)\xi] - \frac{1}{n}[2(1 + \frac{\tau}{3})\xi^4 - \\
& -3(2p+1)\xi^2 + p(p+1)] + O(n^{-3/2})\chi = 0. \quad (\text{A.4})
\end{aligned}$$

The notations are [10]

$$\omega = [3(1 + v_2)]^{1/2}, \quad \sigma = 4(1 - v_3), \quad \tau = \frac{15}{2}(v_4 + \frac{3}{5}),$$

$$v_k = \frac{2}{(k+1)!} x^{k-1} \frac{d^k v}{dx^k} \Big|_{x=x_0(\nu)} \quad (\text{A.5})$$

Eqs. (A.3) define the x_0 dependence on ν and the initial term of $1/n$ -expansion for the "reduced" energy ϵ . Eq. (A.4) with $n \rightarrow \infty$ gives

$$\epsilon^{(1)} = (2p+1)(\omega - 1)(\nu/x_0)^2, \quad (\text{A.6})$$

and $d^2\chi/d\xi^2 + 2(\bar{E} - U)\chi = 0$, where

$$\bar{E} = (p+1/2)\omega + \frac{1}{2}(\frac{x_0}{\nu})^2(\frac{\epsilon^{(2)}}{n} + \frac{\epsilon^{(3)}}{n^2} + \dots),$$

$$\begin{aligned}
2U(\xi) = & \omega^2\xi^2 - n^{-1/2}[\sigma\xi^3 - 2(2p+1)\xi] + \\
& + \sum_{s=2}^{\infty} (-1)^s n^{-s/2} [(s+3)(1-v_4)\xi^{s+2} -
\end{aligned}$$

$$-(2p+1) \cdot (s+1)\xi^s + p(p+1)(s-1)\xi^{s-2}] \quad (\text{A.7})$$

The coefficients of anharmonic terms are small at $n \rightarrow \infty$, which provides the applicability of the perturbation theory (PT). If the perturbed hamiltonian is

$$H = 1/2(p^2 + \omega^2 x^2) + g(\alpha x^3 + \alpha_1 x) + g^2(\beta x^4 + \beta_1 x^2 + \beta_2),$$

$$g \rightarrow 0, \quad (\text{A.8})$$

then

$$\begin{aligned}
E_p = & (p+1/2)\omega - g^2 \left\{ \frac{1}{8}[30p(p+1) + 11] \frac{\alpha^2}{\omega^4} + \frac{\alpha_1^2}{2\omega^2} + \right. \\
& + \frac{3(2p+1)\alpha\alpha_1}{2\omega^3} - \frac{3\beta}{4\omega^2}[2p(p+1) + 1] - \\
& \left. - \frac{2p+1}{2\omega}\beta_1 - \beta_2 \right\} + O(g^4). \quad (\text{A.9})
\end{aligned}$$

Here E_p is the energy of the p^{th} oscillator state. The simplest case $\alpha_1 = \beta_1 = \beta_2 = 0$ is considered in ref.[22], p.132. The general case is easily obtained with the help of the shift. $x \rightarrow x - c$, and the change of oscillator frequency, $\omega \rightarrow \omega + \Delta\omega$, where

$$c = \frac{g\alpha_1}{\omega^2} + O(g^3), \quad \Delta\omega = -g^2 \left(\frac{3\alpha\alpha_1}{2\omega^2} - \frac{\beta_1}{\omega} \right) + \dots$$

The equation [10] for $\epsilon^{(2)}$ follows immediately from the above equations (here $g = n^{-1/2}$). The next coefficients $\epsilon^{(3)}$ can be obtained in a similar way, though the calculations become more cumbersome. For this purpose one can use recurrence relations which are very convenient for a computer.

Now let us discuss eqs. (8), (9). The asymptotics of the higher orders of PT for the anharmonic oscillator is well known

[12,13]. For instance, for the energy of the ground state E_0 we have

$$E_0(g) \underset{g \rightarrow 0}{\sim} \frac{1}{2}\omega + \sum_{k=1}^{\infty} C_k g^k, \quad (\text{A.10})$$

where $V(x) = 1/2\omega^2 x^2 + gx^\nu$ and

$$C_k \sim \text{const.} (k\alpha)! a^k k^\beta, \quad k \rightarrow \infty,$$

$$\alpha = \frac{1}{2}(\nu - 2), \quad \beta = -\frac{1}{2},$$

$$a = -\frac{1}{2}\omega^{-\frac{\nu-2}{2}} \left[\Gamma\left(\frac{2\nu}{\nu-2}\right) / \Gamma^2\left(\frac{\nu}{\nu-2}\right) \right]. \quad (\text{A.11})$$

The term $\alpha n^{-s/2} \xi^{s+2}$ in (A.7) leads to contribution $\sim n^{-(k-1)}$ in the energy when it is taken in the q^k order PT, $q = 2(k-1)/s$. The corresponding contribution to asymptotics of the coefficient $\varepsilon^{(k)}$ is $\sim (k-1)! \omega^{-(1+4/s)k}$, see eq. (A.11). If ω is finite, all the contributions ($s = 1, 2, \dots$) are of the same order of magnitude and are essential. However, if frequency $\omega \rightarrow 0$ (or $\nu \rightarrow \nu_*$, see eq.(9) and Fig.7), the contribution from $s = 1$, i.e., from the cubic term in (A.7), dominates. The cubic anharmonic oscillator was considered in detail by Alvarez [13]. Using his results, we obtain that

$$\varepsilon^{(k)} \sim k! a^k k^{-3/2} \text{ as } k \rightarrow \infty, \quad (\text{A.12})$$

$$a = 15\sigma_4^2/8\omega^3 = A \left(\frac{\nu_* - \nu}{\nu_*} \right)^{-3/4}, \quad \nu \rightarrow \nu_*, \quad (\text{A.13})$$

where A is the coefficient defined in eq.(9a), and $\sigma_* = \sigma(\nu_*)$. This result is valid for an arbitrary potential $V(r)$ and shows that $a = a(\nu)$ has the power singularity at $\nu = \nu_*$. It explains the lack of convergence of $1/n$ -expansion for $\nu \sim \nu_*$. [7-9].

In the case of the potential (10) we have

$$C = [1 + 4\lambda - (1 + 4\lambda)^{1/2}]^{1/4}, \quad \sigma_* = \frac{1}{6} C^4,$$

$$A = \frac{5}{96} [1 + 4\lambda - (1 + 4\lambda)^{1/2}]^{3/4}. \quad (\text{A.14})$$

In particular, for the Yukawa ($\lambda = 1$) potential $\omega^2 = 1 - x_0^2/(1 + x_0)$,

$$x_* = \frac{1}{2}(1 + \sqrt{5}), \quad \nu_* = x_*^2 e^{-x_*} = 0.839\ 962\dots,$$

$$A = \frac{5}{96} (5 - \sqrt{5})^{3/4} = 0.111\ 646\dots \quad (\text{A.15})$$

and for the exponential potential: $\lambda = 2, \omega = (3 - x_0)^{1/2}$,

$$x_* = 3, \quad \nu_* = 27e^{-3} = 1.34425\dots,$$

$$A = 2^{-17/4} \cdot 3^{-1/4} \cdot 5 = 0.19967\dots \quad (\text{A.16})$$

(A.15) is in accordance with our numerical calculations of the parameter $a(\nu)$ near $\nu = \nu_*$ (Fig.3).

