27

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BEAM DYNAMICS IN MODIFIED BETATRON: Preprint ITEP 8-99/

E.R.Mustafin, P.R.Zenkevich - M., 1999 - 24p.

«Modified betatron» is an accelerator using betatron method of acceleration, whose magnetic system includes usually solenoidal longitudinal magnetic field, bending magnetic field (which, in principle, can have additional gradient) and quadrupole spiral windings. Theoretical analysis of transverse beam dynamics in this machine should be based on Teng's theory which permits to consider transverse motion as projection of two uncoupled modes. It is shown that the direct use of the theory is impossible due to a presence of «break points»; it is suggested a method to avoid these problems, named «advanced Teng's algorithm» (ATA). The main physical effects in the modified betatron (matching, influence of field errors, collective effects and so on) are considered by use of «projecting functions» and other characteristics of the linear transverse motion. It is shortly described the corresponding numerical code and its application to the positron accumulator «LEPTA» which is now under construction in JINR.

«Модифицированный бетатрон» представляет собой кольцевой ускоритель, использующий бетатронный метод ускорения, магнитная система которого включает соленоидную обмотку, поворотное магнитное поле (возможно, с дополнительным градиентом) и спиральное квадрупольное поле. Анализ поперечного движения в такой машине выполнен на основе теории Тенга, которая позволяет рассматривать движение по каждой из поперечных степеней свободы как сумму проекций колебаний двух несвязанных мод. Показано, что прямое применение этой теории невозможно из-за наличия «особых точек», в которых параметры нормальных мод терпят разрыв; предложен метод, позволяющий избежать этих затруднений («улучшенный алгоритм Тенга»). Основные физические проблемы (согласование, влияние ошибок магнитного поля, коллективные эффекты и т. д.) рассмотрены с помощью «проектирующих функций» и других параметров линейного движения, полученных в рамках этой теории. Кратко описана программа, основанная на «улучшенном алгоритме Тенга», а также ее применение к расчету позитронного накопителя «LEPTA», который в настоящее время сооружается в ОИЯИ (Дубна, Россия).

Fig. - 3, ref. - 5 name.

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1. INTRODUCTION

Recently it was suggested [1] to use «modified betatron» in experiments for positronium generation, for high energy electron cooling and so on. A specific feature of the particle dynamics in such kind of machine is a strong coupling of the transverse oscillations which appears due to longitudinal magnetic field. For the simplest case of the «symmetrical» modified betatron with constant parameters and equal gradients for both degrees of freedom (field index n=0.5) it is possible to find analytical solution describing a transverse particle motion.

A theory of the particle dynamics in the modified betatron with arbitrary magnetic structure can be built using of Teng's theory [2] where it is suggested a method to find the parameters of the uncoupled modes which describe the transverse motion of the particles. In investigation of this method we have discovered some circumstances which make difficult its direct use. We have found a way to overcome these problems named «advanced Teng's algorithm» (ATA) and to use it for an analyze of particle dynamics in modified betatron with arbitrary magnetic system.

A paper consists of four chapters: 1)theory of the «symmetrical» modified betatron; 2)description of the physical principles ATA; 3)analysis of beam dynamics in non-symmetrical modified betatron by use of ATA, which includes investigation of main characteristics of the transverse individual particle motion as well as the main limitations due to the collective effects; 4) short description of the numerical code used ATA and the results its application to the ring for positron accumulation («LEPTA») which is now under construction in JINR (Dubna) [3].

2. SYMMETRICAL MODIFIED BETATRON

a. Equations of motion and frequencies of transverse oscillations

Equations of transverse motion of the particle in a section of the modified betatron have the following form:

$$v_x' - \Omega_L v_y + \Omega_x^2 x - \Lambda\,[x\cos(2kz) + y\sin(2kz)] = 0$$
$$v_y' + \Omega_L v_x + \Omega_y^2 y - \Lambda\,[x\sin(2kz) - y\cos(2ks)] = 0 \qquad (1)$$

Here x and y are, correspondingly, the horizontal and vertical transverse coordinates, z is a longitudinal coordinate, $\Omega_L = B_z / (BR)$, ((BR) is a magnetic rigidity in T m, B_z is a longitudinal component of magnetic induction), $\Lambda = G/(BR)$, v_x and v_y are derivatives from x and y on z,

$$\Omega_x^2 = (1 - n)/R^2 - (\Omega_x^{sc})^2, \qquad (\Omega_x^{sc})^2 = \frac{P}{a_x(a_x + a_y)}$$

$$\Omega_y^2 = n/R^2 - (\Omega_y^{sc})^2 \qquad (\Omega_y^{sc})^2 = \frac{P}{a_y(a_x + a_y)} \tag{2}$$

In Eq.(2) field index

$$n = \frac{1}{BR}\frac{dB_y}{dx}R^2 \tag{3}$$

and perveance P is defined by:

$$P = \frac{4I}{(\beta\gamma)^3 I_A} \tag{4}$$

Here I is the beam current, I_A is the Alfven current (for electrons and positrons I_A=1.7 10^4A), β and γ are relativistic parameters, a_x and a_y are mean squared beam sizes on corresponding degrees of freedom. In the symmetrical modified betatron a_x=a_y, n=0.5, $\Omega_x^2 = \Omega_y^2 = \Omega^2$ and all other parameters are constant; then the Eq.(1) can be written as:

$$v_x' - \Omega_L v_y + \Omega^2 x - \Lambda[x\cos(2kz) + y\sin(2kz)] = 0$$

$$v_y' + \Omega_L v_x + \Omega^2 y - \Lambda[x\sin(2kz) - y\cos(2ks)] = 0 \tag{5}$$

$$\Omega^2 = \frac{1}{R^2}(0.5 - \frac{PR^2}{2a^2}) \tag{6}$$

Let us introduce complex variable $u = x + iy$; then our system of two uniform equations for real variables can be written as one complex uniform equation:

$$u'' + i\Omega_L u' + \Omega^2 u - \Lambda u^* \exp.(i2kz) = 0 \tag{7}$$

Changing the independent variable as $u = r\exp.(ikz)$ we obtain

$$r'' + 2i(k + \Omega_L/2)r' - [k(k + \Omega_L) - \Omega^2]r - \Lambda r^* = 0 \tag{8}$$

Let us look for a solution of Eq.(8) as

$$r = C_1\exp.(-i\nu z) + C_2\exp.(i\nu z) \tag{9}$$

Then we obtain:

$$\Delta_1 C_1 - \Lambda C_2^* = 0$$

$$-\Lambda C_1 + \Delta_2 C_2^* = 0 \tag{10}$$

Here

$$\Delta_{1,2} = -\nu^2 \pm 2\nu(k + \Omega_L/2) - \Omega_2^2 = 0 \tag{11}$$

$$\Omega_2^2 = k(k+\Omega_L) - \Omega^2 \tag{12}$$

Eq.(10) is a system of uniform linear equations which has a non-trivial solution only if its determinant is equal to zero. Using this condition we find the following dispersion equation:

$$\nu^4 - 2\,\Omega_1^2\nu^2 + \Omega_2^4 - \Lambda^2 = 0 \tag{13}$$

where

$$\Omega_1^2 = k(k+\Omega_L) + \Omega_L^2/2 + \Omega^2 \tag{14}$$

The roots of the equation are defined by:

$$\nu_{1,2} = \sqrt{\Omega_1^2 \pm \sqrt{\Omega_1^4 + \Lambda^2 - \Omega_2^4}} \tag{15}$$

