ITEP-42-93

INSTITUTE OF THEORETICAL
AND EXPERIMENTAL PHYSICS

ITEP-42-93

V.V.Goloviznin, S.V.Molodtsov,
A.M.Shigirev

FERMILAB
OCT 05 1993
LIBRARY

The classical approach to the problem of two-body interacting through Yang-Mills field.

Fermilab Library
0 1160 0031579 0

MOSCOW 1993

УДК 517.530.145 М-15

THE CLASSICAL APPROACH TO THE PROBLEM OF TWO-BODY INTERACTING THROUGH YANG-MILLS FIELD: Preprint 42-93/

Golovizin V.V.*), Molodtsov S.V., Shigirev A.M.**) - M.,1993 - 44p.

В работе рассматривается нерелятивистское приближение в задаче двух тел, взаимодействующих посредством неабелевого калибровочного поля для случая SU(2) группы. Показано, что задача может быть сведена к решению системы квазилинейных уравнений эллиптического типа. Приведены свойства регулярных решений. В найденном самосогласованном поле рассмотрено поведение кварковых полей источников. Обсуждены результаты численных расчетов. На основе полученных точных решений анализируется задача двух тел.

In non-relativistic approximation SU(2) gauge group Yang-Mills equations with two quasistatical sources are shown to be reduced to the set of elliptic equations analogous to that of non-linear electrodynamics. The regular solutions of obtained equations are discussed. We give an interpretation of behavior of quark field in such gluon configuration. The results of numerical calculations are discussed and the two-body problem is analyzed.

Fig. - 6, ref - 20

© Институт теоретической и экспериментальной физики, 1993

*) Fakultät für Physik, Universität Bielefeld, D-48 Bielefeld 1, Germany

**) NPO "ELTEST", Moscow 117334, Russia

Introduction

The present paper is devoted to a consideration of a system consisting of two classical non-relativistic particles interacting through a classical non-abelian field; to some extent such a system is a model of a bound state of two heavy quarks. It is well-known from electrodynamics that at small particles velocities a regular self-consistent scheme of an approximate description of the dynamics of the particles and surrounding fields can be elaborated with the help of an expansion in v/c. In our considerations we will try to follow electrodynamics as close as possible. As it was shown in [1, 2], for the SU(2) gauge group the problem of two bodies can be considerably simplified by some special substitution. Restricting ourselves with the zeroth-order terms in v/c and neglecting the signal retardation (this is equivalent to a restriction of the size of the system under consideration) we can reduce the Yang-Mills equations to a set of elliptic equations. The latter is much simpler for analysis and numerical calculations. The chromoelectric and chromomagnetic potentials appear to be time-dependent; this dependence describes a rotation of some isotopic vector field with the same frequency at each point of the configuration space.

Classical solutions of the Yang-Mills equations with an external source are being actively studied since the mid-70's [3, 6]. We will touch shortly the main results concerning the problem under consideration.

In [3] Sikive and Weiss have shown the existence of a variety of Coulomb-like solutions. Such solutions were found to accompany nearly any configuration of static sources. Gauge invariance appears to be enough to construct an operator (generally speaking, singular) transforming any static charge density into a charge density aligned with some given direction in isotopic space.

Then the nonlinear terms in the Yang-Mills equations immediately disappear due to a disappearance of the commutators contribution, and the equations themselves become a set of independent Maxwell equations. In contrast to electrodynamics, one can even change the "sign" of a charge switching between attraction and repulsion; transformations between configurations with different values of Casimir generators (for example, total isospin and energy) are also possible. Having used this property, Sikive and Weiss constructed fully screened solutions which could possess arbitrarily small energy. It seems quite natural from here that arbitrary initial fields must evolve (when a radiation is taken into account) into a solution with the smallest possible energy – a screened one.

In the same year a work by Khriplovich [2] appeared in which a set of equations leading to "gluostatic fields" was proposed for the case of two static colored sources. Some properties of the solutions were discussed, a singularity was shown to exist in the case of point-like sources and qualitative changes in the solutions at large α_s were noted. Unfortunately, this work was poorly cited. One of the reasons may be a "shadowing" by the works [3] which established a general method of a gauge transformation for obtaining solutions of Yang-Mills equations.

In Mandula's work [4] an instability was found which leads to the effect of a full screening of a singular Coulomb source when its charge exceeds some critical one $Q_l \geq l + 1/2, l = 1, 2, \ldots$. Such solutions were interpreted as a falling of massless gluons onto a center similar to the falling of electron-positron pairs, created from the vacuum, in quantum electrodynamics. The consideration was based on a substitution $[e, \bar{e}] = \lambda\, \bar{e}$ in a linearized set of Yang-Mills equations with a Coulomb source $A_0 = q\epsilon/r$. Such a substitution selects some modes of field fluctuations $\bar{e}$ which define solution vectors. An essential point in this consideration was the assumption that these eigenvalues λ were real. As it will be seen later, in our notations this condition is equivalent to the existence of real isotopic vectors $\tilde{b}$ such that the identity $\tilde{a} \times \tilde{b} = \lambda \tilde{b}$ is valid. But the latter is impossible because of the orthogonality of vectors $\tilde{a} \times \tilde{b}$ and $\tilde{b}$. That is why no such modes appear in our calculations. Creation and annihilation operators with necessary projective properties may exist only for imaginary λ's.

Jackiw, Jacobs and Rebbi [5] have shown that Coulomb-like solutions of Yang-Mills equations similar to monopole ones may exist in a field of nontrivial topology. The energy of such solutions is lower than the energy

of a Coulomb source. Axially-symmetric solutions possessing an additional topological invariant were obtained. Such solutions remain of great physical interest up to now (see e.g.[6]).

In the present work moveable sources are considered. For this reason gauge transformations used in [3] become time-dependent and the problem is to extract radiative contributions. We will follow electrodynamics and will find, first, a modification of the Yang-Mills equations which corresponds to electro- and magnetostatics and gives a description of the system evolution when the radiation has already gone away. Having obtained such field configurations, we will then consider a motion of the particles which create the field itself. We expect such subsequent approximations to remove the non-uniqueness in correspondence between a configuration of charges and their interaction energy which was noted by Sikive and Weiss. Taking radiation into account, we could trace an evolution of our system into a state with the lowest energy.

In a number of articles [7] a stochastic behaviour of solutions of the Yang-Mills equations was predicted. In the above works nonlinear equations of field dynamics were studied for the case when they were independent of spacial variables. We also note that a consideration of SU(N) gauge group instead of SU(2) and continuous charge distributions instead of point-like sources give more and more complicated picture of a motion of charges in an isotopic space which can, generally speaking, become stochastic.

The paper is organized as follows: In Sect.1 SU(2) gluostatics equations are derived for the case of two point-like sources. Their regular solutions are analyzed and some general properties are discussed in Sect.2. In Sect.3 an interpretation of the obtained solutions for quark fields is given. Sect.4 is devoted to the results of numerical calculations. In Sect.5 the dynamics of a system of interacting particles and the radiation from it are considered.

1 "Gluostatics" equations

Let us now start from the Yang-Mills equations with an external source:

$$D^\mu F^a_{\mu\nu} = -J^a_\nu, \tag{1.1}$$

where:

$$D^\mu = \partial^\mu - igA^\mu,$$

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{g}{i}[A_\mu, A_\nu],$$

$$A_\mu = A^a_\mu T_a,$$

T_a — is the algebra generator.

In the case under consideration $J^a_\nu = \bar{\psi}_k \gamma_\nu T^a_{kl} \psi_l$ is the current density of the external source, ψ is the wave function of a matter field. The four-spinor ψ satisfies the following Dirac equation:

$$i\gamma^\mu \partial_\mu \psi_k - g\gamma^\mu A^a_\mu T^a_{kl} \psi_l - m\psi_k = 0. \tag{1.2}$$

One should note that in the case of electrodynamics we could describe the motion of the matter (i.e. the sources of the field) by a subsequent transformation from the relativistic Dirac equation and a corresponding wave-function ψ to the non-relativistic Pauli equation and a non-relativistic wave-function φ. Then a quasiclassical expansion at $\hbar \to 0$ gives Hamilton-Yacobi equation for an action and continuity equations for amplitudes. These ones describe the dynamics of the sources of an electromagnetic field. Together with Maxwell equations (1.1) they provide a self-consistent description of the system "matter plus field". As a result, we could find classical trajectories of the motion of two massive non-relativistic objects interacting through an electromagnetic field both in the zero th order in v/c and with higher orders taken into account.

Turning now to Yang-Mills fields, one sees that according to the self-consistency condition

$$D^\mu J^a_\mu = 0, \tag{1.3}$$

a charge which is the source of a field A_μ has to variate in time. Hence, if we consider a set of point-like objects we can not characterize them with some fixed "charges" similar to electrical ones. Instead, we need to consider the "intensity" of a source as some new dynamical variable. Thus, one must necessarily take into account the charge's motion in an isotopic space; at the same time, for heavy particles its variation due to a mechanical motion can be neglected.

Let the SU(2) algebra be considered. Then $T_a = \frac{1}{2}\sigma_a$, $[T_a, T_b] = i\varepsilon_{abc}T_c$, where $\sigma_a (a = 1, 2, 3)$ is a Pauli matrix. One can easily see that a commutator $\frac{1}{i}[A, B]$ in this case can be represented as the product of two vectors in an

isotopic space: $\tilde{A} \times \tilde{B}$, where $\tilde{A} = (A^1, A^2, A^3)$. Eq.(1.1) will be rewritten now in the following form:

$$\partial_\mu \partial^\mu \tilde{A}_\nu - \partial_\nu \partial^\mu \tilde{A}_\mu + g\partial^\mu \tilde{A}_\mu \times \tilde{A}_\nu + g\tilde{A}^\mu \times (\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu) + g^2 \tilde{A}^\mu \times \tilde{A}_\mu \times \tilde{A}_\nu = \tilde{J}_\nu.$$

The condition (1.3) becomes:

$$\partial_\mu \tilde{J}^\mu + g\tilde{A}_\mu \times \tilde{J}^\mu = 0.$$

For small velocities, to the zero th order in v/c some terms in Eq.(1.3) may be neglected:

$$\frac{\partial \tilde{\rho}}{\partial x^0} + g\tilde{\varphi} \times \tilde{\rho} = 0. \tag{1.4}$$

where $\tilde{\rho}$ and $\tilde{\varphi} = \tilde{A}^0$ are isovectors of a charge and a chromoelectric potential, respectively.