Finally, for the Hulthén potential:

$$f(x) = x/(e^x - 1),$$

$$x_0^2 e^{2x_0} (e^{2x_0} - 1)^{-2} = \nu, \quad \omega = (3 - x_0 \coth \frac{x_0}{2})^{1/2} \quad (\text{A.17})$$

The value x_0 is determined from the equation: $e^{-x} = (1 - \frac{x}{3}) / (1 + \frac{x}{3})$, whence

$$\nu_* = 1.5231, \quad C = (x_*^2 - 3)^{1/4} = 1.3807,$$

$$A = 0.13709 \quad (\text{A.18})$$

The numerical values of ν_{cr} , ν_* , C and A (i.e. the coefficients in eq.(9)) are given in Table 1. Here ν_{cr} corresponds to $\epsilon^{(0)} = 0$, i.e. to the point at which a highly excited level with $l = n - 1 \gg$ crosses the boundary $E = 0$ and escapes to the continuous spectrum. The values of ν_{cr} for the potentials from sect.3 are:

$$\nu_{cr} = \begin{cases} 2 \cdot (\lambda/e)^4, & \text{for (10)} \\ 2 \cdot \exp(-\beta^{-1}), & \text{for (11)} \end{cases}$$

($e = 2.718\dots$). So, $\nu_{cr} < \nu_*$ for all the cases considered.

Appendix B

Calculating the integral in eq.(14a), we obtain

$$a = c/\phi(z), \quad (\text{B.1})$$

$$\phi(z) = z + \frac{z^3}{3(1-z^2)} - \text{Arth}z =$$

$$= \frac{2}{15}z^5(1 + \frac{10}{7}z^2 + \frac{5}{3}z^4 + \dots), \quad z \rightarrow 0, \quad (\text{B.2})$$

where $c = 1/2$ for the Stark effect in a hydrogen, while the dependence of z and energy $\epsilon^{(0)}$ on the reduced electric field, $F \equiv \nu^2 = n^4 \epsilon$, is determined parametrically, $\epsilon^{(0)} = -(1 + 3\tau^2)(1 - \tau^2)^2$ and eqs.(17). For the spherical model (see eq.(12) with $N = 1$ and $g < 0$) we get: $c = 1/4$,

$$z = [(1 - 3\tau)/(1 - \tau)]^{1/2}, \quad \epsilon^{(0)} = -(1 + 3\tau)(1 - \tau), \quad (\text{B.3})$$

$$\tau(1 - \tau)^2 = F$$

Note that in both cases ⁶⁾ the root $\tau = \tau(F) \rightarrow 0$ at $F \rightarrow 0$ should be chosen, and $\tau = 1/3$ corresponds to the collision of two classical solutions (stable and unstable equilibrium points). It occurs at $F = F_*$, where $F_* = 2^{12} \cdot 3^{-9} = 0.2061$ for the Stark problem [8], $F_* = 2^2 \cdot 3^{-3} = 0.1481$ is the case of its spherical model. At $F \rightarrow F_*$ we put $\tau = \frac{1}{3}(1 - t)$ and obtain from eq.(B.3) that

$$t^2 + \frac{1}{3}t^3 = \frac{4}{3}f, \quad z = \left[\frac{3}{2}t/(1 + \frac{1}{2}t)\right]^{1/2}, \quad (\text{B.4})$$

$f = 1 - F/F \rightarrow 0$, while eqs.(17) yield

$$t^2 + \frac{1}{6}t^3 + \dots + 2^{-12}t^9 = \frac{8}{9}f, \quad z = \frac{3}{2}t^{1/2}(1 + \frac{1}{2}t)^{-1} \quad (\text{B.5})$$

From eqs.(B.1)-(B.4) we obtain

$$z = (3f)^{1/4}[1 - 2 \cdot 3^{-3/2}f^{1/2} + 0(f)]$$

Hence, eq.(18) follows, where

$$A = 5/8 \cdot 3^{1/4} = 0.4749,$$

$$b = -20/7 \cdot 3^{5/2} = -0.5499, \quad b_1 = -0.6699 \quad (\text{B.6})$$

(in the case of the spherical model). Analogous calculations for the Stark effect give

$$A = 5/2^{3/4} \cdot 3^{3/2} = 0.5722, \quad b = -0.4770 \quad (\text{B.7})$$

These values ⁷⁾ are in agreement with the curves in Fig.2.

The parameter a of the asymptotics (8) in the weak field region can be calculated, when substituting expansions of $\epsilon^{(0)}$ and $J(F)$ at $F \rightarrow 0$ into eqs.(14). In such a way, for an arbitrary (n_1, n_2, m) state in a hydrogen atom, we obtain

$$a(F) = \frac{3}{2}F(1 - \frac{3}{2}\rho F \ln F + kF + \dots), \quad (\text{B.8})$$

where $k = \frac{3}{2}\{3 + (3\ln 2 - 2)\rho - \frac{1}{2}[(\rho + \mu) \ln(\rho + \mu) + (\rho - \mu) \ln(\rho - \mu)]\}$, $\rho = (2n_2 + |m| + 1)/n = 1 - (n_1 - n_2)/n$ and $\mu = m/n$. In particular, $\rho = 1$ and $k = 3.579$ for the $(0, 0, n - 1)$ states with $n \gg 1$, $\rho = 1$ and $k = 4.619$ for the ground state, $n = 1$.

Appendix C

Here we derive the formulae of sect.4.

Proceeding to the elliptic coordinates ξ, η, φ [22]

$$\xi = (r_1 + r_2)/R, \quad \eta = (r_1 - r_2)/R,$$

$$1 \leq \xi < \infty, \quad -1 \leq \eta \leq 1, \quad 0 \leq \varphi \leq 2\pi,$$

and fulfilling the scaling transformation

$$\vec{x}' = \lambda \vec{x}, \quad \vec{R}' = \lambda \vec{R}, \quad E = \lambda \epsilon, \quad \lambda = \frac{1}{m^2 - 1} \approx n^{-2},$$

$$\psi = [(\xi^2 - 1)(1 - \eta^2)]^{1/2} \chi_1(\xi) \chi_2(\eta) \exp(i m \varphi), \quad (\text{C.1})$$

we arrive at the one-dimensional Schrödinger equations for the functions χ_i ,

$$\lambda^2 \frac{d^2 \chi_1}{d\xi^2} + 2(\epsilon - U(\xi)) \chi_1 = 0, \quad (\text{C.2})$$

(λ is analogous to the Planck constant \hbar , $\lambda \rightarrow 0$), where the effective energy ϵ and effective potentials are

$$\epsilon = \frac{1}{4} \epsilon \vec{R}'^2,$$

$$U(\xi) = \frac{1}{2(\xi^2 - 1)^2} - \vec{R}' \frac{Z_+ \xi + \beta}{\xi^2 - 1}, \quad (\text{C.3})$$

$$V(\eta) = \frac{1}{2(1 - \eta^2)^2} + \vec{R}' \frac{Z_- \eta - \beta}{1 - \eta^2},$$

$Z_{\pm} = \frac{1}{2}(Z_1 \pm Z_2)$, and β is the separation constant. Further we put $Z_+ = 1, Z_- = 0$, which corresponds to H_2^+ ion.

1) $\bar{R} > R_*$ = 1.299 038 ...

The potential $V(\eta)$ is shown at Fig.5, i.e. it reduces to the case of double-well potential [23], which has symmetric and antisymmetric states with different energies, but identical $1/n$ -expansions. In this case the Borel sum of series (1) has imaginary part $\Gamma/2$, which corresponds to the tunnelling from one vacuum to another [24,25] (calculation of large orders of perturbation theory is considered in [23]).