All the roots are real (the motion is stable) if

$$|\Lambda| \le \Omega_2^2 \tag{16}$$

A complete solution of Eq.(8) can be written as:

$$r = C_1 \exp.(-i\nu_1 z) + \kappa_1 C_1^* \exp.(i\nu_1 z) + C_2 \exp.(-i\nu_2 z) + \kappa_2 C_2^* \exp.(i\nu_2 z) \tag{17}$$

Here

$$\kappa_s = \Lambda / \Delta_2(\nu_s), \tag{18}$$

Let us introduce new dimensionless variables $Q = \Omega R$, $\tilde{\Lambda} = \Lambda R^2$, $m = kR$. Using these variables we can write a complete solution of Eq.(7) in the following form:

$$u = \sum_s C_s \exp.[i(-1)^s Q_s \frac{z}{R}] + k_s C_s^* \exp.\{i[2m + (-1)^{s-1} Q_s \frac{z}{R}]\} \tag{19}$$

Here positive «betatron frequencies» $Q_{1,2}$ are defined by:

$$Q_1 = \nu_1 R - m, \qquad Q_2 = m - \nu_2 R \tag{20}$$

Let us introduce vector

$$X^N = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \tag{21}$$

Separating imaginary and real parts in Eq.(19), we can write:

$$X_j^N = \sum_s \sqrt{w_s \Phi_s^j(z)} \cos(Q_s \frac{z}{R} + \Psi_s^j + \alpha_s) \tag{22}$$

Here $w_s = |C_s|^2$, $\alpha_s = (-1)^s \arg(C_s)$;

$$\Phi_s^{\,j} = (T_{1,s}^{\,j})^2 + (T_{2,s}^{\,j})^2 \tag{23}$$

$$\Psi_1^{\,j}(z) = \arccos.\left(\frac{T_{1,s}^{\,j}}{\sqrt{\Phi_s^{\,j}}}\right) \qquad \text{if } T_{2,s}^{\,j} \geq 0$$

$$\Psi_1^{\,j}(z) = 2\pi - \arccos.\left(\frac{T_{1,s}^{\,j}}{\sqrt{\Phi_s^{\,j}}}\right) \qquad \text{if } T_{2,s}^{\,j} < 0 \tag{24}$$

Coefficients $T_{1,s}^{\,j}$ and $T_{1,s}^{\,j}$ are defined by:

$$T_{1,s}^{\,1}(z) = 1 + k_s \cos(2m\frac{z}{R}), \qquad T_{2,s}^{\,1}(z) = (-1)^{s-1} k_s \sin(2m\frac{z}{R})$$

$$T_{2,s}^{\,2}(z) = \frac{1}{R}(-1)^{s-1}\{-Q_s(-1)^{s-1} + k_s[2m - (-1)^{s-1}Q_s]\cos(2m\frac{z}{R})\}$$

$$T_{1,s}^{\,2}(z) = \frac{1}{R} k_s[(-1)^{s-1}Q_s + 2m]\sin(2m\frac{z}{R})$$

$$T_{1,s}^{\,3}(z) = (-1)^s[-1 + k_s\cos(2m\frac{z}{R})], \qquad T_{2,s}^{\,3}(z) = k_s \sin(2m\frac{z}{R}),$$

$$T_{2,s}^{\,4}(z) = \frac{1}{R}(-1)^s\{Q_s + k_s[2m(-1)^s - Q_s]\cos(2m\frac{z}{R})\}$$

$$T_{1,s}^{\,4}(z) = \frac{1}{R} k_s[Q_s + (-1)^{s-1}2m]\sin(2m\frac{z}{R}) \tag{25}$$

If $\Lambda_q \to 0$, then $k_s \to 0$, x and $y' \to \cos(2m\frac{z}{R})$, y and $x' \to \sin(2m\frac{z}{R})$.

For strong solenoidal field ($Q_L >> m$, $Q_L^2 >> \Lambda_q$) approximate expressions for the frequencies can be written as

$$Q_1 \approx Q_L + \frac{Q_0^{\,2}}{Q_L} + \frac{\Lambda_q^{\,2}}{4Q_L(Q_L+m)(Q_L/2+m)}$$

$$Q_2 \approx \frac{Q_0^{\,2}}{Q_L} + \frac{\Lambda_q^{\,2}}{4Q_L(Q_L/2+m)m} \tag{26}$$

Using Eq.(26) it is easy to estimate «chromaticities» $\Gamma_{1,2}$, i.e. derivatives from frequencies on momentum deviation $z=\Delta p/p$:

$$\Gamma_1 \approx -Q_L, \qquad \Gamma_2 \approx -\frac{\Lambda_q^{\,2}(Q_L+m)}{4mQ_L(Q_L/2+m)^2} \tag{27}$$

For $\Lambda=0$ and $Q_0=0$ $\kappa_{1,2}=0$, $Q_1 = Q_L$, $Q_2=0$, and

$$u = C_1 \exp\left(-iQ_L \frac{z}{R}\right) + C_2$$

This is a standard solution for Larmor oscillations of a particle in the solenoidal field. For small spiral focusing gradient the motion consists from two components: fast rotation with frequency near the cyclotron one (this mode appears due to the angular spread and its amplitude for our conditions is much smaller than the amplitude of the slow mode) and slow rotation with small frequency; amplitudes and phases of both modes are modulated with wavelength equal to the spiral step.

b. Stationary orbit and dispersion function

Let us consider one particle oscillations forced by a harmonics of the electromagnetic field which doesn't depend on u (for generalization of the problem, with exponential dependence on time). In this case right hand side (RHS) in Eq.(7) is defined by:RHS $= a_n \exp[i(n\frac{s}{R} - m\frac{s}{R} - \omega t)]$. Taking into account that $\frac{d}{ds} = \frac{\partial}{\partial s} + \frac{1}{v}\frac{\partial}{\partial t}$, we can find the following solution of Eq.(7):

$$u = \frac{a_n}{\Delta(\lambda_n)} \{\Delta_2(\lambda_n) \exp[i(n\frac{z}{R} - \omega t)] + \Lambda \exp[-i(n\frac{z}{R} - 2m\frac{z}{R} - \omega^* t)]\} \quad (28)$$

Here $\lambda_n = (n - m - \omega/\Omega)R$, Ω is the revolution frequency, determinant $\Delta_2(\lambda_n)$ is defined by substitution λ_n in Eq.(11),

$$\Delta(\lambda_n) = (\lambda_n^2 - \nu_1^2)(\lambda_n^2 - \nu_2^2) \quad (29)$$

The resonance appears if

$$\omega/\Omega = n + Q_1, \quad \text{or } \omega/\Omega = n + 2m + Q_1$$
$$\omega/\Omega = n - Q_2, \quad \text{or } \omega/\Omega = n + 2m - Q_2 \quad (30)$$

We see that besides the main harmonics it appears the harmonics shifted on $2m$. Amplitude of the stationary orbit (for $\omega = 0$) is defined by:

$$|u_n| = |a_n| \frac{1}{\Delta(\lambda_n)} |[\Delta_2(\lambda_n) \pm \Lambda]| \quad (31)$$

Analysis of these expressions has shown that the more dangerous case corresponds to main harmonics. For strong solenoidal field ($Q_L >> m$, $Q_L^2 >> \Lambda_q$) amplitude of the forced oscillations for resonant harmonics is defined by

$$|u_n|_{max} \approx |a_n| (Q_L \varepsilon)^{-1} \quad (32)$$

where ε is a deviation of the frequency from the most dangerous resonance.