Let two point-like sources of intensity $\tilde{P}$ and $\tilde{Q}$ be placed at the points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively. We need to describe "slow" motions of the sources and the corresponding motion of a color field, in which a possible radiation should not present. According to this, we will neglect the signal retardation within the physical system under consideration.

In behalf of comfort we rewrite the Yang-Mills equations in terms of $\tilde{\varphi}$ and $\tilde{\boldsymbol{A}}$

$$\begin{aligned}
&\Delta\tilde{\varphi} - \nabla\dot{\tilde{\boldsymbol{A}}} + g\nabla\tilde{\boldsymbol{A}} \times \tilde{\varphi} + g(\tilde{\boldsymbol{A}} \times [\nabla\tilde{\varphi} - \dot{\tilde{\boldsymbol{A}}}]) + g^2(\tilde{\boldsymbol{A}} \times \tilde{\boldsymbol{A}}) \times \tilde{\varphi} = \\
&\qquad\qquad = \tilde{P}\delta(\boldsymbol{x} - \boldsymbol{x}_1) + \tilde{Q}\delta(\boldsymbol{x} - \boldsymbol{x}_2), \\
&-\ddot{\tilde{A}}_i + \nabla_j\nabla_j\tilde{A}_i - \nabla_i\nabla_j\tilde{A}_j + \\
&+g\nabla_j[\tilde{A}_j \times \tilde{A}_i] + g\tilde{A}_j \times [\nabla_j\tilde{A}_i - \nabla_i\tilde{A}_j] + \\
&+\nabla_i\dot{\tilde{\varphi}} + g\tilde{\varphi} \times \nabla_i\tilde{\varphi} - g\dot{\tilde{\varphi}} \times \tilde{A}_i - 2g\tilde{\varphi} \times \dot{\tilde{A}}_i - \\
&-g^2\tilde{\varphi} \times \tilde{\varphi} \times \tilde{A}_i + g^2\tilde{A}_j \times \tilde{A}_j \times \tilde{A}_i = 0, \\
&\qquad\qquad i, j = 1, 2, 3;
\end{aligned} \tag{1.5}$$

dot means a time-derivative and (...) is a scalar product in Minkovski space.

The current density of external sources in the vector part of the set (1.5) was neglected because it is of the first order in v/c and is necessary only when taking a radiation into account (see Sect.5). Nevertheless, one sees that a vector (or "magnetic") potential $\tilde{\boldsymbol{A}}$ is nonzero: it is generated by

the term $\nabla_i\dot{\varphi} + g\tilde{\varphi} \times \nabla_i\tilde{\varphi}$. In the present publication we will consider the only case of a topologically trivial vacuum $\tilde{\varphi}_{vac} = 0$, $\tilde{A}_{vac} = 0$. A possible influence of instantons will not be considered as well. It seems therefore quite reasonable to suppose that the chromoelectric potential is simply a sum of the contributions of both sources:

$$\tilde{\varphi} = \pi_1\tilde{P} + \pi_2\tilde{Q}, \tag{1.6}$$

where π_1 and π_2 are some new functions of coordinates. The substitution (1.6) naturally appears, for example, when trying to solve (1.5) with a perturbation theory in g.

In the vector equation of the set (1.5) the substitution (1.6) leads to the terms of isotopic structure $\tilde{P} \times \tilde{Q}$, and only to them. In turn, such a field $\tilde{A}$ generates only $\tilde{P}$ and $\tilde{Q}$ components in the scalar equation. Hence we can search for the field $\tilde{A}$ in the following form:

$$\tilde{A} = \mathbf{a}\tilde{P} \times \tilde{Q}, \tag{1.7}$$

where $\mathbf{a}$ is some new vector function. One sees that for SU(2) gauge group the charges of the two particles together with their commutator give a suitable natural basis for this problem.

Let us now make the assumption that the limiting values

$$\lim_{\mathbf{x}\to\mathbf{x}_1} \pi_2(\mathbf{x}) \qquad \text{and} \qquad \lim_{\mathbf{x}\to\mathbf{x}_2} \pi_1(\mathbf{x}),$$

of the functions defined by the substitution (1.6) exist and do not depend on a path along which this limit is taken. Then the self-consistency condition (1.4) becomes

$$\begin{aligned} \dot{\tilde{P}} &= -g\,\tilde{\varphi}(\mathbf{x}_1) \times \tilde{P} = g\,\overset{*}{\pi}_2\,\tilde{P} \times \tilde{Q}, \\ \dot{\tilde{Q}} &= -g\,\tilde{\varphi}(\mathbf{x}_2) \times \tilde{Q} = -g\,\overset{*}{\pi}_1\,\tilde{P} \times \tilde{Q}, \end{aligned} \tag{1.8}$$

where $\overset{*}{\pi}_1 = \pi_1(\mathbf{x}_2)$ and $\overset{*}{\pi}_2 = \pi_2(\mathbf{x}_1)$. It will be shown later that the above assumption is not trivial and is closely related to the very general problem of a point-like charge definition in non-abelian gauge theories.

Equations (1.8) imply that the new functions π_1, π_2 and $\mathbf{a}$ depend upon time only through the positions of the sources $\mathbf{x}_1$ and $\mathbf{x}_2$. In this case,

$\dot{\pi}_1, \dot{\pi}_2$ and $\dot{a}$ can be omitted as the terms of the first order in v/c. Equations (1.8) can be easily seen to describe a precession of isotopic vectors $\tilde{P}$ and $\tilde{Q}$ around some constant vector $\tilde{\Omega} = \dot{\pi}_1\tilde{P} + \dot{\pi}_2\tilde{Q}$. A precession frequency is $g|\tilde{\Omega}|$. Both the lengths of the vectors $\tilde{P}$ and $\tilde{Q}$ and the scalar product of them are conserved ($\dot{\tilde{P}}^2 = \dot{\tilde{Q}}^2 = (\tilde{P}\tilde{Q}) = 0$). This is the very property which enables us to introduce new quasistatical potentials π_1, π_2 and a in a noncontradictory way.

It is interesting to note that such a substitution is possible for all other three-dimensional Lee algebras. In some cases, equations (1.8) can lead not only to oscillating solutions but also to infinitely increasing charges. However, such algebras seem not to have "physical" representations such that $T^+ = T$.

With the substitutions (1.6)-(1.7), equations (1.5) become (simple but long transformations may be seen in [1]):

$$\begin{aligned} \boldsymbol{D}\boldsymbol{D}(\pi - \overset{*}{\pi}) &= \delta, \\ rot\ rot\ \boldsymbol{a} &= \boldsymbol{j}, \end{aligned} \tag{1.9}$$

where :$\boldsymbol{D}_{kl} = \nabla\delta_{kl} + g\ \boldsymbol{a}\ C_{kl}; k, l = 1, 2;$

$$\boldsymbol{j} = g\ (\pi - \overset{*}{\pi})J\boldsymbol{D}(\pi - \overset{*}{\pi})$$

$$\pi^T = ||\pi_1, \pi_2||, \quad \overset{*}{\pi}{}^T = ||\ \overset{*}{\pi}_1, \overset{*}{\pi}_2\ ||, \quad \delta^T = ||\delta(\boldsymbol{x} - \boldsymbol{x}_1), \delta(\boldsymbol{x} - \boldsymbol{x}_2)||.$$

In the above C and J are 2×2-matrix:

$$C = \left\| \begin{matrix} -(\tilde{P}\tilde{Q}) & -(\tilde{Q}\tilde{Q}) \\ (\tilde{P}\tilde{P}) & (\tilde{P}\tilde{Q}) \end{matrix} \right\| \qquad J = \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\|$$

One can easily see that $\dot{C} = 0$, so that potentials π and $\boldsymbol{a}$ are quasistatical indeed. This proves the self-consistency of quasistatical substitutions (1.6)-(1.7).

The set (1.9) is consistent if $div\boldsymbol{j} = 0$. Indeed, applying ∇ to the last equation of the set (1.9) we have:

$$div\ \boldsymbol{j} = (\pi - \overset{*}{\pi})J(\Delta\pi + 2g\ C\boldsymbol{a}\nabla\pi + g\ C(\nabla\boldsymbol{a})(\pi - \overset{*}{\pi})).$$

Taking into account that C^2 is diagonal and using the scalar equation of the set (1.9), one finds the relation

$$div\ \boldsymbol{j} = (\pi - \overset{*}{\pi})J\delta = 0,$$

which shows that the set (1.9) is consistent outside the sources. As usual, one may fix a gauge by an additional condition. In the following we will use the Coulomb gauge $div\ \boldsymbol{a} = 0$ (that is, $div\ \vec{A} = 0$) which is analogous to QED case.

The set of equations (1.9) has an obvious physical sense. A gluon field originated from the two point-like sources is charged itself. For this reason, some self-consistent picture of a charges' and currents' motion takes place. But the total invariant charge of the system is conserved.

2 Properties of solutions.

In the present paper we will consider only "regular" solutions of (Eqs. 1.9). In our interpretation it means that the vector field has no singularities and the column of potentials is the sum of a "Coulomb" singular part and some less singular function in the vicinity of the charges (we will further call these solutions regular though this name may be incorrect in a strict sense):

$$\begin{aligned} \pi_i' &= \pi_{i0} + \pi_i, \\ \pi_{i0} &= -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_i|}, \end{aligned} \tag{2.1}$$

where π_i's are less singular than π_{i0},'s in the vicinity of the points $\boldsymbol{x}_i$ and are vanishing at least as r^{-1} at large distances (no conditions are imposed on their derivatives) [8, 15].