When $n \rightarrow \infty$ and $n_{\xi} = n_{\eta} = 0$, the four equations follow

$$U(\xi_0) = V(\eta_0) = e,$$

$$U'(\xi_0) = V'(\eta_0) = 0, \quad (\text{C.4})$$

the solution of which is

$$\bar{R} = \frac{8\xi_0^3}{(\xi_0^2 - 1)(\xi_0^2 + 1)^2}, \quad \epsilon_0 = -\frac{[(3\xi_0^2 - 1)(\xi_0^2 + 1)]^2}{32\xi_0^6},$$

$$\eta_0 = \left(\frac{3\xi_0^2 - \xi_0^4}{3\xi_0^2 - 1}\right)^{1/2} = [1 - (-2e)^{-1/2}]^{1/2}, \quad (\text{C.5})$$

$$e = -\frac{1}{2}\left(\frac{3\xi_0^2 - 1}{\xi_0^4 - 1}\right)^2, \quad \beta\bar{R} = \sqrt{-2e}$$

(ξ_0, η_0 are elliptic coordinates of the classical orbit, $1 < \xi_0 \leq 3^{1/2}, -\infty < e \leq -1/2$). Since $V(\eta) - V(\eta_0) = -e[(\eta^2 - \eta_0^2)/(1 - \eta^2)]^2$, then

$$a^{-1} = 2 \int_{-\eta_0}^{\eta_0} |p_{\eta}| d\eta = 2^{5/2} \int_0^{\eta_0} [V(\eta_0) - V(\eta)]^{1/2} d\eta =$$

$$= 4 \left(\frac{\eta_0}{1 - \eta_0^2} - \text{Arth } \eta_0 \right) \quad (\text{C.6})$$

(instanton contribution). We put $\tau = \xi_0^{-2} \geq 1/3$ and arrive at eqs.(24), where $\zeta \equiv \eta_0$. In particular, at $\tau = 1/3$

$$\xi_0 = 3^{1/2}, \quad \eta_0 = 0, \quad \bar{R} = R_* = 3^{3/2}/4, \quad e = -1/2 \quad (\text{C.7})$$

and $\epsilon_0 = -32/27$. Note that at $\eta \rightarrow 0$

$$V(\eta) - V(0) = \begin{cases} -(\sqrt{-2e} - 1)\eta^2 + \dots, & e < -1/2 \\ \frac{1}{2}\eta^4 + \dots, & e = -1/2 \end{cases} \quad (\text{C.8})$$

So, the effective potential changes its shape at $\bar{R} = R_*$.

2) $0 < \bar{R} < R_*$. In this case $-1/2 < e < 0$,

$$V(\eta) - V(0) = \frac{\eta^2}{(1 - \eta^2)^2} \left[\frac{1}{2} + e(1 - \eta^2) \right] =$$

$$= \frac{e\eta^2(\eta_1^2 - \eta^2)}{(1 - \eta^2)^2} = (e + 1/2)\eta^2 + (e + 1)\eta^4 + \dots$$

So, the equilibrium point is $\eta_0 = 0$ and the turning points are complex, $\eta_1^2 = 1 + (2e)^{-1} < 0$, where $e = \frac{1}{4}\epsilon_0\bar{R}^2$, as before. Therefore,

$$a^{-1} = 2^{3/2} \int_0^{\eta_1} \frac{\eta(\eta_1^2 - \eta^2)^{1/2}}{1 - \eta^2} d\eta = -2(\text{Arth } \bar{\zeta} - \zeta), \quad (\text{C.9})$$

where $\zeta = (2e+1)^{1/2} \equiv (1-3\tau)^{1/2}(1-\tau)^{-1}$, see eqs.(23). Contrary to the previous case, the equilibrium orbit is symmetrical ($\tau_1 = \tau_2, \eta_0 = 0$), so

$$\tau \equiv \zeta_0^{-2} = \cos^2 \alpha, \quad R = 2r_1 \cos \alpha,$$

and $3^{1/2} \leq \zeta_0 < \infty, 1/3 \geq \tau > 0$.

3) Compare our variables with that of ref.[20], where large-order dimensional perturbation theory for H_2^+ is considered. Note that

$$l = \frac{D-3}{2}, \quad n = l+1 = \frac{D-1}{2} \rightarrow \infty$$

($n_r = 0$), the dimensional scaling factor in [20] is

$$f = \frac{1}{6}D(D-1) \approx \frac{2}{3}n^2, \quad \tilde{R} = R/f \approx \frac{3}{2}\bar{R} \quad (\text{C.10})$$

and the expansion parameter $\delta \equiv 1/D \approx 1/2n$. So, $\epsilon^{(k)} \approx 2^{-(k+2)}E_k^r$ at $k \rightarrow \infty$ (see eq.(3b) in [20]). Let the Borel transform $F(\delta)$,

$$\sum_{k=0}^{\infty} E_k^r \delta^k = \int_0^{\infty} e^{-t} F(\delta t) dt,$$

has the singularity closest to the origin of the type $F(\delta) \propto (\delta_0 - \delta)^{-\rho}$. Then, the coefficients $F_k = E_k^r/k! \propto \delta_0^{-k} k^{\rho-1}$ and $\epsilon^{(k)} \propto \text{const} \cdot k!(2\delta_0)^{-k} k^{\rho-1}, k \rightarrow \infty$. Comparison with eq.(8) gives

$$a = 1/2\delta_0, \quad \sigma = \rho - 2 \quad (\text{C.11})$$

It was shown numerically [20] that $\delta_0 < 0$ and $\rho = -1/2$, if $\tilde{R} < R$, which corresponds to a square-root branch point of the Borel transform at $\delta = \delta_0$. The agreement between calculation results of ref.[20] and our formula (23) is excellent, see Table 1.

Footnotes

1) See ref. [18]. The numerical methods we have used will be described in detail elsewhere.

2) When $f(x) = e^{-x}$ and $x(e^x - 1)^{-1}$ in eq.(4), we obtain the Yukawa and Hulthen potentials, frequently used in nuclear physics; $f(x) = x \exp(-x^2)$ corresponds to the Gaussian potential, $f(x) = 1 - x^2$ to the funnel potential (3), etc. In fact, an arbitrary central potential $V(r)$ can be written in the form of eq.(4), if the condition $0 < f(0) < \infty$ is ignored.

3) Note that

$$\omega^2 = 1 - \frac{x^2 f''}{f - x f'} \equiv x \frac{d \ln \nu}{dx},$$

where $x = x_0(\nu)$, eq. (6), and $\nu = x^3 f''(x)$.

4) Large orders of the usual perturbation theory E_k (in powers of the coupling g) may increase as $(k\alpha)!$ with an arbitrary value of α . For example [12],

$$E_k \propto (k\alpha)! a^k k^\beta, \quad \alpha = \frac{1}{2}(N-2)$$

for $D=1$ anharmonic oscillator, $V(x) = \frac{1}{2}x^2 + gx^N$.

On the other hand, in the case of $1/n$ -expansion $\alpha = 1$ and $\epsilon^{(k)} \propto k!$ for all the problems considered.

5) For example, when calculating complex energies of quasi-stationary states (n_1, n_2, m) for the Stark effect in hydrogen [8,9].

6) The dependence of the variable z on F/F_* is shown in Fig.8 and is quite similar in both cases.

7) The analytical value of the coefficient b for the Stark problem is $b = -85/63 \cdot 2^{3/2} \approx -0.47702$.

A few additional remarks.

1) After this work was completed and submitted for publication, we became acquainted with the paper by López-Cabrera, Goodson, Herschbach and Morgan [20], in which large-order dimensional perturbation theory for H_2^+ is developed. Using the advanced numerical technique, including the Borel transformation and Padé-Borel summation, the authors showed that the coefficients $\epsilon^{(k)}$ grow as $k!$, $k \rightarrow \infty$, and determined the dependence of the Borel parameter $\tilde{\delta}_0 = I/2a$ on the scaled distance \tilde{R} ($\tilde{\delta}_0$ is the position of the singularity in the Borel function $F(\tilde{\delta})$ closest to the origin). However, the analytical formulae (22)-(24) are not given in ref. [20]. The numerical values of $\tilde{\delta}_0$ from [20] are in an excellent agreement with our eq.(23), see Table 2.