By use of Eqs.(28,29) we can calculate dispersion function (Ψ - function) and momentum compaction factor α. substituting $a_n = (\Delta p/p)/R$, $n = 0$ and $\omega = 0$. After simple algebra we obtain:

$$\Psi(s) = R\,(Q_0^2 + \frac{\Lambda_q^2}{4m^2 + 2mQ_L - Q_0^2})^{-1}[1 + \frac{\Lambda_q \exp(2imz/R)}{4m^2 + 2mQ_L - Q_0^2}] \tag{33}$$

We see that in modified betatron due to a coupling of the transverse modes it appears a vertical component of the dispersion function. A momentum compaction factor is defined by

$$\alpha - (Q_0^2 + \frac{\Lambda_q^2}{4m^2 + 2mQ_L - Q_0^2})^{-1} \tag{34}$$

c. Beam transverse size and temperature

For our purposes it is necessary to get small particle temperature. Let us assume that the beam is matched, and therefore phases of the complex amplitudes a_s have uniform distribution in interval $[0, 2\pi]$; then using Eq.(19) and averaging on phases and amplitudes, we obtain:

$$\langle x^2\rangle, \langle y^2\rangle = 0.5\langle w_1\rangle\,[1 \pm 2k_1\cos(2m\frac{z}{R}) + k_1^2] + 0.5\langle w_2\rangle\,[1 \mp 2k_2\cos(2m\frac{z}{R}) + k_2^2]$$

$$\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle = 0.5\langle w_1\rangle\,(1 + k_1^2) + 0.5\langle w_2\rangle\,(1 + k_2^2) \tag{35}$$

Here symbol < > means averaging on particle ensemble. Correspondingly, for mean squared derivatives we have:

$$\langle (x')^2\rangle, \langle (y')^2\rangle = 0.5\langle w_1\rangle\frac{1}{R^2}[Q_1^2 \pm 2k_1Q_1(Q_1 + 2m)\cos(2m\frac{z}{R}) + k_1^2(Q_1 + 2m)^2] +$$

$$0.5\langle w_2\rangle\frac{1}{R^2}[Q_2^2 \mp 2k_2Q_2(Q_2 - 2m)\cos(2m\frac{z}{R}) + k_2^2(Q_2 - 2m)^2]$$

$$\langle (r')^2\rangle = \frac{1}{2R^2}\{\langle w_1\rangle[Q_1^2 + k_1^2(Q_1 + 2m)^2] + \langle w_2\rangle\,[Q_2^2 + k_2^2(Q_2 - 2m)^2]\} \tag{36}$$

Using Eqs.(35, 36) we find, that

$$\frac{\langle (r')^2\rangle R^2}{\langle r^2\rangle} = \frac{C[Q_1^2 + k_1^2(Q_1 + 2m)^2] + [Q_2^2 + k_2^2(Q_2 - 2m)^2]}{C(1 + k_1^2) + (1 + k_2^2)} \tag{37}$$

Here $C = \langle w_1\rangle/\langle w_2\rangle$. It is seen that $\langle (r')^2\rangle$ reaches minimal value if $C = 0$, i.e. amplitude of the «fast» solenoidal mode is equal to zero ($\langle w_1\rangle = 0$); then

$$(<(r')^2>)_{\min} = <r^2> \frac{[Q_2^{\,2} + k_2^{\,2}(Q_2 - 2m)^2]}{1 + k_2^{\,2}} \tag{38}$$

Minimal transverse energy $(T_\perp)_{\min}$ is expressed by

$$(T_\perp)_{\min} = T_0 \frac{\langle r^2 \rangle}{R^2} \frac{[Q_2^{\,2} + k_2^{\,2}(Q_2 - 2m)^2]}{1 + k_2^{\,2}} \tag{39}$$

where T_0 is kinetic energy of the particle.

ADVANCED TENG'S ALGORITHM.

a. Brief description of Teng's theory.

Theory developed in paper [2] permits to use Courant- Snyder representation for coupled transverse motion. Let us write the equations of motion in Hamilton's form with the Hamiltonian

$$H = \frac{1}{2}(P_x^2 + P_y^2) + L\,(P_x\,y - P_y\,x) + \frac{1}{2}F\,x^2 + K\,xy + \frac{1}{2}\,G\,y^2 \tag{40}$$

where P_x and P_y are moments conjugated with x and y, L, F, K and G are periodic in φ with period $2\pi R$ (R is an average beam radius). In our case

$$L = 0.5\,\Omega_L, \qquad F = \Lambda\cos(2kz) + \Omega_x^2 + 0.25\Omega_L^2$$

$$K = \Lambda\,\sin(2kz), \qquad G = -\Lambda\cos(2kz) + \Omega_y^2 + 0.25\Omega_L^2 \tag{41}$$

Using Eqs(38,39) we obtain the following equations of motion:

$$x' = \frac{\partial H}{\partial P_x} = P_x + Ly, \qquad y' = \frac{\partial H}{\partial P_y} = P_y - Lx,$$

$$P_x' = -\frac{\partial H}{\partial x} = LP_y - Fx - Ky, \qquad P_y' = -\frac{\partial H}{\partial y} = -LP_x - Kx - Gy. \tag{42}$$

It easy to see (excluding $P_{x,y}$) that this system coincides with Eq.(1).

Teng's theory permits to find canonical transformation R to new coordinates V of the form

$$V = R^{-1}\,X \tag{43}$$

where

$$X = \begin{pmatrix} x \\ P_x \\ y \\ P_y \end{pmatrix} \qquad V = \begin{pmatrix} u \\ P_u \\ v \\ P_v \end{pmatrix} \tag{44}$$

In Eq.(43) R is 4x4 matrix depending on s, the motions described by u and v are decoupled. Solutions for each new variable can be written in the Courant-Snyder phase-amplitude form, containing periodic parameters analogous to usual Twiss parameters. The periodic parameters are related to the single period matrix in a fashion similar to one-dimensional case. However, our analysis have shown that the direct use of the theory is impossible due to discontinuities of some of parameters. To avoid these problems we suggest to use some new «projecting functions» which are continuos (together with these derivatives). Moreover, we introduce new «direct» method derivation of phases of the uncoupled modes. The corresponding procedure is described below.

b. Calculation of «single-period transfer matrix»

Let us introduce the transfer matrix $T(z_2, z_1)$ connecting the state of the motion at z_2 to the state of the motion at z_1 according to

$$X(z_2) = T(z_2, z_1)\, X(z_1) \tag{45}$$

For periodic system, the «single-period» transfer matrix defined by

$$T(z) = T(z + L_{\text{per}}, z) \tag{46}$$

is also periodic.