Due to the behaviour expected at infinity, is seems quite natural to use a free Green function of a Poisson equation and to rewrite the expression (1.9) in the form of integral equations

$$\begin{aligned} \pi(\boldsymbol{x}) &= -\int dy G(\boldsymbol{x}, \boldsymbol{y}) \left(2gC\boldsymbol{a}(\boldsymbol{y})\nabla\pi'(\boldsymbol{y}) + g^2C^2\boldsymbol{a}^2(\boldsymbol{y})(\pi'(\boldsymbol{y}) - \bar{\pi}')\right), \\ \boldsymbol{a}(\boldsymbol{x}) &= -\int dy G(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{j}(\boldsymbol{y}), \end{aligned} \tag{2.2}$$

where $G(\boldsymbol{x}, \boldsymbol{y})$ is the Green function of the linear equation

$$\Delta G(\boldsymbol{x}, \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}).$$

One may search for a solution of the set (2.2) in the form of iterative series in a coupling constant g. The iterative series are expected to converge in regions

with excluded ε-vicinities of the points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ at small enough coupling constant, as a consequence of Fredholm theorems for the equations with a polar kernel [9]. In the first iteration, the vector field $\boldsymbol{a}$ originates from a current of two point-like sources

$$\boldsymbol{j}_0 = \boldsymbol{j}((\pi_{10} - \dot{\pi}_{10})\nabla\pi_{20} - (\pi_{20} - \dot{\pi}_{20})\nabla\pi_{10}). \tag{2.3}$$

Expanding the current density in the Legendre polynomials one may calculate arising integrals and obtain the vector field $\boldsymbol{a}$ (see Appendix A). Particularly, in the points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and on the line connecting this points, the field $\boldsymbol{a}$ is constant and is directed along this line. On the whole, the vector field looks like the field of a constant magnet with the poles at the points $\boldsymbol{x}_1, \boldsymbol{x}_2$. Current lines of the first iteration vector field are shown in Fig.1 (see also Appendix B).

Let us get convinced of one important property of the solutions of the set (1.9).

Assertion 1. Solutions of the set (1.9) for the current density $\boldsymbol{j}$ without sources and flows in the whole R^3-space possess a scaling property: if a solution for the charges located at a distance 1 is known , then the solution for the charges located at a distance d can be expressed through the first one by formulae

$$\begin{aligned} \pi(\boldsymbol{x}; d) &= d^{-1}\pi(\boldsymbol{x}/d; 1), \\ \boldsymbol{a}(\boldsymbol{x}; d) &= d^{-1}\boldsymbol{a}(\boldsymbol{x}/d; 1), \end{aligned} \tag{2.4}$$

(in case of D dimensions one should replace d^{-1} by d^{-D+2}).

Proof. Indeed, the set (1.9) may be transformed into the set of integral equations (2.2)); from the latter one sees that the Green function is multiplied by a factor d^{-1} $[d^{-D+2}]$ under such a transformation of scale, the volume element is replaced by d^3 $[d^D]$, the iterative series of the current $\boldsymbol{j}(\boldsymbol{x}; d)$ are transformed into $d^{-3}\boldsymbol{j}(\boldsymbol{x}/d; 1)$ $[d^{-2D+3}]$ (we suppose a convergence of the series here). Substituting these transformations back into the integral equations (2.2) we see that the property (2.4) is valid.

Let us now turn to the behavior of the solutions of the set (1.9) near the charges. To be specific, we choose a coordinate frame so that $\boldsymbol{x}_1 = 0$ and consider solutions in the vicinity of this point. Then one may omit non-singular terms in the set (1.9) after the replacement $\pi_i' \to \pi_{i0} + \pi_i$ (remind

that we are interested in the solutions for which $\dot{\pi}_i = const$) and investigate a linearized set of equations.

Expanding the vector field in vector spherical functions one can show [8, 15] that the solution with nonzero main term ($l = 1$) is quite different from the solutions vanishing at $x = 0$.

2.1 Nonzero vector field.

Let us first consider the case when the vector field is nonzero in the vicinity of the sources. If the charges are placed on the z-axis then the solutions of the set (1.9) in the vicinity of the point $x_1 = 0$ can be written in the following form:

$$\begin{aligned} \pi_1(x) &\simeq -g/4\pi\ a\mu(\tilde{P}\tilde{Q}) + K_1 r\mu + \ldots, \\ \pi_2(x) &\simeq +g/4\pi\ a\mu(\tilde{P}\tilde{P}) + K_2 r\mu + \ldots, \end{aligned} \tag{2.5}$$

where a is the module of the vector field $\mathbf{a} = (0, 0, a)$, $\mu = \cos\theta, \theta$ is an angle counted from the z-axis, $r = |x|$, $K_{1,2} = const$. From here one sees that the limiting value $\lim_{x\to 0} \pi_2(x)$ is dependent on a path along which the limit is taken.

To clarify a physical meaning of the singularity (2.5) in the solution of Eqs.(1.9), let us consider the following model problem . Let the field $\mathbf{a}$ be identical to the magnetic field created by a solenoid. Placing the charges inside the solenoid, we consider only the scalar part of the set (1.9) and determine the column of solutions π.

The field inside a solenoid is approximated by a constant field $\mathbf{a} = (0, 0, a)$. This field satisfies the condition $rot\ \mathbf{a} = 0$; it means that the vector field can be expressed through gradient of some scalar function: $\mathbf{a} = \nabla f$. Using this fact we write the column π in the form of the product of a phase 2×2 matrix-function F and a column y: $\pi = Fy$. The matrix F is then determined by the equations

$$\begin{aligned} D\pi &= F\nabla y, \\ F(0) &= I. \end{aligned}$$

Resolving these equations we obtain

$$F^{\pm 1} = \left\| \begin{matrix} \cos\frac{z}{\lambda} \pm \frac{(\tilde{P}\tilde{Q})}{\Lambda}\sin\frac{z}{\lambda} & \pm\frac{(\tilde{Q}\tilde{Q})}{\Lambda}\sin\frac{z}{\lambda} \\ \mp\frac{(\tilde{P}\tilde{P})}{\Lambda}\sin\frac{z}{\lambda} & \cos\frac{z}{\lambda} \mp \frac{(\tilde{P}\tilde{Q})}{\Lambda}\sin\frac{z}{\lambda} \end{matrix} \right\|, \tag{2.6}$$

where F^{-1} is an inverse matrix, λ is a parameter arising in the problem and having the dimension of a length (!): $\lambda = (g\,|a|\,\Lambda)^{-1}, \Lambda = |\tilde{P} \times \tilde{Q}|$. For the column y we obtain Poisson equation

$$\Delta y - F^{-1} g^2 C^2 a^2 \overset{*}{\pi} = F^{-1}\rho.$$

The column ρ in the right-hand side of this equation represents a charge density. For δ-like charges it is easy to write an explicit expression for π neglecting the contribution of an "influx" charge density near the axis:

$$\begin{aligned} \pi_1 &= -\frac{\Lambda\cos((z-z_1)/\lambda) + (\tilde{P}\tilde{Q})\sin((z-z_1)/\lambda)}{4\pi\Lambda|x-x_1|} - \frac{(\tilde{Q}\tilde{Q})\sin((z-z_2)/\lambda)}{4\pi\Lambda|x-x_2|} \\ \pi_2 &= -\frac{\Lambda\cos((z-z_2)/\lambda) - (\tilde{P}\tilde{Q})\sin((z-z_2)/\lambda)}{4\pi\Lambda|x-x_2|} + \frac{(\tilde{P}\tilde{P})\sin((z-z_1)/\lambda)}{4\pi\Lambda|x-x_1|}. \end{aligned} \tag{2.7}$$

It is easy to see that near the point of charge localization this solution has a singularity similar to that of (2.5). The solution obtained with the help of a phase factor allows to perform a simple regularization: point-like sources can be replaced by spheres with a finite radius. One can see now that the singularity of (2.5)-type is involved by the fact that the scalar part current $j = g\,\pi J\nabla\pi$ penetrates through the spheres. Then it is possible to extend a solution of the whole set (1.9) inside the spheres in a continuous way so that the field a has its initial divergent less form. Thus we see that the singularity (1.9) is an inherent property of the definition of δ-like sources for the scalar part of the set (1.9) provided that the vector field penetrates inside the charges.

One should also note that the expression for the current is invariant with respect to the phase transformation mentioned above:

$$j = g\,\pi J D\pi = g\,y J\nabla y.$$

So the contributions of all iterations are effectively resumed in the scalar part of the set (1.9) near the axis.

A more accurate consideration of the behaviour of solutions of the model problem for the set (1.9) on the basis of its transformation into a set of integral singular equations with a weak singularity was done in [10].

Thus, if the vector field is not equal to zero on the charge then the solutions have the singularity (2.5). This singularity leads us to a contradiction with initial premises of the derivation of the set (1.9) from Yang - Mills equations. Indeed, the values (2.5) determine the frequency of a charges precession in an isotopic space. It means that one supposed a charge density to have the same value on the whole surface of the sphere in the small radius limit. The singularity (2.5) shows that there is a disalignment of charge rotation phases. However, this solution is a satisfactory approximation for small charges and can be used for the investigation of a quark wavepacket evolution (the disalignment is of the order g). From numerical calculations one will see that the solution is applicable, for example, to systems consisting of heavy quarks.

The obtained solution can be improved consistently by a perturbative theory for the phase. At the same time, a number of terms which we need to take into consideration increases rapidly and one should obtain an information in the framework of the approximation (1.9). Such a possibility exists if these equations are applied to the extended sources – for example, to the rings arranged one over the other. In this case the potential has only a logarithmic singularity near the source and we have an uncertainty $\sin \mu r \ \ln r$ instead of $\sin \mu r / r$. It is convenient to use this fact in numerical calculations.

2.2 Zero vector field

Let us now consider another type of local solutions, for which the vector field vanishes near the sources. We will search for the solutions of linearized equations with selected singular terms in the following form:

$$\begin{aligned} \pi_i(\boldsymbol{x}) &= \pi_i^l(r) P_l(\mu), \\ a_1(\boldsymbol{x}) &= a_1^l(r) P_l(\mu), \\ a_2(\boldsymbol{x}) &= a_2^l(r) P_l^{(1)}(\mu), \\ i &= 1,2, \quad l = 1,2,\ldots, \end{aligned} \tag{2.8}$$

where $P_l(\mu)$ are Legendre polynomials, $P_l^{(1)}(\mu)$ are associated Legendre polynomials.