Note that $\sigma' + I = -1.5$ in eq.(8) corresponds to the square-root singularity of the Borel transform, $F(\tilde{\delta}) \sim (\tilde{\delta}_0 - \tilde{\delta})^{1/2}$ (if $\tilde{R} < R_*$, $\tilde{\delta}_0 < 0$). In this case, the quadratic Padé analysis is very appropriate [20] and gives the correct values of $\tilde{\delta}_0$ and $a(\tilde{R})$ with 5-8 significant figures, as is seen from Table 2.

2) Let us give some expansions in the two-centre Coulomb problem

At $\tilde{R} \rightarrow R_*$

$$\epsilon^{(0)} = \epsilon_* \cdot \begin{cases} 1 - \frac{1}{2}h + \frac{7}{16}h^2 - \frac{113}{288}h^3 + \dots, & h \rightarrow +0 \\ 1 - \frac{1}{2}h + \frac{1}{4}h^2 - \frac{1}{9}h^3 + \dots, & h \rightarrow -0 \end{cases} \quad (I')$$

where $\epsilon_* = -32/27$ and $h = (\tilde{R} - R_*)/R_*$. The singularity of $a(\tilde{R})$ is not symmetrical, which differs from the case of a spherically symmetric potential, eq.(9):

$$a(\tilde{R}) = A_{\pm} |h|^{-3/2} \begin{cases} 1 + \frac{3}{20} h + \dots, & h \rightarrow +0 \\ 1 + \frac{13}{20} h + \dots, & h \rightarrow -0 \end{cases} \quad (2')$$

where $A_+ = 3^{-1/2}$ and $A_- = -(2/3)^{1/2}$. From eq.(23) it follows that

$$a = -\frac{1}{2} [1 - (1 - \ln 2)]^{-1} + O(\Lambda^{-3}), \quad (3')$$

where $\Lambda = \ln(1/\tilde{R})$ and $R \rightarrow 0$.

At $R \rightarrow \infty$ we put $z = 1 - 2t$, $t \rightarrow 0$ in eq.(24), then

$$\frac{t(1-t)^2}{(1-2t)^{3/2}} = z^3 + z = \rho, \quad \epsilon^{(0)} = -\frac{(1-t^2)^2}{2(1-2t)}$$

with $z = t(1-2t)^{-1/2}$, $\rho = \frac{1}{R} = \frac{h^2}{R}$. If $z = \rho y$, then

$$y = 1 - \rho^2 y^3,$$

$$\epsilon^{(0)} = -\frac{1}{2} (1 + 2\rho y^{-1/2} - 2\rho^3 y^{3/2} + \rho^4 y^3 + \rho^6 y^6) \quad (4')$$

Using the Lagrange's formula from the theory of analytic functions, we finally obtain *):

*) Note that the coefficients c_k can be calculated for arbitrary k . For instance,

$$c_{2k} = (-1)^{k+1} \frac{(3k-3)!}{k!(2k-1)!}, \quad k \geq 2.$$

$$y_0' = 1 + \nu \sum_{k=1}^{\infty} \frac{\Gamma(3k+\nu)}{k! \Gamma(2k+\nu+1)} (-\rho^2)^k =$$

$$= 1 - \nu \rho^2 + \frac{1}{2} \nu(\nu+5) \rho^4 - \dots,$$

$$\epsilon^{(0)}(R) = \sum_{k=0}^{\infty} c_k \rho^k = -\frac{1}{2} - \rho + \frac{1}{2} \rho^3 -$$

$$-\frac{1}{2} \rho^4 - \frac{3}{8} \rho^5 + \rho^6 + \frac{143}{16} \rho^7 - 3\rho^8 + \dots,$$

$$a = \frac{1}{2} \rho [1 + \rho \ln \frac{2}{\rho} + O((\rho \ln \rho)^2)] \quad (6')$$

(here y_0 is the root of eq.(4'), which $\rightarrow 1$ at $R \rightarrow \infty$).

On the other hand, the expansion of the electronic energy for the two Coulomb centers, $Z_1 = Z_2 = 1$, is well known:

$$\epsilon(R) = -\frac{1}{2} - \rho + \frac{1}{2} (1 - n^{-2}) \rho^3 -$$

$$- (1 + n^{-1}) \left[\frac{1}{2} \left(1 + \frac{5}{4n}\right) \rho^4 + \frac{3}{8} \left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{n^2}\right) \rho^5 - \right.$$

$$\left. - \left(1 + \frac{14}{n} - \frac{5}{n^2} - \frac{28}{n^3}\right) \rho^6 + O(\rho^7) \right], \quad \rho \rightarrow 0 \quad (7')$$

(see ref.[26] and eq.(4.59) in [27] with $n_1 = n_2 = 0$, $m = n - 1$). At $n \rightarrow \infty$ the six first terms of the expansions (5') and (7') coincide.

3) For the potential

$$V(z) = -\frac{1}{2} - \frac{g}{2} z^2 \quad (8')$$

we have [21]

$$a = \frac{1}{2} \left[\frac{3^{1/2} (1 - \frac{1}{2} z^2)}{(1 - z^2)^{1/2}} \operatorname{arctg} \frac{3^{1/2} z}{2(1 - z^2)^{1/2}} - \operatorname{Arth} \frac{3z}{z^2 + 2} \right]^{-1} \quad (9')$$

where

$$z = \left(\frac{1 - 4x_0^2}{1 - x_0^2} \right)^{1/2} = (4 - 3y_0)^{1/2}, \quad x_0 = \nu y_0, \\ \nu = h^2 g^{1/3} \text{ and } y_0 - \nu^3 y_0^4 = 1.$$

Here $x_* = 2^{-2/3}$, $\nu_* = 3 \cdot 2^{-2/3} = 0.4727$ and $z \rightarrow 0$,
when $\nu \rightarrow \nu_*$. Since $a = \frac{10}{3} z^{-5} + O(z^{-3})$ at
 $z \rightarrow 0$, we obtain from eq.(9') that

$$a(\nu \rightarrow \nu_*) \approx A \left(1 - \frac{\nu}{\nu_*}\right)^{-5/4}, \quad A = 5/3 \cdot 2^{11/4} = 0.2478, \quad (10')$$

which is in accordance with the general eq. (12) in ref. [21].

4) Analogously, for the Zeeman effect in hydrogen we have [28]

$$z_0 + \frac{1}{4} B^2 z_0^4 = 1, \quad z_0 = \begin{cases} 1 - \frac{1}{4} B^2 + \dots, & B \rightarrow 0 \\ (B/2)^{1/2}, & B \rightarrow \infty \end{cases}$$

where z_0 is the radius of the classical electron orbit. Let
 $\nu = (B/2)^{2/3}$, $x_0 = \nu z_0$, then $x_0 + x_0^4 = \nu$. It can be
shown that

$$a^{-1} = \frac{3^{1/2} (z^2 - 2)}{(z^2 - 1)^{1/2}} \left(\operatorname{Arth} \frac{2(z^2 - 1)^{1/2}}{3^{1/2} z} \pm \frac{\pi i}{2} \right) - \\ - 2 \left(\operatorname{Arth} \frac{z^2 + 2}{3z} \pm \frac{\pi i}{2} \right), \quad (11')$$

where

$$z = (4 - 3z_0)^{1/2}, \quad 1 < z < 2 \quad (12')$$

($z=1$ at $B \rightarrow 0$, $z=2$ at $B \rightarrow \infty$). In this case the asymptotical parameter $a(B)$ is complex, so the series (I) is alternating in sign. The dependence of the parameter a on B is shown in Fig. 9.