Let us assume that the beginning of the period corresponds to a point $z = 0$. Matrix $T(z)$ may be calculated by direct multiplication of the «section matrices» of the period which could be calculated if the parameters of Hamiltonian are constant in each section (the corresponding expressions are given in Appendix). Matrix $T(z)$ is connected with matrix $T(0)$ by

$$T(z) = T(z, 0)\, T(0)[T(z,0)]^{-1} \tag{47}$$

Let us mark that there is a simple coupling between motion in canonical variables X and a motion in standard variables which is defined by vector X^N. In these variables equation of the motion can be written at the following form:

$$X^N(z_2) = T^N(z_2, z_1) X^N(z_1) \tag{48}$$

Non-canonical matrix of the motion $T^N(z_2, z_1)$ is connected to the canonical matrix of the motion $T(z_2, z_1)$ by the expression

$$T^N(z_2, z_1) = [M(z_2)]^{-1}\, T(z_2, z_1)\, M(z_1) \tag{49}$$

Here

$$M(z_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & L(z_1) & 0 \\ 0 & 0 & 1 & 0 \\ -L(z_1) & 0 & 0 & 1 \end{pmatrix} [M(z_2)]^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -L(z_2) & 0 \\ 0 & 0 & 1 & 0 \\ L(z_2) & 0 & 0 & 1 \end{pmatrix} \quad (50)$$

These equations can be useful for checking of the numerical code and derivation of the matrices for separate focusing sections with constant parameters.

c. Calculation of «characteristic matrices» of the normal modes

A $2n*2n$ matrix T which satisfies the condition

$$\tilde{T}ST = S \quad (51)$$

is called «symplectic». Here $\tilde{T}$ is transposed matrix (i.e., a matrix with $\tilde{T}_{i,j} = T_{j,i}$), S is $2n*2n$ generalization of unit «symplectic» matrix

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (52)$$

The Jacobian matrix of the canonical transformation has this property; from the other side, matrix $T(z_2, z_1)$ gives the canonical transformation from $X(z_1)$ to $X(z_2)$ and therefore it is symplectic. The symplectic condition (49) gives $n(2n - 1)$ relations between $(2n)^2$ elements of the matrix. In our case n=2, and, consequently, the single-period matrix $T(z)$contains 10 independent parameters. Our purpose is to express all Courant-Snyder variables through these parameters. Let us represent the matrix $T(z)$ in the following form:

$$T = \begin{pmatrix} M & n \\ m & N \end{pmatrix} \quad (53)$$

Here M , m, N and n are 2x2 matrices. From the other side, matrix $T(z)$ can be written in the following «symplectic rotation» form, which for two dimensional motion is

$$T(z) = \begin{pmatrix} I\cos\phi & D^{-1}\sin\phi \\ -D\sin\phi & I\cos\phi \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I\cos\phi & -D^{-1}\sin\phi \\ D\sin\phi & I\cos\phi \end{pmatrix} \quad (54)$$

or

$$T(z) = RUR^{-1} \quad (55)$$

Here A, B and D are 2x2 symplectic matrices, each of them requires 3 parameters; the tenth is the angle ϕ. These ten parameters are expressed through ten independent parameters of matrix $T(z)$ by the following expressions:

$$\cos\mu_1 - \cos\mu_2 = \frac{1}{2}Tr(M-N)\left\{1 + \frac{2|m| + Tr(nm)}{[\frac{1}{2}Tr(M-N)]^2}\right\}$$

$$\cos 2\phi = \frac{1/2Tr(M-N)}{\cos\mu_1 - \cos\mu_2}$$

$$D = -\frac{m + Sn\tilde{S}}{(\cos\mu_1 - \cos\mu_2)\sin 2\phi}$$

$$A = M - D^{-1}m\tan\phi \qquad B = N + Dn\tan\phi \tag{56}$$

Analysis shows that this system has «break points» when $Tr(M-N)=0$. Let this condition is satisfied for $z = z_{br}$;moreover, let us denote all the parameters corresponding to the left side of this point (with $z = z_{br} - \delta$; where δ is an infinitely small number) by indice - , and the parameters corresponding to the right side (with $z = z_{br} + \delta$) by indice +. Then we can write:

$$(\cos\mu_1 - \cos\mu_2)^- = -(\cos\mu_1 - \cos\mu_2)^+, \Phi^- = -\Phi^+, D^- = D^+$$

$$A^- = M - D^{-1}m\tan\Phi^-, B^- = N + Dn\tan\Phi^-$$

$$A^+ = M + D^{-1}m\tan\Phi^-, B^+ = N - Dn\tan\Phi^- \tag{57}$$

These breaks result in the corresponding breaks in parameters of the normal modes. In order to confine the invariant mode amplitude let us introduce discrete function $N(z)$, which is equal to a number of the breaks in interval [0, z]. Then we can introduce matrix A^s, which is defined by

$$A^1 = A,\ A^2 = B \quad \text{if } N(z) \text{ is even}$$

$$A^1 = B,\ A^2 = A \quad \text{if } N(z) \text{ is odd} \tag{58}$$

To confine the continuos change of parameters it is necessary to change also the matrix R and to introduce new matrix R^s:

$$R^s = R, \qquad \text{if } N(z) \text{ is even}$$

$$R^s = \begin{pmatrix} D^{-1}\sin\phi & I\cos\phi \\ I\cos\phi & -D\sin\phi \end{pmatrix} \qquad \text{if } N(z) \text{ is odd} \tag{59}$$

d. Calculation of Twiss parameters for the uncoupled modes

If matrices A^s are known, the values of corresponding Twiss parameters can be found from the following standard equations:

$$A^s = I\cos\mu_s + J_s \sin\mu_s; J_s = \begin{pmatrix} \alpha_s & \beta_s \\ -\gamma_s & -\alpha_s \end{pmatrix}; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{60}$$

Using this equation, we obtain

$$\beta_s = A_{1,2}{}^s / \sin\mu_s, \alpha_s = (A_{1,1}{}^s - \cos\mu_s) / \sin\mu_s, \gamma_s = (1+\alpha_s{}^2)/\beta_s \tag{61}$$

Then we can write the solutions for uncoupled modes in standard Courant-Snyder form:

$$u = \sqrt{w_1\beta_1}\ \cos\Psi_1,\ P_u = -\sqrt{\frac{w_1}{\beta_1}}\ (\sin\Psi_1 + \alpha_1\cos\Psi_1)$$

$$v = \sqrt{w_2\beta_2}\ \cos\Psi_2,\ P_v = -\sqrt{\frac{w_2}{\beta_2}}\ (\sin\Psi_2 + \alpha_2\cos\Psi_2) \tag{62}$$

In Eqs(62) phases Ψ_s are defined by

$$\Psi_s(z) = \delta\Psi_s(z) + \gamma_s \tag{63}$$

Here w_s and γ_s are invariants of motion for separate particles, $\delta\Psi_s(z)$ are functions characterizing the corresponding normal modes. Analysis of these formulae have shown that β_s, γ_s are continuos functions, but α_s and $\delta\Psi_s(z)$ have jumps in break points.

e. Calculation of the phase functions for the normal modes

Let us mark that phase functions $\delta\Psi_s(z)$ are not defined by the procedure described above. In Teng's paper it is given the following expression for $\Psi_{1,2}(z)$ (which does not coincide with standard Courant-Snyder theory):

$$\Psi_{1,2}(s) = \int_0^s \frac{1 \mp bL\tan\phi}{\beta_{1,2}} ds' \tag{64}$$

However, numerical calculations have shown that this expression is not valid for break points(where phases have jumps), and therefore it is not applicable to the calculation of the phase functions. To avoid this problem we introduce special procedure, based on calculations of trajectories for «probe particles». The first step in the procedure is a calculation of Twiss parameters for a point z=0. Then we find $X(0)$ for two particles:1)$w_1 = 1$, $w_2 = 0$; 2) $w_2 = 1$, $w_1 = 0$ (for both particles γ_s=0). Then for both particles we have found vectors $X(z)$ by use of Eq.(45). Then, using equations (62), we can write the following evident expressions for the phases:

$$\Psi_1 = \arccos\left(\frac{u}{\sqrt{\beta_1}}\right) \qquad \text{if } \sin\Psi_1 > 0$$

$$2\pi - \arccos\left(\frac{u}{\sqrt{\beta_1}}\right) \qquad \text{if } \sin\Psi_1 \le 0 \tag{65}$$