The condition $div\mathbf{a} = 0$ allows to express one of the field components through another, for example, $\mathbf{a}_2$:

$$a_2^l = -\frac{1}{l(l+1)r}\frac{\partial}{\partial r}r^2 a_1^l\,,\ l \neq 0.$$

The linearized set (1.9) is transformed to

$$\begin{aligned}
&\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\pi_1^l - \frac{l(l+1)}{r^2}\pi_1^l - 2\frac{g}{4\pi}\,(\tilde{P}\tilde{Q})\frac{a_1^l}{r^2} = 0,\\
&\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\pi_2^l - \frac{l(l+1)}{r^2}\pi_2^l + 2\frac{g}{4\pi}\,(\tilde{P}\tilde{P})\frac{a_1^l}{r^2} = 0,\\
&\frac{1}{r^2}\frac{\partial^2 r^2}{\partial r^2}a_1^l - \frac{l(l+1)}{r^2}a_1^l + (\frac{g}{4\pi})^2\,(\tilde{P}\tilde{P})\frac{a_1^l}{r^2} - \frac{g}{4\pi}\,\frac{1}{r}\frac{\partial}{\partial r}\pi_2^l - \frac{g}{4\pi}\,\frac{\pi_2^l - \overset{*}{\pi}{}_2^l}{r^2} = 0.
\end{aligned} \tag{2.9}$$

It is a set of Euler equations. We search for its solution in the form of the column

$$X = \left\| \begin{matrix} \pi_1 - \overset{*}{\pi}_1 \\ \pi_2 - \overset{*}{\pi}_2 \\ a_1 \end{matrix} \right\| = r^{\sigma}\chi,$$

where χ is an eigenvector of the characteristic equation of an eigenvalue problem

$$\left\| \begin{matrix} S-L & 0 & -2\frac{g}{4\pi}\,(\tilde{P}\tilde{Q}) \\ 0 & S-L & 2\frac{g}{4\pi}\,(\tilde{P}\tilde{P}) \\ 0 & -\frac{g}{4\pi}\,(\sigma+1) & (\sigma+1)(\sigma+2) - L + (\frac{g}{4\pi})^2\,(\tilde{P}\tilde{P}) \end{matrix} \right\| \chi = 0, \tag{2.10}$$

with $L = l(l+1)$, $S = \sigma(\sigma+1)$.

We obtain the 6-order equation from here:

$$(\rho^2 - \rho - L)((\rho^2 - \rho - L)(\rho^2 + \rho - L + \Pi) + 2\rho\Pi) = 0,$$

where $\rho = \sigma + 1$, $\Pi = (g/4\pi)^2\,(\tilde{P}\tilde{P})$. Removing trivial roots $(\rho^2 - \rho - L) = 0$, we have the 4-order equation:

$$(\rho^2 - \rho - L)(\rho^2 + \rho - L + \Pi) + 2\rho\Pi = 0. \tag{2.11}$$

By a direct substitution one can verify that $\rho_1 = l, \rho_2 = -l-1$ are roots of the equations (2.11). Two other roots are determined from the 2-order equation $\rho^2 - \rho - L + \Pi = 0$:

$$\rho_{3,4} = \frac{1}{2} \pm \frac{((2l+1)^2 - 4\Pi)^{1/2}}{2}.$$

For the powers of the characteristic equation one has:

$$\sigma_1 = l-1,\ \sigma_2 = -l-2,\ \sigma_{3,4} = -\frac{1}{2} \pm \frac{((2l+1)^2 - 4\Pi)^{1/2}}{2}. \qquad (2.12)$$

(similar relations has been obtained in [2, 4]).

We neglect the roots σ_2, σ_4 because of the condition of a regularity of solutions. The root σ_1 satisfies the regularity condition only at $l \geq 1$. For the root σ_3 the regularity requirement leads to the series of critical values of the charge $\tilde{P}$

$$\Pi \leq l(l+1).$$

The case of $l = 1, \sigma_1 = 0$ has already been considered in the section 2.1. The only degenerated case to be considered is $l = 1, a_1 = a_2 \equiv 0$. In this case, as it is easy to see from the set (2.9), regular solutions are absent.

From the above consideration follows

Assertion 2. Solutions which vanish on the charges and have a local structure described by the Legendre polynomial of the first order, $l = 1$, are possible only at $g|\tilde{P}|/4\pi < \sqrt{2}$

One can expect that the assertion 1 is also valid for such solutions since the topology of the vector field corresponds to that of the field obtained with a perturbative theory. The property (2.4) may, however, be not fulfilled for large charges.

At small g the screened vector field near the source is approximately described by the expression

$$a_1^1 = a_0 - a_0\frac{r_0^3}{r^3} + \ldots,\ a_2^1 = \frac{a_0}{2} - \frac{a_0}{2}\frac{r_0^3}{r^3} + \ldots, r \geq r_0.$$

From here one sees that a characteristic length at which the field noticeably changes is of the order of the size of a source. Unfortunately, we have no analytical methods to trace the whole field topology in the case under consideration and to describe changes in the structure of solutions at large

charges. One of the most powerful methods of an investigation of nonlinear equations, the method of an inverse scattering problem seems not to work here. Even a generalization of the scattering problem for the vector part of the set (1.9) is not a trivial step. Therefore a numerical investigation of the system is rather actual. Some numerical results will be given in Sect.4.

To conclude this section, we will discuss a Coulomb limit of the problem when the isotopic vectors $\vec{P}, \vec{Q}$ become parallel. One should note that although the expansion of the field in three basic vectors $\vec{P}$, $\vec{Q}$, and $\vec{P} \times \vec{Q}$ is no longer possible,nevertheless, the main set (1.9) is formally valid with the only simplification that the square of the matrix C^2 is equal to zero. Then the scalar part of the set has the form:

$$\Delta\pi + 2gC\boldsymbol{a}\,\nabla\pi = \delta. \tag{2.13}$$

These equations lead to

$$\Delta(\pi_1\vec{Q} + \pi_2\vec{P}) = \delta_1\vec{Q} + \delta_2\vec{P},$$

and we obtain immediately

$$\pi_1\vec{Q} + \pi_2\vec{P} = -\frac{\vec{Q}}{4\pi|\boldsymbol{x} - \boldsymbol{x}_1|} - \frac{\vec{P}}{4\pi|\boldsymbol{x} - \boldsymbol{x}_2|},$$

That is to say, only the sum of the solutions of the set (2.13) has a Coulomb form . This is valid also for the model problem in a solenoid field.

Thus, we have considered some properties of the regular solutions of the equations obtained. It has been shown that point-like sources may be defined in two different ways because a vector field is present in the problem. Some restrictions on the solutions are found.

3 Matter fields in a given self-consistent gluon field

We need now to interpret the obtained oscillating solutions with respect to quark fields. As it is well-known, one must introduce "large" χ and "small" φ components of a spinor ψ to reduce the relativistic Dirac equation (1.2)

to the Pauli equation

$$i\dot{\varphi}_k = \left[\frac{(p\delta_{km} - gA_{km})(p\delta_{ml} - gA_{ml})}{2m} - \frac{g}{2m}\sigma\, rot\, A_{kl} + gA^0_{kl}\right]\varphi_l, \qquad (3.1)$$

where $p = \nabla/i$. To further simplify the problem we will consider the two limiting cases: (i) when the particles are infinitely heavy and (ii) when the fields $\vec{A}$ and $\vec{A}^0$ are spatially uniform. For more detailed consideration, see [11].

When the mass m is infinitely large one may neglect the contributions of the momentum p and the vector field $\vec{A}$ in Eq.(3.1). Introducing an isotopic column $\varphi^T = \|\varphi_1, \varphi_2\|$ one can rewrite eq.(3.1) as

$$\dot{\varphi} = ig\Pi\varphi, \qquad (3.2)$$

where matrix Π is defined in the following way: $\Pi = (\pi_1 P_k + \pi_2 Q_k)T_k$, T_k is a group generator.

Let us also define three orthogonal isotopic vectors $\vec{E}_i$, $i = 1, 2, 3$ such that $\vec{E}_i \times \vec{E}_j = \varepsilon_{ijk}\vec{E}_k$ where ε_{ijk} is an antisymmetric tensor, $\varepsilon_{123} = 1$. Then one can express the vectors $\vec{P}$ and $\vec{Q}$ through initial data $\vec{P}'$, $\vec{Q}'$, and phase ξ as

$$\vec{P} = P_\perp[\cos(\omega x_0 + \xi)\vec{E}_1 + \sin(\omega x_0 + \xi)\vec{E}_2] + P_\parallel \vec{E}_3,$$
$$\vec{Q} = Q_\perp[\cos(\omega x_0 + \xi)\vec{E}_1 + \sin(\omega x_0 + \xi)\vec{E}_2] + Q_\parallel \vec{E}_3,$$

where $\omega = g|\vec{\Omega}|$, $\vec{\Omega} = \overset{*}{\pi}_1 \vec{P}' + \overset{*}{\pi}_2 \vec{Q}'$; $P_\parallel = (\vec{P}'\vec{E}_3)$; $Q_\parallel = (\vec{Q}'\vec{E}_3)$; $P_\perp, Q_\perp$ are components of the vectors $\vec{P}', \vec{Q}'$ transversal to the axis of precession. In such a notation, the matrix Π can be rewritten as

$$\Pi = \begin{Vmatrix} a & \overset{*}{b}\exp(-i\omega x_0) \\ b\exp(+i\omega x_0) & -a \end{Vmatrix},$$

where $a = \frac{D_\parallel}{2}$, $b = \frac{\vec{P}_\perp \pi_1(n)e^{i\xi} + \vec{Q}_\perp \pi_2(n)e^{i\xi}}{2}$, $\vec{D} = \pi_1(n)\vec{P} + \pi_2(n)\vec{Q}$. To regularize the problem, we substitute the sources with some small spheres; then the functions $\pi_i(n)$ are defined on the surfaces of these spheres.

From the explicit expression for the matrix Π it is easy to see now that Eq.(3.2) is quite equivalent to the equation of a resonant spin precession in an external time-periodic magnetic field [12]. With the substitution

$$\varphi_1 = \exp(-i\omega x_0/2)v_1\ ,\ \varphi_2 = \exp(i\omega x_0/2)v_2, \qquad (3.3)$$

one has the following equation with constant coefficients for the column v:

$$\dot{v} = i\,V v, \tag{3.4}$$

where

$$V = \begin{Vmatrix} ga + \omega/2 & g\,\overset{*}{b} \\ gb & ga + \omega/2 \end{Vmatrix}. \tag{3.5}$$

Denoting corresponding eigenvalues $U_\pm$, one finds from eq.(3.4) :

$$U_\pm = \pm\frac{g}{2}|\tilde{D} + \tilde{\Omega}|.$$

The solution of the set (3.4) is then given by the following expression:

$$v_1 = \exp(iU_+x_0)\,c_+ + \exp(iU_-x_0)\,c_-$$
$$v_2 = \exp(iU_+x_0)\,d_+ + \exp(iU_-x_0)\,d_-,$$

with $d_\pm = c_\pm \dfrac{P_\perp \pi_1(\mathbf{n})e^{i\xi} + Q_\perp \pi_2(\mathbf{n})e^{i\xi}}{D_\parallel + \Omega \pm |\tilde{D} + \tilde{\Omega}|}$.