The case of a hydrogen atom in both fields, \mathcal{E} and B , can be considered analogously (see eq.(3) in ref. [28], determining the radius $z_0 = z_0(\mathcal{E}, B)$ of the classical equilibrium orbit).

As an example of calculation of the asymptotical parameter

$|a(\mathcal{E}, B)|$, see Fig. 10.

Table I.

#	$f(x)$	γ_{cr}	γ_*	C	A	potential
1	$\exp(-x)$	0.73576	0.83996	1.289	0.1116	Yukawa
2	$\sqrt{e^x - 1}$	1.29522	1.52344	1.381	0.1371	Hulthén
3	$\exp(-x^2/2)$	1.21306	1.58650	1.682	0.2478	
4	$x \cdot \exp(-x)$	1.08268	1.34425	1.565	0.1997	
5	$(1+x)^{-2}$	0.5	0.52815	1.033	0.0574	Tietz
6	$x \cdot \exp(-x^2)$	0.73576	1.08268	2.000	0.4167	Gaussian
7	$1+x^2$	—	0.38490	1.565	0.1997	spherical model
8	—	—	0.45618	0.8660	0.2406	Stark effect

Footnote: $f(x)$ is the screening function in (4), A and C - the coefficients entering eq. (9).

Table 2.

The values of the Borel parameter $\delta_0 = 1/2a$ for the molecular ion H_2^+ ($\tilde{R} < R_*$).

\tilde{R}	$-\delta_0$	method of calculation
0.2	1.720 950	eq.(23)
0.4	1.062 376	—
0.6	0.705 527	—
	≈ 0.7	ref.[20]
0.9	0.474 800 620	eq.(23)
	0.474 795	ref.[20]
1.0	0.313 841 191	eq.(23)
	0.313 841 21	ref.[20]
1.2	0.197 516 621 52	eq.(23)
	0.197 516 618 76	ref.[20]
1.4	0.112 797	eq.(23)
1.6	0.052 563	—
1.8	0.013 569	—

Footnote: Note that the definition of the scaled internuclear distance \tilde{R} in ref.[20] differs from ours by the factor 3/2, as it is explained in eq.(C.10), so here $R_* = 3^{5/2}/8 = 1.948 557$.

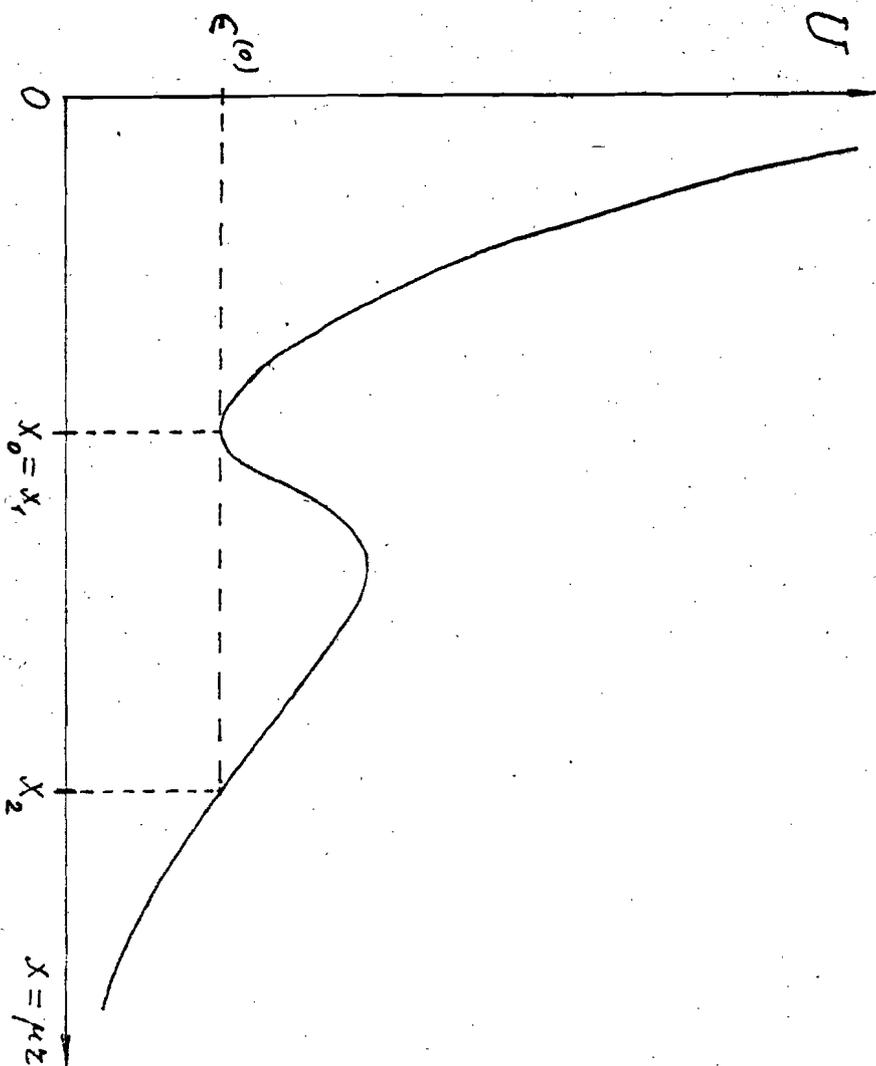


Fig. 1. An effective potential U (qualltative).

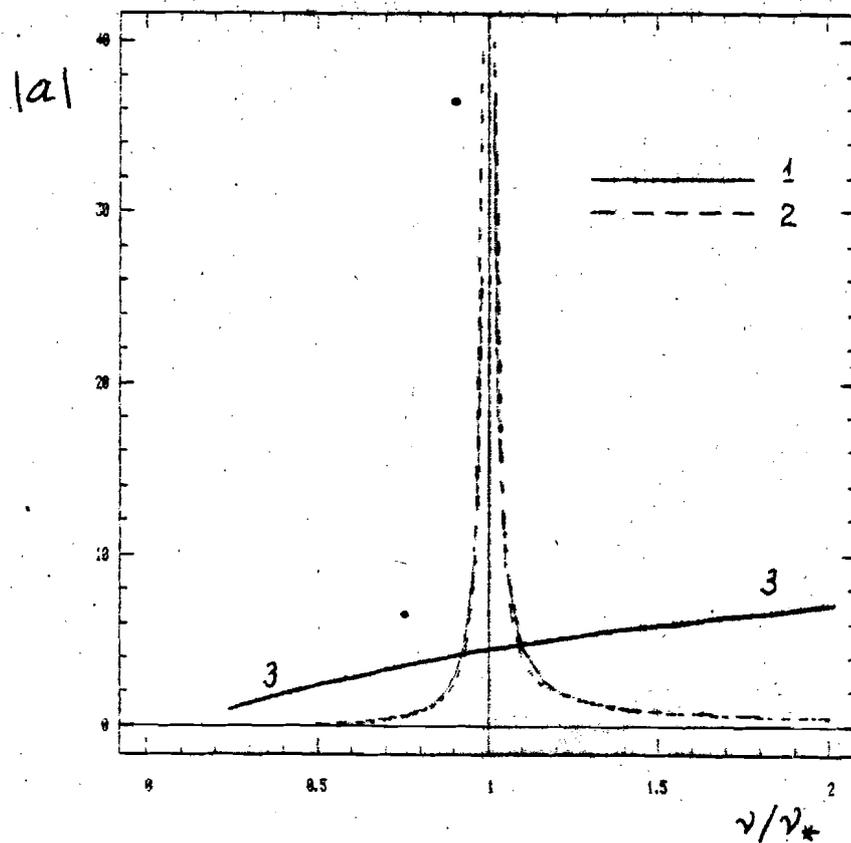


Fig. 2. $|a(\nu)|$ versus ν/ν_* . The curves 1, 2 and 3 correspond to the Stark effect, its spherical model ($g < 0$) and the funnel potential. In the latter case the values of $|a|$ are multiplied by 10^2 .