Here

$$\sin\Psi_1 = -\sqrt{\beta_1}P_u + \alpha_1\frac{u}{\sqrt{\beta_1}}. \tag{66}$$

Expressions for the second mode can be found by the corresponding change of the indices and a change u, P_u on v, P_v.

f. Calculation of the «projecting functions»

For the most part of the applications we are interested not in the normal modes, but in physical variables (coordinates and its derivatives) expressed through invariants of the uncoupled normal modes. Using Eqs(43),(62) we can find, that components of canonical vector X are defined by

$$X_j = \sum_s \sqrt{w_s\beta_s}\,[T_{1,s}{}^j \cos\Psi_s - T_{2,s}{}^j \sin\Psi_s] \tag{67}$$

Here

$$T_{1,s}{}^j = R_{j,1+2s} - R_{j,2+2s}\frac{\alpha_s}{\beta_s},\ T_{2,s}{}^j = \frac{R_{j,2+2s}}{\beta_s} \tag{68}$$

Components of the vector of physical variables X^N are defined by the similar expressions for components with j=1,3 (x,y), but for components with j=2,4 (correspondingly, for x' and y') expressions for T^j are changed:

$$T_{1,s}{}^2 = R_{2,1+2s} + LR_{3,1+2s} - \frac{\alpha_s}{\beta_s}(R_{2,2+2s} + LR_{3,2+2s}),\ T_{2,s}{}^2 = -\frac{R_{2,2+2s} + LR_{3,2+2s}}{\beta_s} \tag{69}$$

$$T_{1,s}{}^4 = R_{4,1+2s} - LR_{1,1+2s} - \frac{\alpha_s}{\beta_s}(R_{4,2+2s} - LR_{1,2+2s}),\ T_{2,s}{}^4 = -\frac{R_{4,2+2s} - LR_{1,2+2s}}{\beta_s} \tag{70}$$

After simple algebra we can write these expressions given Eq.(67) in the following form:

$$X_j = \sum \sqrt{w_s\Phi_s{}^j}\cos(\Psi_s{}^j + \gamma_s) \tag{71}$$

Here

$$\Phi_s{}^j = \beta_s[(T_{1,s}{}^j)^2 + (T_{2,s}{}^j)^2],$$

$$T_{1,s}{}^{j} = R_{j,1+2s} - R_{j,2+2s}\frac{\alpha_s}{\beta_s}, T_{2,s}{}^{j} = -\frac{R_{j,2+2s}}{\beta_s} \tag{72}$$

$$\Psi_s{}^{j} = \Psi_s + \arccos\frac{T_{2,s}{}^{j}}{\sqrt{(T_{1,s}{}^{j})^2 + (T_{2,s}{}^{j})^2}} \qquad \text{if } T_{1,s}{}^{j} \geq 0$$

$$\Psi_s + 2\pi - \arccos\frac{T_{2,s}{}^{j}}{\sqrt{(T_{1,s}{}^{j})^2 + (T_{2,s}{}^{j})^2}} \qquad \text{if } T_{1,s}{}^{j} < 0 \tag{73}$$

g. Calculation of betatron frequencies and periodical components of the «projecting functions» phases

In calculation of the betatron frequencies Q_s it is necessary to take into account, that the phases Ψ_s are defined in interval $(0,2\pi)$, and in real calculations we can observe «jumps» from 2π to zero and back (from zero to 2π). Let $M_1(L_{per})$ is a number of «direct» jumps, and $M_2(L_{per})$ is a number of the «backward» jumps; then we can find betatron frequencies from the following expression:

$$Q_s = \{M_1(L_{per}) - M_2(L_{per}) + \frac{1}{2\pi}[\Psi_s(L_{per}) - \Psi_s(0)]\}N_{per} \tag{74}$$

Here N_{per} is a number of the magnetic periods in the ring.

For some applications it is useful to write the phases of the projecting functions in the following form:

$$\Psi_s{}^{j} = 2\pi Q_s\frac{z}{L} + \delta\theta_s{}^{j} + \gamma_s \tag{75}$$

Here $\delta\theta_s{}^{j}$ is periodic functions with a period equal to a period of the magnetic system.

h. Calculation of the invariant matrix

In accordance with Teng's paper, invariants w_s are expressed through canonical variables by the following formula:

$$w_s = \sum_{i,j}(G_{i,j}{}^{s})^{can} X_i X_j \tag{76}$$

Here matrices $(G^{1,2})^{can}$ are defined by

$$(G^1)^{can} = SR\begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix}R, (G^2)^{can} = SR\begin{pmatrix} 0 & 0 \\ 0 & J_2 \end{pmatrix} \tag{77}$$

However, usually it is more interesting to know the expressions of the invariants through the physical variables which can be written as follows:

$$w_s = \sum_{i,j} G_{i,j}{}^s X_i{}^N X_j{}^N \tag{78}$$

Here matrices G^s are defined by:

$$G^s = \tilde{M}(G_s)^{can}(M) \tag{79}$$

i. Calculation of periodic dispersion function

Dispersion function (in our case, two-dimensional vector) is defined by the following column:

$$\Psi = \begin{pmatrix} \Psi_x \\ \Psi_x{}' \\ \Psi_y \\ \Psi_y{}' \end{pmatrix} \tag{80}$$

This function can be found by use of standard matrix formalism. Let vector $B(z_2,z_1)$ is a particular solution of the following non-uniform system of linear equations for zero initial conditions:

$$v_x{}' - \Omega_L v_y + \Omega_x{}^2 x - \Lambda\,[x\cos(2kz) + y\sin(2kz)] = \frac{1}{R(z)}$$

$$v_y{}' + \Omega_L v_x + \Omega_y{}^2 y - \Lambda\,[x\sin(2kz) - y\cos(2kz)] = 0 \tag{81}$$

Due to a periodicity of the dispersion function its value in initial point ($z = 0$) is defined by:

$$\Psi(0) = B(L_{per},0) + T(L_{per},0)\Psi(0) \tag{82}$$

In intermediate points

$$\Psi(z) = B(z,0) + T(z,0)\Psi(0) \tag{83}$$

These formula permit to find $\Psi(z)$ by use of special computer code. Let us mark that in a presence of the momentum deviation an expression for the invariants becomes the following:

$$w_s = \sum_{i,j} G_{i,j}{}^s X_i{}^N X_j{}^N - \frac{\Delta p}{p}\sum_{i,j} G_{i,j}{}^s(\Psi_i X_j{}^N + \Psi_j X_i{}^N) + (\frac{\Delta p}{p})^2 \sum_{i,j} G_{i,j}{}^s \Psi_i \Psi_j \tag{84}$$

3. MAIN PHYSICAL EFFECTS IN NON-SYMMETRICAL MODIFIED BETATRON

a. Transverse size and temperature of matched beam

As for symmetrical betatron, the matched beam should have uniform distribution on phases γ_s. Squaring given above expressions and averaging on phases we obtain:

$$\langle (X^j)^2 \rangle = \frac{1}{2} \sum_s \Phi_s^{\,j} \langle w_s \rangle, \tag{85}$$

Let us underline that the injected non-matched beam becomes matched due to phase mixing which is connected with linear and nonlinear spread in betatron frequencies of the normal modes.