With the help of the solution obtained we can now select initial data which are consistent with the gluonic fields (the latter were shown earlier to have Z_3 component which is constant in time). This is the case when the solutions of the Yang-Mills and Dirac equations do not contradict each other because time dependence becomes the same. Turning to the explicit form

$$Z_3 = \varphi_1\,\overset{*}{\varphi}_1 - \varphi_2\,\overset{*}{\varphi}_2,$$

and rewriting coefficients $c_\pm, d_\pm$ as

$$c_+ = \rho e^{i\alpha};\ c_- = re^{i\beta};\ d_+ = mc_+;\ d_- = nc_-,$$
$$m = \mu e^{i\sigma};\ n = \nu e^{i\tau}.$$

one has the following expression for Z_3 component:

$$Z_3 = \rho^2(1-\mu^2) + r^2(1-\nu^2) +$$
$$+2\rho r[\cos((U_+ - U_-)x_0 + \alpha - \beta) - \mu\nu\cos((U_+ - U_-)x_0 + \alpha - \beta + \sigma - \tau)].$$

It can be easily seen from the above equation that the non-oscillating Z_3 component corresponds to such a choice of initial data that either ρ or r is

zero. It means a fixing of the projection of an isospin of our particles onto the axis of precession at initial time. All the other initial data describe mixed states which include both isospin projection components oscillating with a frequency different from that of the gluonic field precession.

To be specific, let us put $r = 0$. Then another two charge components become

$$Z_1 = \varphi_2 \dot{\varphi}_1 + \varphi_1 \dot{\varphi}_2 = 2\rho^2 \mu \cos(\omega x_0 + \sigma),$$
$$Z_2 = i\varphi_1 \dot{\varphi}_2 - i\varphi_2 \dot{\varphi}_1 = 2\rho^2 \mu \sin(\omega x_0 + \sigma).$$

The solution shows that the isospin of the particle behaves like a spin in an external magnetic field: it periodically changes its orientation with a frequency $U_\pm$ which is different from $\tilde{\Omega}$. An isospin-flip probability depends on $\tilde{\Omega}$ in a resonant way.

For better physical interpretation, let us turn now to the case of finite masses. We will assume for simplicity that the fields π_1 , π_2 a are nearly uniform in the vicinity of the sources. Then the solution may be found as a superposition of plane waves:

$$\varphi(x, x_0) = \int dp \exp(ipx) \varphi'(p, x_0).$$

In this case eq.(3.4) becomes:

$$\dot{\varphi}' = i V \varphi',$$

where

$$V = \begin{Vmatrix} \alpha_+ & \dot{\gamma} \exp(-i\omega x_0) \\ \gamma \exp(+i\omega x_0) & \alpha_- \end{Vmatrix},$$

$$\alpha_\pm = \frac{p^2}{2m} \pm \frac{g\Lambda_\parallel}{2m}(pa) + \frac{g^2}{2m}\frac{\Lambda^2}{4}a^2 \pm \frac{gc}{2}D_\parallel,$$

$$\gamma = -\frac{g\Lambda_\perp}{2m}(pa)e^{i\zeta} + gc\,\frac{\tilde{P}_\perp \pi_1(n)e^{i\xi} + \tilde{Q}_\perp \pi_2(n)e^{i\xi}}{2}.$$

An isovector $\tilde{\Lambda}$ and a phase $\zeta = \xi - \pi/2$ in the above are defined analogous to those for $\tilde{P}$ and $\tilde{Q}$ (see eq.(3.2)): $\tilde{\Lambda} = \tilde{P} \times \tilde{Q}$, $\Lambda = |\tilde{\Lambda}|$. The explicit form for $\tilde{\Lambda}$ is

$$\tilde{\Lambda} = \Lambda_\perp[\cos(\omega x_0 + \zeta)\tilde{E}_1 + \sin(\omega x_0 + \zeta)\tilde{E}_2] + \Lambda_\parallel \tilde{E}_3,$$

with $\Lambda_{\parallel} = 0$.

Using again the substitution (3.3) for the components of φ':

$$\varphi'_1 = \exp(-i\omega x_0/2)v_1 \ , \ \varphi'_2 = \exp(i\omega x_0/2)v_2,$$

one has the equation similar to (3.4):

$$\dot{v}' = i\,V'v', \tag{3.6}$$

where the matrix V' in the right hand side is time-independent:

$$V' = \begin{Vmatrix} \alpha'_+ & \overset{*}{\gamma} \\ \gamma & \alpha'_- \end{Vmatrix},$$

$\alpha'_\pm = \alpha_\pm \pm \omega/2$.

Solutions of the above equation have the following form

$$v'_1(p) = \exp(iU_+(p)\ x_0)c_+(p) + \exp(iU_-(p)\ x_0)c_-(p)$$
$$v'_2(p) = \exp(iU_+(p)\ x_0)d_+(p) + \exp(iU_-(p)\ x_0)d_-(p),$$

where

$$U_\pm(p) = \frac{p^2}{2m} + \frac{g^2}{2m}\frac{\Lambda^2}{4}a^2 \pm \frac{g}{2}|\tilde{D} + \tilde{\Omega} - \frac{(pa)\tilde{\Lambda}}{mc}|,$$

and

$$d_\pm(p) = c_\pm(p)\,\frac{P_\perp\pi_1(n)e^{i\varphi} + Q_\perp\pi_2(n)e^{i\varphi} - (pa)/mc\tilde{\Lambda}_\perp e^{i\varphi}}{D_\parallel + \Omega \pm |\tilde{D} + \tilde{\Omega} - (pa)/mc\tilde{\Lambda}|}. \tag{3.7}$$

(one can see that the denominator in eq.(3.7) is never singular because the vector $(pa)/mc\tilde{\Lambda}$ is orthogonal to $\tilde{D} + \tilde{\Omega}$). In accord with [13], for fermions (in contrast to bosons) only oscillating solutions are possible.

Again, we can easily select initial data which provide constant Z_3 component of the charge as it was made in the previous case $m \to \infty$. Equations of this type appear in the problem of fermion pairs creation by a time-dependent external field [13]. In this sense our solutions might be interpreted as a creation of the quarks with a given isospin projection by an external gluon field.

4 Numerical results

Numerical calculations on the basis of a finite differences method were performed to solve nonlinear equations (1.9). Due to axial symmetry, all the quantities may be found as functions of ρ and z only. Correspondingly, the functions π_1 , π_2 and a_ρ , a_z were considered as defined on a nonuniform lattice inside some "large" cylinder. Because of the scaling property (2.4) it was enough to make calculations for a fixed distance between particles. The particles themselves were approximated by some "small" charged cylinders. Such an approximation is quite sufficient to simulate "point-like" sources and does not influence the results. The height and the radius of this "particle" was chosen to be 0.1 (in the units for which the distance between the sources is equal to 1) while the size of the outer ("infinitely large") cylinder was equal to 10.

Vector function $\boldsymbol{a}$ was found as a solution of the following boundary problem:

$$\begin{aligned} rot\, rot\, \boldsymbol{a} &= \boldsymbol{j}, \\ \boldsymbol{a}|_G &= 0, \end{aligned} \tag{4.1}$$

where G means boundary condition at "infinity", that is, on the surface of the large cylinder.

In contrast, the function π_1 is obtained as a result of solving two different boundary problems. First of them is the Neumann problem outside the sources:

$$\begin{aligned} &\boldsymbol{DD}(\pi_1 - \overset{*}{\pi}_1) = 0, \\ &\left.\frac{\partial \pi_1}{\partial \boldsymbol{n}_1}\right|_{G_1} = C, \\ &\left.\frac{\partial \pi_1}{\partial \boldsymbol{n}_2}\right|_{G_2} = 0, \\ &\pi|_G = 0, \end{aligned} \tag{4.2}$$

where $\boldsymbol{n}_i$; $i = 1, 2$ are unit vectors orthogonal to the surfaces of "small" cylinders G_i, $C$is a constant providing that the total charge of the cylinder is equal to unity. $\overset{*}{\pi}_1$ is taken to be an average of π_1 over the surface of

the second source. The second problem is the Dirichlet problem inside the sources:

$$\boldsymbol{DD}(\pi_1 - \tilde{\pi}_1) = 0,$$
$$\pi_1(\boldsymbol{n}_1) = \pi'_1(\boldsymbol{n}_1),$$
$$\pi_1(\boldsymbol{n}_2) = \pi'_1(\boldsymbol{n}_2),$$

where $\pi'_i(\boldsymbol{n}_i)$ are the functions found from the above Neumann problem. Equations for the function π_2 are similar except in so far as the indexes are correspondingly interchanged.

One sees that the boundary conditions (4.1),(4.2) correspond to the case of penetration of the vector field into the sources (see Sect. 2.1). Let us call them P-type boundary conditions. The case considered in Sect. 2.2 corresponds to boundary conditions of another type (let us call them F-type conditions) when the vector field flows around the sources. In this case the vector field outside the sources can be expressed through given longitudinal $(\boldsymbol{n}\boldsymbol{a})$ and transversal $\boldsymbol{n} \times \boldsymbol{a}$ components and their derivatives on the surfaces of the sources. The F-type conditions will not be considered in the present publication.

One could also try to apply another boundary conditions at the surfaces of small cylinders. For instance, Dirichlet problem seems to be more natural for the functions π_i outside the sources. But a more detailed consideration [10] shows that in this case serious difficulties appear which do not allow to properly approximate the sources themselves.

One more way is to determine a charge density with the help of the discontinuity of a derivative of the functions π_i at the surfaces of the sources:

$$\left[\frac{\partial \pi_1}{\partial \boldsymbol{n}_1}\right] = \frac{\partial \pi_1^+}{\partial \boldsymbol{n}_1} - \frac{\partial \pi_1^-}{\partial \boldsymbol{n}_1} = C, \tag{4.3}$$

where $\partial \pi_1^+ / \partial \boldsymbol{n}_1$, $\partial \pi_1^- / \partial \boldsymbol{n}_1$ mean derivatives from outside and inside the surface, respectively. This way seems to be the most correct one, in another cases a gluonic component introduces some uncontrolled contribution into a charge density (see eq.(5.2)). But under such conditions the problem becomes equivalent to diffraction problems and requires much more time for computations. Estimates show that (4.2) gives quite sufficient approximation of the problem.