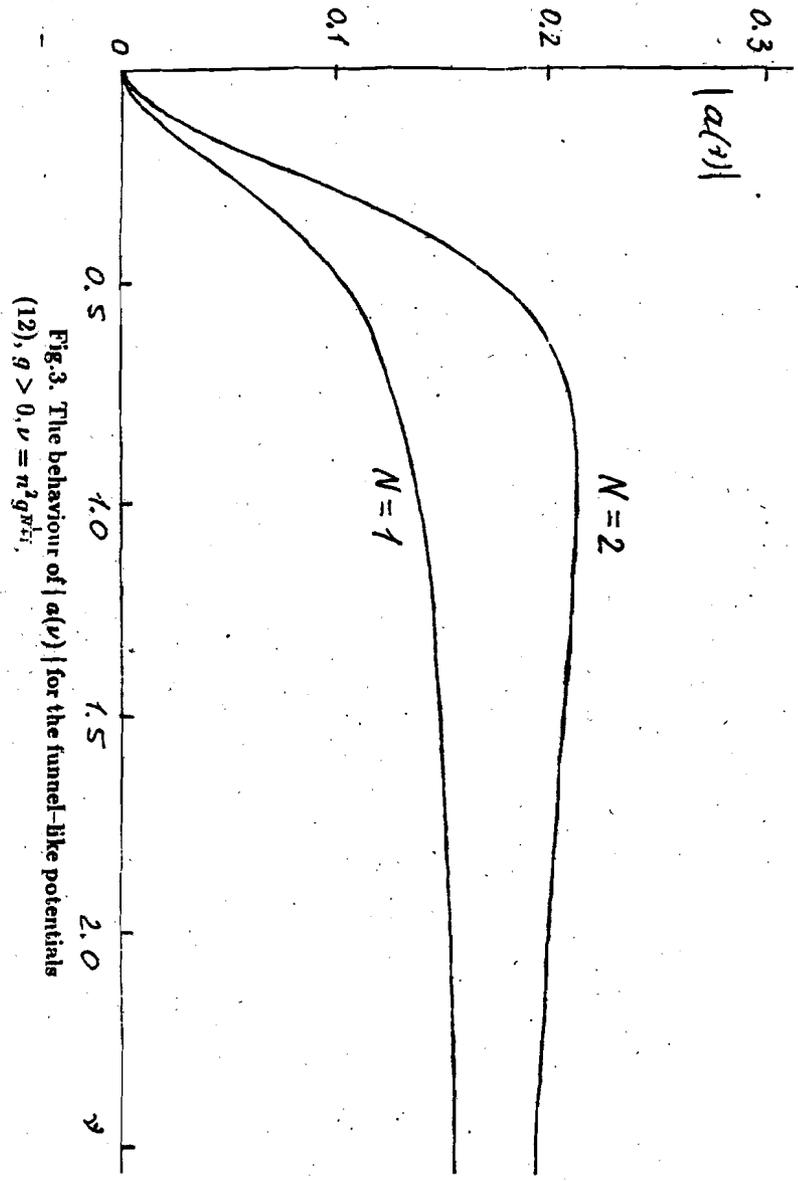


Fig.3. The behaviour of $|a(\nu)|$ for the funnel-like potentials (12), $g > 0, \nu = n^2 g^{N/2}$.

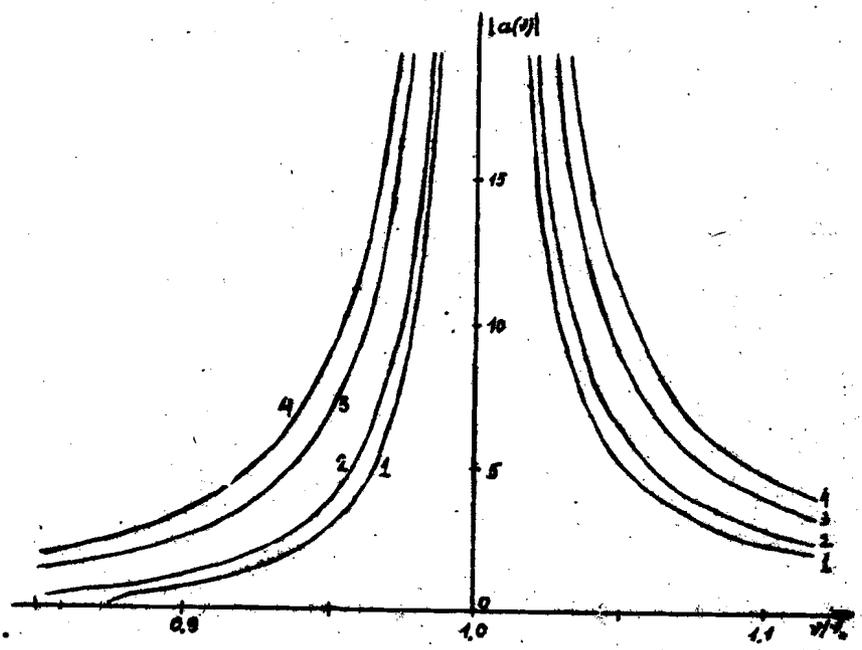


Fig.4. The same as in Fig.2 at $\nu \approx \nu_c$ for the Yukawa potential ($\nu_c = 0.8400$, curve 1), the Hulthén potential ($\nu_c = 1.5234$, curve 2) and funnel potentials (12) with $N=1$ and 2 (curves 3 and 4, $\nu_c = 0.3849$ and 0.4725 , correspondingly). Analytical equations (7),(8), etc. were used for calculation of the curves.

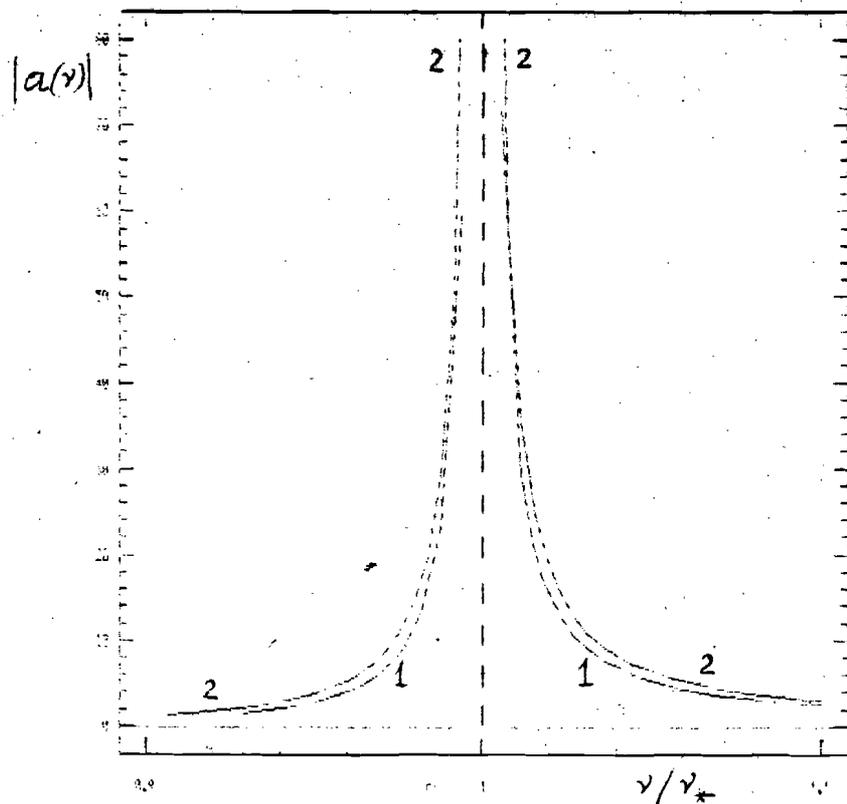


Fig.5. The same as in the preceding Figure for the Yukawa (1) and Hulthén (2) potentials (the results of numerical calculations [13]).