In order to find $<w_s>$ we should substitute in Eq.(84) values of all parameters in a point z=0 and make an averaging on the ensemble. The second member in right side of the Eq.(84) is equal to zero since $<\frac{\Delta p}{p}>=0$. The first member could be simplified if we have injection with radial symmetry (in this case $\langle (x_0)^2 \rangle = \langle (y_0)^2 \rangle = 0.5 \langle (r_0)^2 \rangle = a^2$, $\langle x_0 y_0 \rangle = 0$, $\langle (x_0')^2 \rangle = \langle (y_0')^2 \rangle = b^2$, $\langle x_0' y_0' \rangle = 0$). Then we obtain:

$$\langle w_s \rangle = a^2 (G_{1,1}^{\,s} + G_{3,3}^{\,s}) + b^2 (G_{2,2}^{\,s} + G_{4,4}^{\,s}) + \sigma_p^{\,2} \sum_{i,j} G_{i,j} A_i A_j \tag{86}$$

Substituting found values of $\langle w_s \rangle$ in Eq.(85) we can calculate beam size and temperature.

b. Invariant growth due to Coulomb scattering

Coulomb scattering of the circulating particles results in a growth of transverse invariants for the normal modes. By use of Eq. (84) we obtain:

$$\langle \Delta w_s \rangle = 0.5 < (G_s^{\,2,2} + G_s^{\,4,4}) > \langle \Delta(\theta^2) \rangle \tag{87}$$

Here sign <> means averaging of the corresponding elements on the ring's length; r. m. s. value of the squared scattering angle of $\langle \Delta(\theta^2) \rangle$ is defined by:

$$\langle \Delta(\theta^2) \rangle = \left(\frac{E_{scat}}{pc\beta}\right)^2 \frac{\Delta t}{X_r} \tag{88}$$

Here E_{scat} = 25 MeV, X_r is a radiation length of the residual gas in g/cm^2, Δt is a length of the particle inside the gas (as well in g/cm^2).

c. Forced oscillations

Let us consider oscillations forced by an external electromagnetic field which doesn't depend on transverse coordinates but has an arbitrary dependence on longitudinal variable and sinusoidal dependence on time (account of this effect it is necessary for investigation of the dipole oscillations stability). In this case additional member in Hamiltonian can be written as follows:

$$\delta H = \mathrm{Re}[f(z)x \exp.(-\mathrm{i}\omega t)] \tag{89}$$

Here $f(z)$ is a known complex function of z. Substituting x from Eq.(71) we obtain:

$$\delta H = \mathrm{Re}[f(z) \sum \sqrt{w_s \Phi_s^{\,1}} \cos \Psi_s^{\,1} \exp.(-\mathrm{i}\omega t)] \tag{90}$$

Let us consider the first member in Hamiltonian. Representing cosine as a sum of two exponents, taking into account Eq.(75) and limiting ourselves by analysis of the second (negative) exponent, we obtain:

$$\delta H = 0.5 \sqrt{w_s} \ \sqrt{\Phi_s^{\,1}(z)}\, \mathrm{Re}\{ f(z) \exp[-\mathrm{i}\delta\theta_s^{\,1}(z)] \exp(-\mathrm{i}Q_s \frac{z}{R} - \mathrm{i}\gamma_s - \mathrm{i}\omega t)\} \tag{91}$$

Let us expand periodical function

$$F(z) = f(z) \sqrt{\Phi_s^{\,1}(z)} \ \exp[-\mathrm{i}\delta\theta_s^{\,1}(z]$$

in complex Fourrier series on variable z:

$$F(z) = \sum_{n=-\infty}^{n=\infty} c_n \exp(i\omega_n z) \tag{92}$$

Then we obtain:

$$\delta H = 0.5 \sqrt{w_s} \ \mathrm{Re}\ [\exp(-\mathrm{i}Q_s \frac{z}{R} - \mathrm{i}\gamma_s - \mathrm{i}\omega t) \sum_{n=-\infty}^{n=\infty} c_n \exp(i\omega_n z)] \tag{93}$$

Separating one member of the series and taking into account that $t = z / v$ (v is a particle longitudinal velocity), we obtain:

$$\delta H = g\sqrt{w_s} \ \cos[(-Q_s + n - \frac{\omega}{\Omega_{rev}}) \frac{z}{R} - \gamma_s + \alpha_n] \tag{94}$$

Here $g = 0{,}5|c_n|$, $\alpha_n = \arg(c_n)$. Taking into account that w_s and γ_s are the canonical conjugated variables (since u, P_u are the canonical conjugated variables as well) we obtain the following pair of equations:

$$\frac{dw_s}{dz} = 2\sqrt{w_s}\, g \sin(\chi)$$

$$\frac{d\chi}{dz} = \frac{g}{\sqrt{w_s}}\cos(\chi) + \varepsilon \tag{95}$$

Here $\varepsilon = (-Q_s + n - \frac{\omega}{\Omega_{rev}}) / R$, $\chi = (-Q_s + n - \frac{\omega}{\Omega_{rev}})\frac{z}{R} - \gamma_s + \alpha_n$.

This pair of equations can be written as a complex one:

$$\frac{da}{dz} = ig - i\varepsilon a \tag{96}$$

Here $a = \sqrt{w_s}\exp.(i\chi)$. Stationary solution has the following form:

$$a = -g / \varepsilon \tag{97}$$

The resonance appears if

$$\frac{\omega}{\Omega_{rev}} = n - Q_1 \tag{98}$$

We see that this theory coincides with standard theory for uncoupled oscillations; a single difference consists in more complicated formula for the resonance width. Using these formula, we can derive the following expression for forced oscillations:

$$x_f(z) = \sqrt{\Phi_s^1(z)}\ \mathrm{Re}\{\frac{g}{n - Q_s - \frac{\omega}{\Omega_{rev}}}\cos\,[n\frac{z}{R} + \alpha_n + \delta\theta_s^1(z)]\exp.(-i\omega t)\} \tag{99}$$

e. Influence of arbitrary order field perturbations

Let us consider an arbitrary order perturbation of the Hamiltonian, which can be written in the following form:

$$\delta H = \sum_m f_m(z) x^m y^{p-m} \tag{100}$$

Substituting x and y (Eqs.(71)), considering only one mode and neglecting all low order terms, we obtain:

$$\delta H = (w_s)^{\frac{p}{2}} \frac{1}{2^{p-1}} \sum_m f_m(z)(\Phi_s^1)^{\frac{m}{2}}(\Phi_s^3)^{\frac{p-m}{2}} \cos[p(Q_s\frac{z}{R} + \gamma_s) + m\delta\theta_s^x + (p-m)\delta\theta_s^y] \tag{101}$$

and expand complex periodic function

$$F(z) = \frac{1}{2^p}\sum_m f_m(z)(\Phi_s^1)^{\frac{m}{2}}(\Phi_s^3)^{\frac{p-m}{2}} \exp.\{-i[m\delta\Psi_s^x + (p-m)\delta\Psi_s^y]\} \tag{102}$$

in Fourrier series on variable z:

$$F(z) = \sum_n c_n \exp.(in\frac{z}{R}) \tag{103}$$

Then, omitting all members besides the resonant one (with number n) we obtain:

$$\delta H = g(w_s)^{\frac{p}{2}} \cos(p\chi) \tag{104}$$

Here $g = |c_n|$, $\chi = \frac{1}{p}[(n - pQ_s)\frac{z}{R} - p\gamma_s]$. We see that this Hamiltonian has a standard form of one-dimensional non-linear resonance and can be analyzed by standard procedure.