The above problem (4.1)-(4.2) was solved by iterations with an empirical choice of optimal iteration steps [14]. The equations for the functions π_i were solved first at fixed $\boldsymbol{a}$, then $\boldsymbol{a}$ was obtained at fixed π_i, and then the procedure was reiterated until a sufficient accuracy was reached.

We have considered strong coupling constant values in the range interesting for heavy quarkonium physics: $\alpha_s \geq 0,1$, $\alpha_s = g^2/4\pi$. Typical curves are shown in Fig. 2-4. Let us discuss shortly the results obtained.

One might expect from the set (1.9), that for a small coupling constant the main contributions will be given by the Coulomb solution for the scalar part of the set and the first iteration of the vector field $\boldsymbol{a}$ which is linear in α_s. This linear dependence should be changed when the gluon field contribution (see eq.(5.2)) becomes comparable with that of the Coulomb part. In Fig.5 one sees, indeed, a linear growth of the field $\boldsymbol{a}$ at small α_s (the field at the axis of the system is shown, each curve corresponds to a given value of the angle α between the vectors $\bar{P}$ and $\bar{Q}$). The small coupling approximation is seen to remain valid for the longest time at $\alpha = \pi/2$, until approximately $\alpha_s = 0,6$.

The gluon field contribution can be estimated as (see Appendix C)

$$V = \int_{R^3} \left[-\pi \frac{JC}{2} (\delta + g^2 \Lambda^2 \boldsymbol{a}^2 \ \mathring{\pi}) + g^2 \Lambda^2 \boldsymbol{a}^2 \ \mathring{\pi} \ \frac{JC}{2} \ \mathring{\pi} + g \ \frac{\Lambda^2}{2} (\boldsymbol{a} \boldsymbol{j}) \right] dx, \tag{4.4}$$

$$\boldsymbol{j} = (\pi - \mathring{\pi}) J \ \boldsymbol{D}(\pi - \mathring{\pi}), \ \Lambda^2 = -C^2,$$

the second term behind δ-function represents an "induced" charge density. The regularized Coulomb contribution into a potential energy is given by

$$V_{coul} = -(\mathring{\pi}_1 + \mathring{\pi}_2)/2 \ (\bar{P}\bar{Q}). \tag{4.5}$$

Some numerical results are shown in Table 1.

α	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
α_s	0,2	0,25	0,5	0,6
g	1,585	1,772	2,587	2,746
A_s	$8,475\cdot10^{-3}$	$9,398\cdot10^{-3}$	$1,809\cdot10^{-2}$	$2,121\cdot10^{-2}$
$-\overset{*}{\pi}_1$	$7,322\cdot10^{-2}$	$7,248\cdot10^{-2}$	$7,905\cdot10^{-2}$	$7,594\cdot10^{-2}$
$-\overset{*}{\pi}_2$	$7,361\cdot10^{-2}$	$7,291\cdot10^{-2}$	$7,940\cdot10^{-2}$	$7,617\cdot10^{-2}$
$-V_{coul}$	$5,191\cdot10^{-2}$	$5,140\cdot10^{-2}$	0	0
V_{gluon}	$9,892\cdot10^{-5}$	$1,225\cdot10^{-4}$	$3,305\cdot10^{-3}$	$7,138\cdot10^{-3}$
$-V_1$	$1,237\cdot10^{-6}$	$1,892\cdot10^{-6}$	$2,915\cdot10^{-4}$	$6,340\cdot10^{-4}$
$-V_1'$	$1,968\cdot10^{-6}$	$2,409\cdot10^{-6}$	$9,278\cdot10^{-5}$	$1,682\cdot10^{-4}$
V_2	$3,644\cdot10^{-5}$	$5,597\cdot10^{-6}$	$1,692\cdot10^{-3}$	$3,987\cdot10^{-3}$
V_2'	$5,799\cdot10^{-6}$	$7,126\cdot10^{-6}$	$5,387\cdot10^{-4}$	$1,057\cdot10^{-3}$
V_3	$9,651\cdot10^{-5}$	$1,188\cdot10^{-4}$	$1,904\cdot10^{-3}$	$3,786\cdot10^{-3}$
V_3'	$1,536\cdot10^{-4}$	$1,513\cdot10^{-4}$	$6,062\cdot10^{-4}$	$1,004\cdot10^{-3}$
V_4	$9,286\cdot10^{-5}$	$1,132\cdot10^{-4}$	$6,308\cdot10^{-4}$	$7,591\cdot10^{-4}$
V_4'	$2,343\cdot10^{-4}$	$2,555\cdot10^{-4}$	$5,033\cdot10^{-4}$	$5,529\cdot10^{-4}$

Here

$$V_{gluon} = V_1 + V_2 + V_3,$$

$$V_1 = -\int dx(g^2\,\tilde{A}^2/2)\,\pi\,JC\,\overset{*}{\pi}\,a^2,\ V_2 = \int dx(g^2\,\tilde{A}^2/2)\,\overset{*}{\pi}\,JC\,\overset{*}{\pi}\,a^2,$$

$$V_3 = \int dx(\tilde{A}^2/2)\,(aj),\ V_4 = \int dx(g\,\tilde{A}^2/2)\,a(\pi - \overset{*}{\pi})\,J\,\nabla\pi,$$

$$V_i' = \frac{2}{g^2\tilde{A}^2}V_i,\ i = 1,2,3,\ V_4' = \frac{2}{g\tilde{A}^2}V_4.$$

One sees that the difference $V - V_{coul}$, as it was expected, becomes noticeable at relatively large α_s. For larger α_s, more accurate boundary conditions are necessary.

It was already mentioned that the P-conditions must be considered as approximate ones because they lead to different precession frequencies at different points of the source. To estimate how important it is, in Fig.6 we show the values of π_i on the surface of our "small" cylinder and corresponding values of $\overset{*}{\pi}$ for $\alpha_s = 0,3$ and zero angle between $\vec{P}$ and $\vec{Q}$. One sees that even

for this choice of parameters the difference in frequencies is not too large. In Fig. 7 one can also see the vector field component A_z for different values of a coupling constant.

Let us now estimate the critical value g_c of a coupling constant g when the "induced" gluon field contribution into an energy-momentum tensor becomes comparable with that of the Coulomb part. Extrapolating linear dependence $a' = ga$ for the whole g-range $(0, \infty)$ and considering a typical gluon contribution (see Appendix C)

$$E = \int d\boldsymbol{x}\ (rot\ \boldsymbol{a})^2 (\bar{P} \times \bar{Q})^2 = g^4\ \frac{V_{gluon}}{(4\pi)^4}\ ,$$

one has

$$g_c = 4\pi \left(\frac{1}{2}\ \frac{\sin(\alpha)}{\cos(\alpha)^2}\ \frac{1}{V_{gluon}} \right)^{1/2}.$$

For $\alpha = \pi/4$, for example, it gives

$$g_c \simeq 4,3,$$

and

$$\alpha_s \simeq g_c^2/4\pi \simeq 1,5.$$

One should note that in this range of g the gluonic contribution is always positive. It therefore shifts discrete energy levels up. It is reasonable to expect that at some g close to g_c the bound states could disappear at all.

5 The mechanical motion of the two particles and radiation

We are ready now to consider the mechanical motion of our coloured charges. Corresponding non-relativistic Hamilton function is given by

$$H = p_1^2/2m_1 + p_2^2/2m_2 + V, \tag{5.1}$$

where V is an interaction energy:

$$V = \int_{R^3} T_{00} d\boldsymbol{x}.$$

Simple transformations (see Appendix C) lead to the following expression:

$$V = \int_{R^3} \left[-\pi \frac{JC}{2} (\delta + g^2 \Lambda^2 a^2 \overset{*}{\pi}) + g^2 \Lambda^2 a^2 \overset{*}{\pi} \frac{JC}{2} \overset{*}{\pi} + g \frac{\Lambda^2}{2} (aj) \right] dx, \tag{5.2}$$

$$j = (\pi - \overset{*}{\pi}) J \, D(\pi - \overset{*}{\pi}).$$

One should note two important points. First, to find the field contribution into the momentum of the system *particles + fields* one has to symmetries the energy-momentum tensor in a standard way similar to electrodynamics. Then the "interaction" terms cancel out, and the above contribution becomes the sum of the contributions of "free" particles and fields.

Second, in electrodynamics, due to a linearity of the Maxwell equations, the Lorenz force for a point-like particle, moving in an external field produced by another particle, coincides with the force obtained from the two-bodies problem. One sees that this is not the case for non-abelian theories. If one identifies the changes of the energy of the field with the changes of the kinetic energy of the particles then the Lorenz force is given by an expression different from that obtained by Wong [16]. The force in this case should be related not to an individual particle but to the whole system of the particles and the field [17], [18] (the authors acknowledge rather helpful discussions with A.I.Alekseev [19].)

Using the self-similarity property (2.4), one can see that the potential energy of interaction is Coulomb-like:

$$V(d) = 1/d \, V(1)$$

where d is the distance between the charges. Excluding self-action energy (the procedure is quite similar to electrodynamics because $\tilde{P}^2 = const$, $\tilde{Q}^2 = const$), one has the following expression for the interaction energy:

$$V = -\frac{\overset{*}{\pi}_1 + \overset{*}{\pi}_2}{2} \frac{(\tilde{P}\tilde{Q})}{|x_1 - x_2|} + \frac{g^2}{2} \frac{I \, (\tilde{P} \times \tilde{Q})^2}{|x_1 - x_2|} = \frac{K}{|x_1 - x_2|},$$

where $I = \int_{R^3} \left[-\pi \, JC \, (g^2 \Lambda^2 a^2 \overset{*}{\pi}) + g^2 \Lambda^2 a^2 \overset{*}{\pi} \, JC \overset{*}{\pi} + g \Lambda^2 (aj) \right] dx$, the quantities $\overset{*}{\pi}$, I are taken for the distance between the particles equal to unity: $|x_1 - x_2| = 1$.

The trajectories of mechanical motion are therefore completely described. As for the inverse problem – an evaluation of the coupling constant g from the known trajectories, its solution is, strictly speaking, impossible. One sees from the expression for the interaction energy that the energy can be the same for different values of g because it depends on the angle between charges in an isotopic space. From the precession equation the picture of vector motion in configuration space may be reconstructed when the particles move on the trajectories. e need to know this fact in order to solve the problem of quark-gluon plasma kinetic.