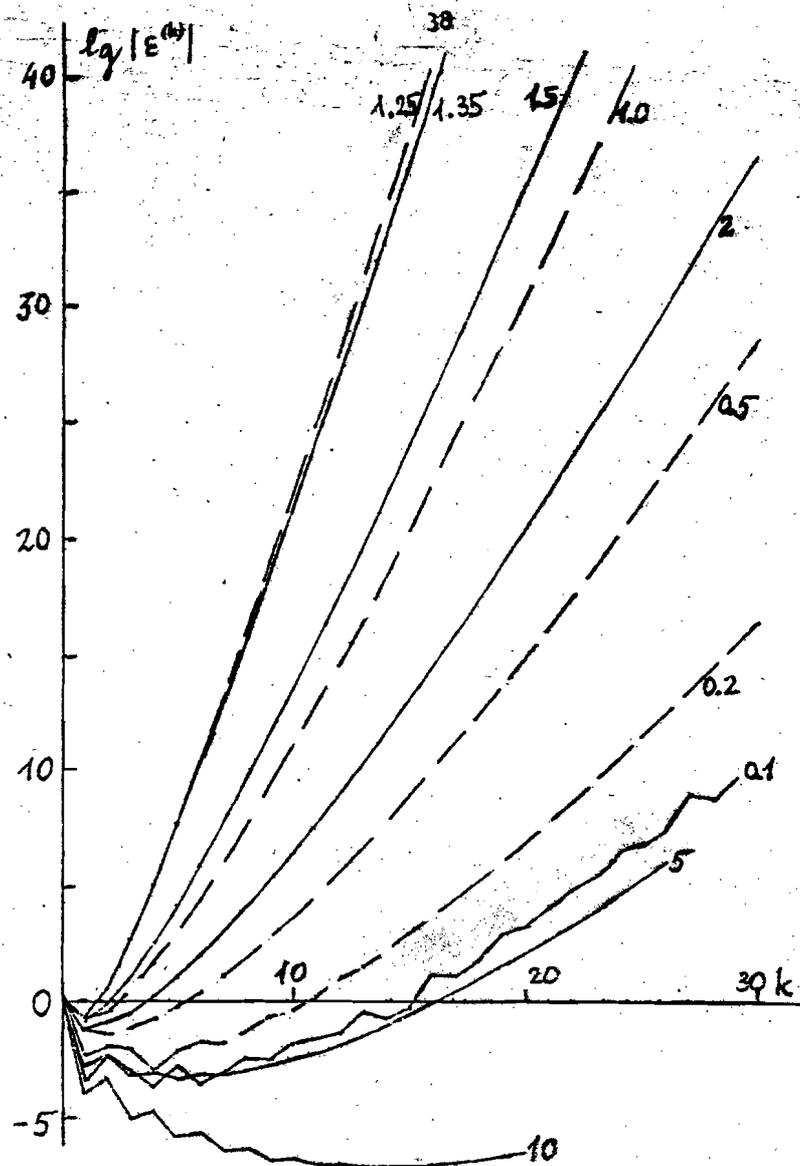


Fig.6. The coefficients of $1/n$ -expansion in the problem of two centres [19], $Z_1 = Z_2 = 1$. The solid lines for $\tilde{R} > R_*$, the dotted ones - for $\tilde{R} < R_* = 1.299$. Note that $|\epsilon^{(k)}(\tilde{R})|$ are extremely increasing at $\tilde{R} \approx 1.3$.

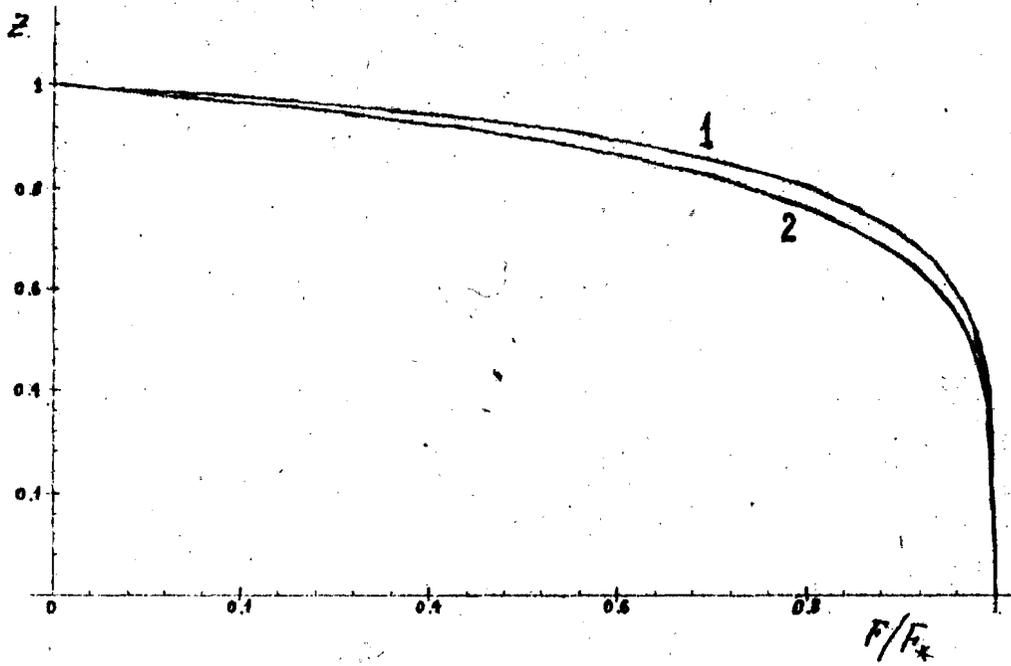


Fig.8. The variable z versus F/F_* for the Stark effect (1) and its spherical model (2).