f. Coulomb shift of the transverse frequencies

Let us assume that $\Delta\nu_{1,2}^{\lim}$ (maximal shifts of the normal modes betatron frequencies) are known. Then we can write:

$$P_{\lim} = \min(\Delta\nu_{1,2}^{\lim} / D_{1,2}) \tag{105}$$

Here $D_{1,2}$ are corresponding derivatives ($D_{1,2} = \frac{\partial\nu_{1,2}}{\partial P}$). Using Eqs.(4) we find the following expression for the maximal current:

$$I_{\lim} = 0.25\ (\beta\gamma)^3\ I_A\ (P)_{\lim} \tag{106}$$

Values $\Delta\nu_{1,2}^{\lim}$ can be defined by examination dynamical aperture and (or) resonance widths, values of derivatives can be calculated numerically or by use of the corresponding analytical procedure. The additional member in Hamiltonian can be written as follows:

$$\delta H = \frac{P}{8(b_x + b_y)}(\frac{x^2}{b_x} + \frac{y^2}{b_y}) \tag{107}$$

Here $b_{x,y} = \langle x^2, y^2 \rangle$. Substituting x and y, taking into account only one mode and averaging on phases, we obtain:

$$\delta H = \frac{Pw_s}{16(b_x + b_y)}(\frac{\Phi_s^{\ 1}}{b_x} + \frac{\Phi_s^{\ 3}}{b_y}) \tag{108}$$

Using Hamilton equations, we obtain:

$$\frac{\partial Q_s}{\partial P} = \frac{L}{32\pi} < \frac{}{(b_x + b_y)}(\frac{\Phi_s^{\ 1}}{b_x} + \frac{\Phi_s^{\ 3}}{b_y}) > \tag{109}$$

g. Longitudinal instability

Dynamics of longitudinal motion in modified betatron is the same as in standard ring, and therefore we can use the usual criterion for longitudinal stability of the electrons and positrons, which can be written as follows:

$$I \leq 4F_{\text{long}} U_e \beta^2 \gamma |\eta| \sigma_f^2 \frac{1}{|Z_n / n|} \tag{110}$$

Here I is the beam current, U_e is «the electron rest voltage» (U_e = 510 keV), β and γ - relativistic parameters, $\eta = \gamma^{-2} - \alpha$, α is a momentum compaction factor, σ_f is a spread in momentum deviation (half width on half height), Z_n is a longitudinal coupling impedance of the beam for a mode with number n, factor F_{long} takes into account the real form of the stability diagram (in standard Keil-Schnell kritherium F_{long}=1).

For non-relativistic beams longitudinal impedance is defined by space-charge component Z_n^{sc} which can be calculated by use of the following expression:

$$Z_n^{sc}/n = Z_0 \beta^1 \gamma^2 [0.5 + \ln(h/b)] \tag{111}$$

Here Z_0 is a free space impedance (Z_0=377 Ω), h is the beam size, b is the chamber radius.

The most critical problems are a choice of reasonable values of σ_f and F_{long}. The maximal threshold corresponds to a case when the following conditions take place: 1)dynamical aperture on the momentum spread σ_0 significantly increases σ_f; 2)beam is «space-charge dominated», i.e. its space charge component of the longitudinal impedance Z_n^{sc} is much more than the last sources of the impedance; 3)the beam distribution has long tails. Then it is possible to show [4] that with good accuracy

$$F_{\text{long}} \approx (\sigma_0)^2 / 2(\sigma_f)^2 \tag{112}$$

h. Transverse dipole instability.

Now let us consider coherent oscillations, when

$$f(z, t) = [\alpha_x \langle x \rangle + \alpha_y \langle y \rangle] \tag{113}$$

Here $\alpha_{x,y}$ are some complex constant (in the further text we will consider a case $\alpha_x = \alpha_y = \alpha$), $\langle x, y \rangle$ are averaged on all particles of the beam. Moreover, let us assume that

$$\langle x, y \rangle = A\varphi_{x,y}(z) \exp.[i(n\frac{z}{R} - \omega t)] \tag{114}$$

Then we can write the additional member in Hamiltonian in the following form:

$$\delta H_s = -\lambda A\sqrt{w_s}[\varphi_x(z)\sqrt{\Phi_s^{\,1}(z)}\cos(Q_s\frac{z}{R}+\delta\theta_s^{\,1}+\gamma_s) +$$

$$\varphi_y(z)\sqrt{\Phi_s^{\,1}(z)}\cos(Q_s\frac{z}{R}+\delta\theta_s^{\,3}+\gamma_s)]\}\exp[i(n\frac{z}{R}-\omega t)] \tag{115}$$

Let us represent cosine as a sum of two exponents and assume, that only one exponent, corresponding to slow wave, is resonant. Then this Hamiltonian could be written as:

$$\delta H_s = -\lambda A\sqrt{w_s}F_s(z)\exp.[i(n-Q_s\frac{z}{R})-\omega t-\gamma_s] \tag{116}$$

Here

$$F_s(z) = \frac{1}{2}[\varphi_x(z)\sqrt{\Phi_s^{\,1}(z)}\exp.(-i\delta\theta_s^{\,1})+\varphi_y(z)\sqrt{\Phi_s^{\,3}(z)}\exp.(-i\delta\theta_s^{\,3})] \tag{117}$$

Expanding $F_s(z)$ in Fourrier series, omitting all members besides a member with zero indice and separating the real part, we obtain

$$\delta H_s = -\lambda AC_0\sqrt{w_s}\cos[(n-Q_s)\frac{z}{R}-\omega t-\gamma_s] \tag{118}$$

$$C_0 = \frac{1}{L_{par}}\int_0^{L_{par}}F_s(z)dz \tag{119}$$

This is a standard Hamiltonian which was analyzed earlier. Corresponding solutions could be written as:

$$x,y = -\frac{\lambda AC_0}{\varepsilon}\sqrt{\Phi_s^{\,1,3}}\exp.(i\delta\theta_s^{\,1,3})\exp.[i(n\frac{z}{R}-\omega t)] \tag{120}$$

Here $\varepsilon = n - Q_s - \frac{\omega}{\Omega}$.

Averaging on $\frac{1}{\varepsilon}$ we obtain:

$$<x>,\ <y> = -\lambda AC_0 I_s\sqrt{\Phi_s^{\,1,3}}\exp.(i\delta\theta_s^{\,1,3})\exp.[i(n\frac{z}{R}-\omega t)] \tag{121}$$

Here dispersion integral

$$I_s = <\frac{1}{\varepsilon}> = \int_{-\infty}^{\infty}\frac{f(u)du}{\varepsilon(u)} \tag{122}$$

Comparing Eq.(121) with Eq.(113,114), we find, that

$$\varphi_{x,y} = \lambda C_0 I_s\sqrt{\Phi_s^{\,1,3}}\exp.(i\delta\theta_s^{\,1,3})] \tag{123}$$

These formulae we can substitute in Eq.(117,119), we obtain standard dispersion equation, which has the following form:

$$1 = \lambda I_s K_s \tag{124}$$

where

$$K_s = \frac{1}{L_{per}} \int_0^{L_{per}} [\Phi_s^1(z) + \Phi_s^3(z)] dz \tag{125}$$

Analysis of this dispersion equation shows that the stability criterion can be written in the form similar to the standard one [4]:

$$I \leq 8 \frac{U_e}{RZ_\perp} \sigma_\Gamma F_\perp \beta \gamma |\eta| \nu_s \Lambda_s S_s \tag{126}$$

In Eq.(126) $S_s = n - \nu_s + \Gamma_s / \eta$, where Γ_s is the ring chromaticity, factor $F_\perp$ takes into account the form of the distribution function and character of the impedance ($F_\perp \geq 1$), and additional factor Λ_s is defined by:

$$\Lambda_s = \frac{K_s Q_s}{2R} \tag{127}$$

Transverse impedance $Z_\perp$ for non-relativistic beam is defined by standard expression:

$$Z_\perp = iRZ_0 \beta^2 \gamma^{-2} (b^{-2} - h^{-2}) \tag{128}$$

4. NUMERICAL CODE ATA AND ITS APPLICATION TO LEPTA

We have written numerical code based on described above algorithm ATA, which calculates all linear characteristics for transverse motion in the ring with strong coupling of the transverse degrees of freedom. Code may be divided on three parts:1) input subroutine; 2)subroutine for calculation of supplementary characteristics; 3) subroutine for calculation of the output functions. As a unit of length in the code it is used cm.