Let us estimate now the gluonic field precession frequency. It can happen that the latter is much higher than a typical frequency of mechanical motion of the charges. Then one could simplify the equations of motion by averaging them over the fast ocsillations. Unfortunately, this is not the case. For heavy quarkonium-like systems the precession frequency is given by $\omega = gc|\tilde{\Omega}|$, where $\tilde{\Omega} = g\,\tilde{\pi}$, $\tilde{\pi} = 1/4\pi\lambda$, λ is a typical quarkonium size. From here one sees that

$$\omega = \alpha_s c/\lambda = v_0/\lambda \simeq T_{\rm orb}^{-1},$$

that is, the frequencies of the precession and the orbital motion are comparable.

In the previous sections we have found the gluonic fields and the motion of the sources to the zero th order in v/c. Following QED scheme, the next terms should describe a radiation. Let us write a gluonic field as

$$\tilde{A}'_\mu = \tilde{A}^0_\mu + \tilde{A}_\mu$$

with $\tilde{A}_\mu$ for small terms of the order $(v/c)^1$. In this approximation Yang-Mills equations become:

$$\begin{aligned}
&\Delta\tilde{\varphi} - \nabla\dot{\tilde{A}} + g\nabla\tilde{A}\times\tilde{\varphi}^0 + g\tilde{A}\times\nabla\tilde{\varphi}^0 = 0,\\
&-\ddot{\tilde{A}} + \Delta\tilde{A} - \nabla\times[\dot{\tilde{\Phi}} + \dot{\tilde{\varphi}} - \nabla\tilde{A}] - g\frac{d}{d\,x_0}[\tilde{\varphi}^0\times\tilde{A}],\\
&-g\tilde{\varphi}^0\times(\dot{\tilde{A}} - \nabla\tilde{\varphi}) + g\tilde{\varphi}\times\nabla\tilde{\varphi}^0 - g^2\tilde{\varphi}^0\times\tilde{\varphi}^0\times\tilde{A} = \tilde{j},
\end{aligned} \tag{5.3}$$

where $\dot{\tilde{\Phi}} = \dot{\pi}_1\tilde{P} + \dot{\pi}_2\tilde{Q}$. For simplicity, we have also neglected $\tilde{A}^0$ contributions as compared to those of $\tilde{\varphi}^0$; this can be justified for quarkonium-like systems (see Sect.4). One should note that the terms containing $\dot{\tilde{P}}$ and $\dot{\tilde{Q}}$ were already

taken into account in the equations describing the zeroth-order field; on the contrary, in the term $\frac{d}{dx_0}[\vec{\varphi}^0 \times \vec{A}]$ one must put $\dot{\vec{\varphi}}^0 = \pi_1 \dot{\vec{P}} + \pi_2 \dot{\vec{Q}}$.

The suitable gauge-fixing condition is now

$$\partial_\mu \vec{A}^\mu - g\vec{A}^0_\mu \times \vec{A}^\mu = 0.$$

In our new notations it means

$$\dot{\vec{\Phi}} + \dot{\vec{\varphi}} - \nabla\vec{A} + g\vec{\varphi}^0 \times \vec{\varphi} = 0. \tag{5.4}$$

Vector equation of the set (5.3) becomes:

$$-\ddot{\vec{A}} + \Delta\vec{A} + g\dot{\vec{\Psi}} \times \vec{A} - 2g\vec{\varphi}^0 \times \dot{\vec{A}} - g^2\vec{\varphi}^0 \times \vec{\varphi}^0 \times \vec{A} = \vec{j}, \tag{5.5}$$

where $\dot{\vec{\Psi}} = \pi_1\dot{\vec{P}} + \pi_2\dot{\vec{Q}}$. The current $\vec{j}$ is given by the expression

$$\vec{j} = \vec{P}v_1\delta(x - x_1) + \vec{Q}v_2\delta(x - x_2)$$

and can be easily shown to satisfy the self-consistency condition (1.3).

We will restrict ourselves with a small coupling case when $g|\dot{\vec{\Psi}}|$, $g|\vec{\varphi}^0| \ll |\vec{A}|$. Then the radiation is described by a linear equation

$$-\ddot{\vec{A}} + \Delta\vec{A} = \vec{j}. \tag{5.6}$$

Its solution may be written as

$$\vec{A}(x,t) = \frac{1}{c}\int \frac{dy}{|x-y|}\vec{j}(y, t - \frac{|x-y|}{c}). \tag{5.7}$$

At large distances R one has, as usual, $|x - y| = R - yn + \ldots$, for phase therefore: $c\,t - R + yn + \ldots$, where $n = \frac{R}{R}$. Eq.(5.7) thus becomes

$$\vec{A}(R,t) = \frac{1}{cR}\int dy\vec{j}(y,\tilde{t}) + \frac{1}{c\,R}\int dy\frac{yn}{c}\frac{d}{d\,t}\vec{j}(y,\tilde{t}) + \ldots, \tag{5.8}$$

where $\tilde{t} = t - R/c$.

In contrast to electrodynamics, one must now take into account that the charges $\vec{P}$ and $\vec{Q}$ are time-dependent. Using the self-consistency condition $\nabla\vec{j} + \dot{\vec{\rho}} + g\,\dot{\vec{\varphi}} \times\vec{\rho} = 0$, it is easy to obtain

$$\frac{\partial}{\partial_\beta}r_\alpha j_\beta = \vec{j}_\alpha + r_\alpha\nabla\vec{j} = \vec{j}_\alpha - r_\alpha\left[\frac{\partial\vec{\rho}}{\partial\tilde{t}} + g\ \dot{\vec{\varphi}} \times\vec{\rho}\right].$$

Strictly speaking, the last term in the above expression is of the next order in g but we keep it to simplify the following calculations because we may now consider only those changes of charge caused by its mechanical motion. Keeping, as usual, only non-vanishing terms at infinity , one arrives at the following prescription:

$$\tilde{j} \Rightarrow \frac{\partial}{\partial t} \tilde{r} \tilde{\rho}.$$

Multi pole terms can be found in a similar way. In particular, the dipole contribution is

$$\tilde{A}(\mathbf{R}, t) = \frac{1}{c\,R} \frac{d}{dt} \int d\mathbf{y}\, \mathbf{y} \tilde{\rho}(\mathbf{y}, \tilde{t}). \tag{5.9}$$

For two point-like particles one has

$$\tilde{A}(\mathbf{R}, t) = \frac{1}{c\,R} \frac{d}{dt} \tilde{d}(\tilde{t}) = \frac{1}{cR} (\tilde{P}\dot{\mathbf{x}}_1 + \tilde{Q}\dot{\mathbf{x}}_2)(\tilde{t}). \tag{5.10}$$

Simple transformations give the following expression for the intensity of radiation from two particles:

$$d\, I_{kk} = \frac{\delta_{kk}}{4\,\pi\, c^3} \mu^2 \left(\frac{\tilde{P}}{m_1} - \frac{\tilde{Q}}{m_2} \right)^2 (\ddot{\mathbf{r}} \times \mathbf{n})^2 d\Omega.$$

$$k = 1, 2, 3$$

Eq.(5.10) enables us to use the results already obtained in electrodynamics for the bremsstrahlung and the radiation of a charge moving along a finite trajectory. On the other hand, it shows also some differences. For example, it is known from electrodynamics that the system consisting of two particles which have the same charge-to-mass ratio does not radiate in dipole approximation. One sees that this is not the case for gluodynamics.

Let us note once more that the only trivial case of a small coupling was considered. In another cases the dipole formula (5.10) has to be revised. In [18] classical self-action of a non-abelian charge and the radiation of a moving colored particle were considered. "Anomalous" modes of radiation were noted to exist in addition to "Coulomb" ones.

6 Conclusions

Thus, we have considered the two-bodies problem for the non-abelian SU(2) theory with the topologically trivial vacuum: $\tilde{\varphi}_{vac} = \tilde{0}$, $\tilde{A}_{vac} = \tilde{0}$. On the physical level of accuracy it his been shown that for a small coupling constant (for the charges $g^2/4\pi|\tilde{P}| < \sqrt{2}$) the obtained picture almost reproduces electrodynamics. The particles are kept with Coulomb forces and the radiation may be described in a similar way.

We have also noted the difficulty consisting in the fact that, in contrast to electrodynamics, point-like charges may be defined in two different ways depending on whether the vector field penetrates inside the sources or not. In the first case the disalignment of charge phases takes place. It breaks the self-consistency of the approach at large enough time intervals. In the second case the solution, obtained with the help of perturbation theory in g, may exist only for small charges.

The charges' behaviour was shown to be a precession in an isotopic space. The frequency of oscillations is comparable with the rotation frequency in Minkowski space (when the coupling constant lies in the range corresponding to heavy quarkonium, i.e. $\alpha_s \approx 0,1$). So we confirm quantitatively the well established opinion that nonphysical "colour" states take place in the classical theory at small coupling constants. It is interesting to note that non-abelian theory, according to our considerations, describes a complicated resonant system, investigation of which may be of its own scientific interest despite the fact that such states are not observable in the real world.

Modern theoretical considerations suppose that the QCD vacuum is populated with the fields of a nontrivial topology. These ones are believed to lead to the observable colour less states (see e.g.[20]). The interaction between these configurations and the ones obtained by us gives an interesting new class of problems which can be studied with the help of perturbation theory. Then these results can be generalized for the continual integral approach of quantum theory. We plan to consider these questions in the following publication.

Appendix A.

First iteration fields.