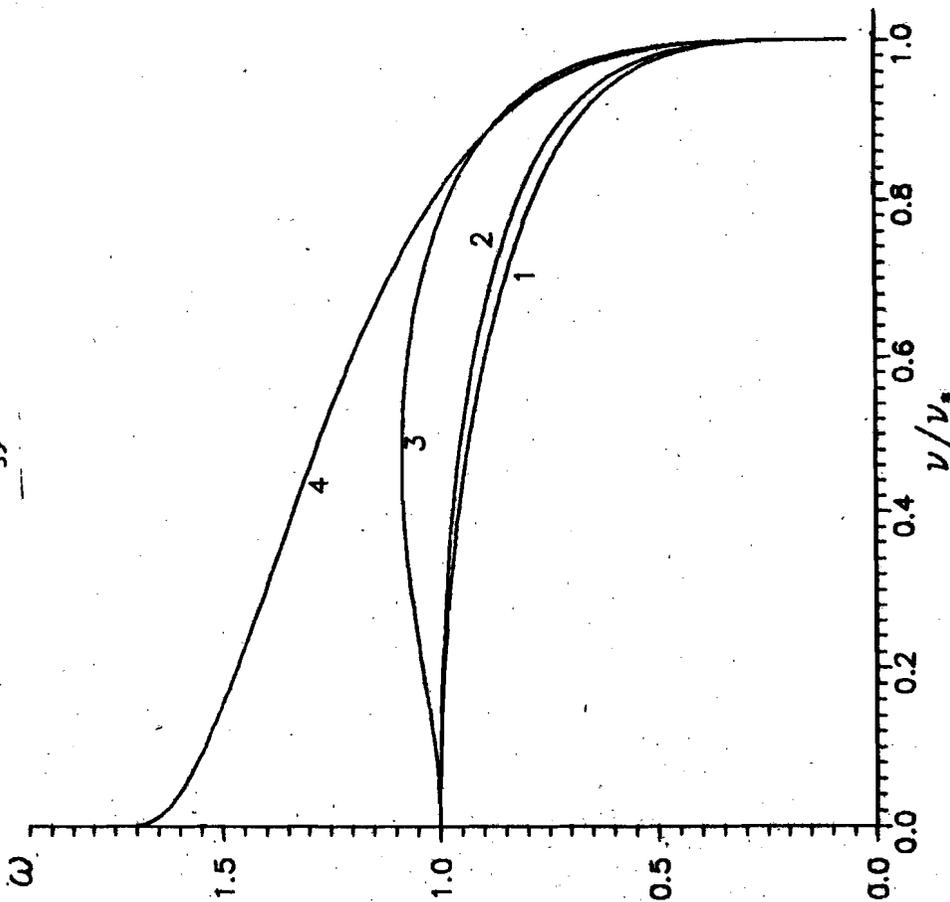


Fig.7. The frequency ω of vibrations around the classical equilibrium point $x_0(\nu)$. The curves are enumerated in accordance with Table 1.

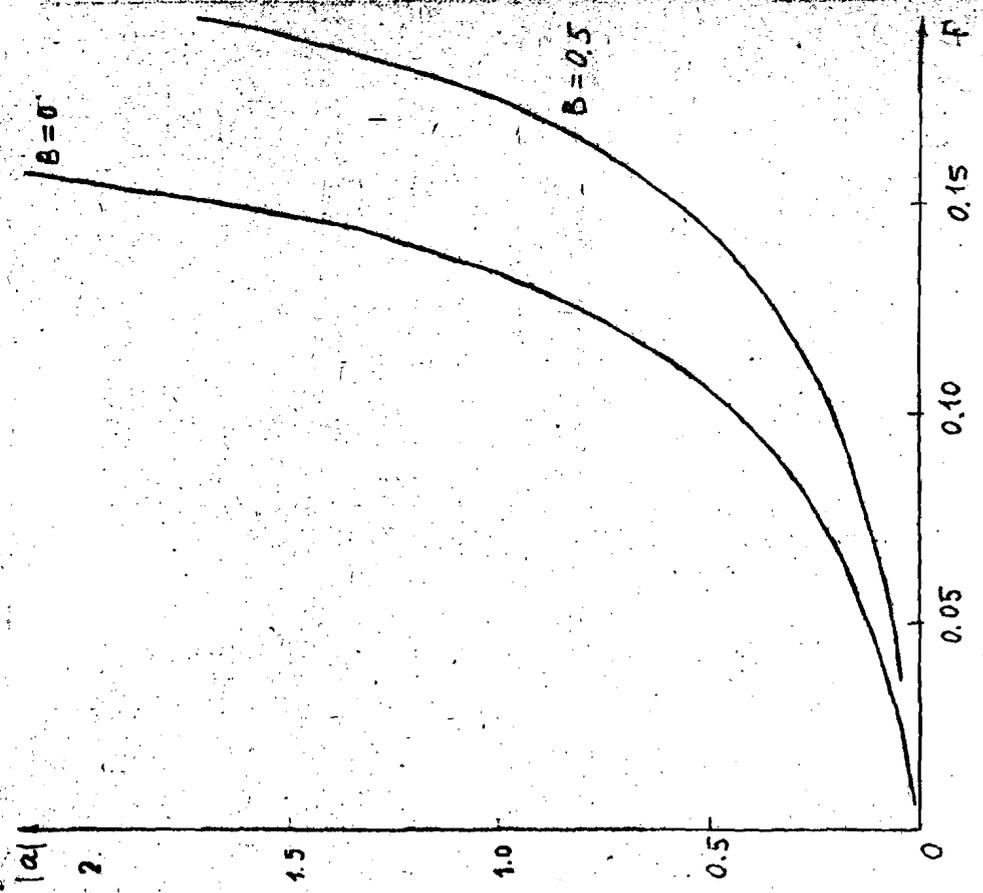


Fig.10. The same as in Fig.10 for a hydrogen atom in parallel fields $\epsilon \parallel B$.

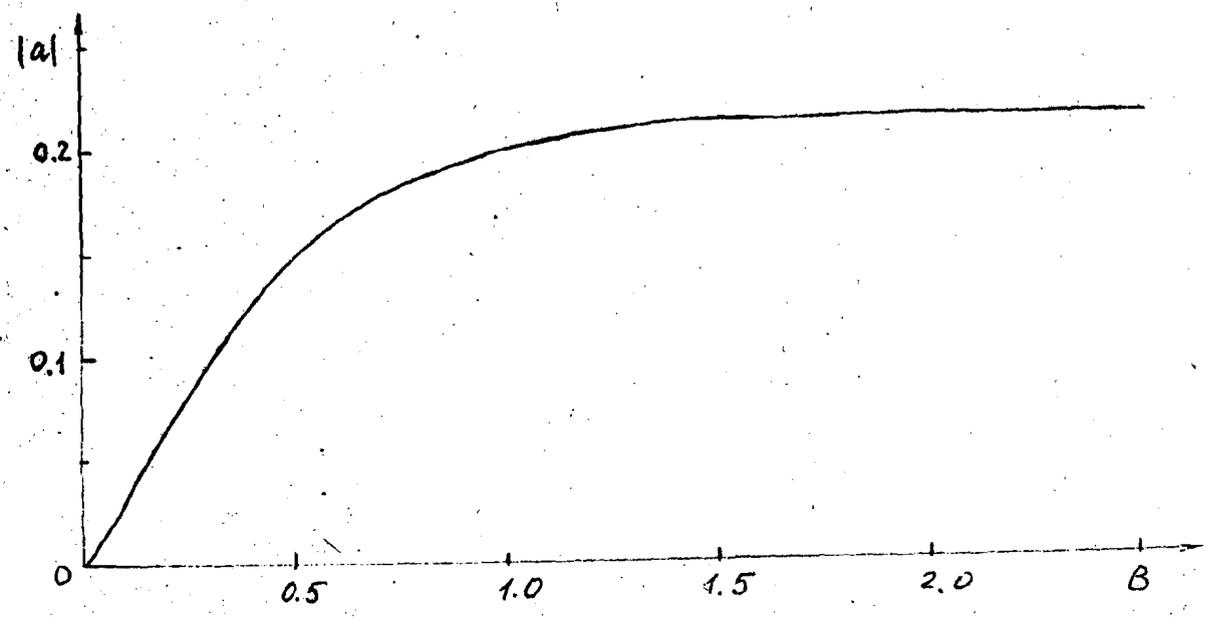


Fig.9. The parameter $|a(B)|$ for the Zeeman effect in hydrogen. Magnetic field B is measured in atomic units, $B_a = m_e^2 e^3 c / \hbar^3 = 2,35 \cdot 10^9 G$.

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