Input subroutine includes optical scheme of the ring, which gives dependence on the longitudinal variable all the coefficients of Hamiltonian (L, F, K and G, see Eq.(40,41)). All coefficients are supposed to be constant at each section of the ring; if the ring includes a part with variable parameter, such part is divided on small sections with constant parameter whose amplitude is changed from section to section to simulate the known law of change the initial parameter.

Calculations in the second and third subroutine are performed for given set of points inside the period. The second subroutine calculate the parameters of the normal uncoupled modes introduced by Teng. This subroutine includes: a)calculation of the

single period transfer matrix; b)calculation characteristic matrices; c)calculation of Twiss parameters; d) calculation of the phase functions.

The results of the second subroutine are used in the third subroutine for calculation of the output functions characterizing the linear motion in the ring. The subroutine includes: a)calculation of the amplitude projecting functions Φ_s^j and phase projecting functions $\delta\theta_s^j$; b)calculation betatron frequencies of the normal modes; c)calculation of the invariant matrix; d)calculation of the dispersion function and momentum compaction factor. Let us mark that functions Φ_s^1 and Φ_s^3 can be considered as a «projecting β-functions», correspondingly, for x and y (these functions has a dimension of our unit of the length - «cm»). Functions Φ_s^2 and Φ_s^4 are projecting functions for derivatives and has a dimension «cm^{-1}». Sometimes it is convenient to use instead of Φ_s^j «amplitude functions» $A_s^j = \sqrt{\Phi_s^j}$; let us underline that functions $A_s^{1,3}$ and $\delta\theta_s^{1,3}$ has a physical sense of «amplitude and phase of projecting Floquet functions».

This numerical code was applied for calculation of the linear parameters for the ring «LEPTA» which is now under construction in JINR (Dubna, Russia). The scheme of the machine is given in a report of A. Sidorin «Peculiarities of LEPTA design», which was made at the second meeting of the INTAS collaboration (Dubna, 13-14 0f November, 1998). The main parameters of the machine are given in Table 1.

Table 1.

Circumference	m	18.12
Positron energy	keV	10
Revolution period	nsec	300
Longitudinal magnetic field	G	400
Major radius of the toroids	m	1.46
Bending magnetic field	G	1.75
Gradient of quadrupole winding	G/cm	10
Positron beam radius	cm	0.5
Number of positrons		1.0 E9
Residual gas pressure	Torr	1.0 E-10

Using numerical code based on described above algorithm ATA, we have calculated for ring LEPTA all the characteristics of the normal modes: 1)betatron frequencies; 2)16 projecting functions: 8 amplitude functions Φ_s^j and 8 phase

functions $\delta\theta_s^j$ (s = 1,2; j = 1,2,3,4); 3)8 diagonal coefficients matrix $G_{j,j}^s$. The most part of these functions are constant (or oscillating with small amplitude) in all ring besides the section with quadrupole winding; however, inside this section amplitude functions for derivatives sharply increased for «slow» mode (in our case slow mode has index «1»). The typical example is given at Fig.1 (dependence A_1^1 on z).

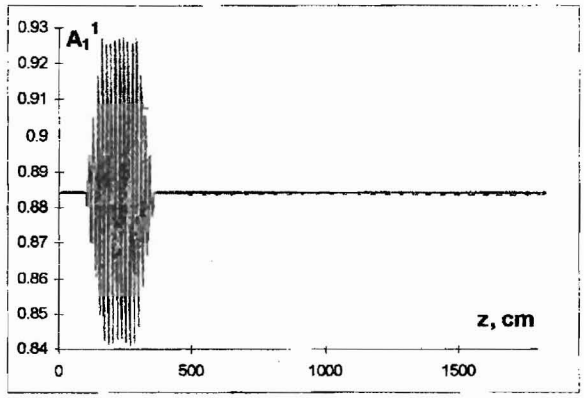

Figure 1. Dependence of amplitude function for the first mode on z

Besides, inside the section with the winding the phases of the projecting functions begins to grow (see Fig.2).

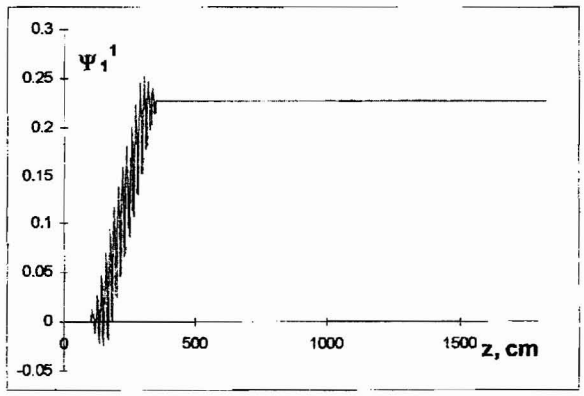

Figure 2. Dependence of Ψ_1^1 on z.

Simultaneously we observe increase and (or) sharp oscillations of some elements of matrix $G_{j,j}^s$ (see Fig.3, where it is plotted function $G_{1,1}^1$).

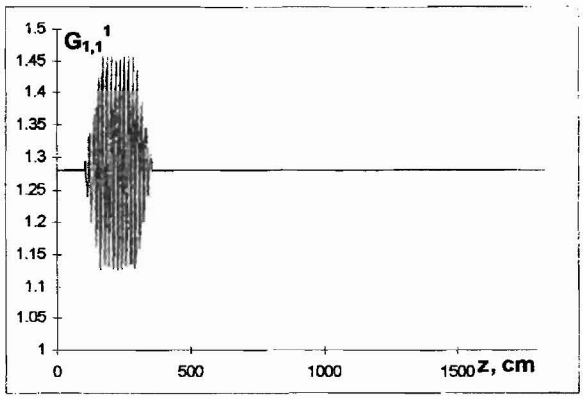

Figure 3. Dependence of $G_{1,1}^{1}$ on z.

5. DISCUSSION

An attractive feature of the symmetrical option of the modified betatron is its simplicity which permits to find analytical formulae describing the transversal motion of the particles. However, real machine (for example, LEPTA) is very far from symmetry, and analysis of dynamics could be made by use of Teng's theory. Our analysis has shown the direct use of this theory is impossible due to a presence of break points, where integral expression for phase advance derived by Teng becomes invalid. We suggested an advanced algorithm which permits to overcome this difficulty. Due to mentioned above jumps Teng's uncoupled modes are very artificial «objects», and in practice the more convenient way is connected with an introduction continuos «projecting functions», which permits to express particle motion through invariants of the Teng's modes (these amplitude and phase functions are similar with amplitude and phase of Floquet functions in standard theory of the uncoupled motion). However, for the coupled motion their number is increased in 4 times (16 instead of 4).

Application of this method permits to find a solution of a number of «classical» problems: analysis of the size and temperature of the matched beam, investigation of the resonances due to field perturbations, analysis of the transverse dipole instability and so on. In conclusion we would like to underline that a lot of the important effects (for example, intra-beam scattering) was omitted in the present paper. We hope that the further development of these ideas will give good insight in physics of so complicated facility as modified betatron.

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