The following integral appears in the first iteration:

$$\boldsymbol{A}(\boldsymbol{x}) = \int_{R^3} \frac{d\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|}\left[\left(\frac{1}{|\boldsymbol{e}-\boldsymbol{y}|}-1\right)\frac{\boldsymbol{y}}{|\boldsymbol{y}|^3}-\left(\frac{1}{|\boldsymbol{y}|}-1\right)\frac{\boldsymbol{y}-\boldsymbol{e}}{|\boldsymbol{y}-\boldsymbol{e}|^3}\right],$$

where $\boldsymbol{e}$ is an unit vector directed to the second charge, $\boldsymbol{e} = (0,0,1)$. Detailed calculations may be found in [8], the result (for $x \leq 1$) is

$$\begin{aligned}
&A_\varphi(\boldsymbol{x}) = 0,\\
&A_z(\boldsymbol{x}) = 4\pi - \frac{2\pi}{3}(x+\xi) - \frac{2\pi}{5}\left[x^2 Y_1^0(\boldsymbol{n}_x) - \xi^2 Y_1^0(\boldsymbol{n}_\xi)\right] +\\
&+\sum_{l=2}^{\infty} 2\pi\left[\left(\frac{x^{l-1}}{2l-1}-\frac{x^{l+1}}{2l+3}\right)Y_l^0(\boldsymbol{n}_x) + (-1)^l\left(\frac{\xi^{l-1}}{2l-1}-\frac{\xi^{l+1}}{2l+3}\right)Y_l^0(\boldsymbol{n}_\xi)\right],\\
&A_\rho(\boldsymbol{x}) = \pi x(\frac{x}{5}-1)Y_1^1(\boldsymbol{n}_x) - \pi\xi(\frac{\xi}{5}-1)Y_1^1(\boldsymbol{n}_\xi) -\\
&-\sum_{l=2}^{\infty}\frac{2\pi}{l(l+1)}(x^l Y_l^1(\boldsymbol{n}_x) + (-1)^l \xi^l Y_l^1(\boldsymbol{n}_\xi)) +\\
&+\sum_{l=2}^{\infty} 4\pi\left(\frac{x^{l-1}}{(2l-1)l}+\frac{x^{l+1}}{(2l+3)(l+1)}\right)Y_l^1(\boldsymbol{n}_x) +\\
&+\sum_{l=2}^{\infty} 4\pi(-1)^l\left(\frac{\xi^{l-1}}{(2l-1)l}+\frac{\xi^{l+1}}{(2l+3)(l+1)}\right)Y_l^1(\boldsymbol{n}_\xi),
\end{aligned}$$

where $x = |\boldsymbol{x}|$, $\xi = |\boldsymbol{x}-\boldsymbol{e}|$. The sources are assumed to lie at $z = 0$ and $z = 1$.

Particularly, at z-axis one has $A_z = 4\pi$ for $z = 0$, $z = 1$ and $z = 0.5$ while A_ρ-component is equal to zero.

Physically observable is a chromomagnetic field $\boldsymbol{H} = rot\ \boldsymbol{a}$, not the vector potential $\boldsymbol{a}$ itself. Due to axial symmetry of the problem, the only H_φ-component is different from zero:

$$H_\varphi = \frac{1}{r}\frac{(1-\mu)^{1/2}}{(1+\mu)^{1/2}}\left(1-\frac{1+r}{(1-2r\mu+r^2)^{1/2}}\right),$$

where r and $\mu = \cos\theta$ are spherical coordinates.

The numerical results for H_φ are shown in Fig.8. An interesting feature is that the sources of a gluonic field appear to lie in some "cavity" which contains almost no magnetic field and is surrounded with a relatively thin layer where the magnetic field sharply increases. This looks like a volcano pit. In quantum theory, one might expect this picture to lead to a noticeable anisotropy. Such a behaviour of the solutions may be related to the confinement problem.

Appendix B.

Current lines of j^0.

The current lines are described with the following set of equations:

$$\frac{d\,x}{d\,s} = j^0_x, \qquad \frac{d\,y}{d\,s} = j^0_y, \tag{B.1}$$

where

$$j^0_x = \frac{x}{|\boldsymbol{x}+\boldsymbol{e}|^3}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{e}|}-1\right) - \frac{x}{|\boldsymbol{x}-\boldsymbol{e}|^3}\left(\frac{1}{|\boldsymbol{x}+\boldsymbol{e}|}-1\right),$$

$$j^0_y = \frac{y+e}{|\boldsymbol{x}+\boldsymbol{e}|^3}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{e}|}-1\right) - \frac{y-e}{|\boldsymbol{x}-\boldsymbol{e}|^3}\left(\frac{1}{|\boldsymbol{x}+\boldsymbol{e}|}-1\right),$$

$$\boldsymbol{x} = (x, y, 0).$$

Let us put $e = |\boldsymbol{e}| = 1/2$, $\boldsymbol{e} = (0, e, 0)$ and define new variables u, v:

$$u = \frac{y+e}{x}, \quad v = \frac{y-e}{x},$$

then

$$x = \frac{2\,e}{u-v}, \quad y = \frac{x(u+v)}{2}, \tag{B.2}$$

and

$$\frac{d\,y}{d\,x} = \frac{u\,dv - v\,du}{dv - du}. \tag{B.3}$$

From eq.(B.1) we have

$$\frac{d\,y}{d\,x} = \frac{u(1+v^2)(1-x(1+v^2)^{1/2}) - v(1+u^2)(1-x(1+u^2)^{1/2})}{(1+v^2)(1-x(1+v^2)^{1/2}) - (1+u^2)(1-x(1+u^2)^{1/2})}. \tag{B.4}$$

where x is a function of u and v (B.2). Comparing eqs.(B.3) and (B.4) one sees that

$$\frac{du}{dv} = \frac{(1+u^2)(1-x(1+u^2)^{1/2})}{(1+v^2)(1-x(1+v^2)^{1/2})}.$$

Substitution of $x(u,v)$ gives

$$\frac{du}{dv} = \frac{(1+u^2)(u-v-2e(1+u^2)^{1/2})}{(1+v^2)(u-v-2e(1+v^2)^{1/2})}.$$

Let us now introduce new variables $u = sh\,\alpha$, $v = sh\,\beta$, $e^{\alpha} = \sigma$, $e^{-\beta} = \tau$. Then the above equation becomes

$$-\frac{d\sigma}{d\tau} = \frac{(\sigma+2\tau-\sigma\tau^2)(1+\sigma^2)}{(\tau+2\sigma-\sigma^2\tau)(1+\tau^2)}. \tag{B.5}$$

We will try to find its integrals in the following form:

$$P/Q = const, \tag{B.6}$$

where P, Q are polynomials in σ and τ:

$$P = \sum_{i,j} a_{ij}\tau^i\sigma^j,\ Q = \sum_{k,l} b_{kl}\tau^k\sigma^l.$$

From eq.(B.5) one has

$$-\frac{d\sigma}{d\tau} = \frac{P\,\partial Q/\partial\tau - Q\,\partial P/\partial\tau}{P\,\partial Q/\partial\sigma - Q\,\partial P/\partial\sigma}. \tag{B.7}$$

The matrix equation for a and b now becomes (numerator):

$$\sum_{K,L} a_{KL}\, b_{A+1-K,B-L}\,(A+1-2K)\tau^A\sigma^B = G_{AB}\tau^A\sigma^B,$$

where G_{AB} is the numerator matrix of eq.(B.5). One can see that for symmetric a and b the above equation is enough to satisfy the corresponding denominator equation as well. If, in addition, the matrix a is diagonal, the following integral of eq.(B.5) can be easily found:

$$C = \frac{\sigma^2+\tau^2+\sigma\tau+\sigma^2\tau^2}{1-\sigma\tau}.$$

Turning now back to the coordinate representation, one has current lines which can be seen in Fig.1.

Appendix C.

Energy-momentum tensor

Let us define chromoelectric and chromomagnetic fields as $\tilde{E}_i = -\tilde{F}^{0i}$, $\tilde{H}_i = -1/2\varepsilon_{ijk}\tilde{F}^{jk}$ where, as usual, $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + g\tilde{A}_\mu \times \tilde{A}_\nu$. Using substitution (1.6) and eq.(1.8) one has the following expression for the vector $\tilde{E}$: $\tilde{E} = E_1\tilde{P} + E_2\tilde{Q}$, where $E = -D(\pi - \overset{*}{\pi})$. $\tilde{F}^{ij}$ becomes now $h\,\tilde{\Lambda}$ where $\tilde{\Lambda} = \tilde{P} \times \tilde{Q}$, $h = rot\ a$.

By definition, an energy-momentum tensor is given by the expression

$$T_{00} = \frac{1}{2}(\tilde{E}^2 + \tilde{H}^2) = \sum_{i=1}^{3} T_{ii}.$$

One sees that $\tilde{E}^2 = E^T\ J\ C\ E$, then

$$T_{00} = \frac{1}{2}\,(E^T\ J\ C\ E + h^2\tilde{\Lambda}^2).$$

The other components of the tensor are

$$T_{0i} = 0\ ,\ T_{ij} = -\tilde{E}_i\tilde{E}_j - \tilde{H}_i\tilde{H}_j + \frac{1}{2}\,\delta_{ij}(\tilde{E}^2 + \tilde{H}^2).$$

Let us consider terms corresponding to the chromoelectric field. Integrating them by parts over the whole space and using the identity $J\ C + C^T\ J = 0$, one obtains

$$E^T\ J\ C\ E = -\pi\ J\ C\ (\delta + g^2 a^2\ \Lambda^2\ \overset{*}{\pi}) - g^2\ \Lambda^2\ a^2\ \overset{*}{\pi}\ \ J\ C\ \ \overset{*}{\pi},$$

where δ represents sources. In the above, the condition $\nabla a = 0$ is also taken into account. The second term in the brackets corresponds to an induced charge density, similar term appears in the scalar part of the set (1.9).

The chromomagnetic contribution can be transformed in the same way, in notations of Sect.1 it becomes

$$h^2\tilde{\Lambda}^2 = g\ \Lambda^2(aj) = g\ \Lambda^2 a(\pi - \overset{*}{\pi})\ J\ D\ (\pi - \overset{*}{\pi})$$

. Putting these terms together, we have (see [8] for more details)

$$V = \int_{R^3} \left[-\pi \frac{JC}{2} (\delta + g^2 \Lambda^2 a^2 \ \mathring{\pi}) + g^2 \Lambda^2 a^2 \ \mathring{\pi} \ \frac{JC}{2} \mathring{\pi} + g \ \frac{\Lambda^2}{2} (aj) \right] dx,$$

$$j = (\pi - \mathring{\pi}) J \ D(\pi - \mathring{\pi}), \ \Lambda^2 = -C^2.$$

The only component of the chromomagnetic field $H = rot \ a$ originated from the first iteration of the current density was found in Appendix A. We will now use it to estimate the induced gluonic contribution to the energy-momentum tensor

$$E = \int dx \ (rot \ a)^2 (\bar{P} \times \bar{Q})^2.$$

We have the following integral:

$$V_{gl} = 2\pi \int_0^\infty r^2 \ dr \int_{-1}^1 d\mu \ h^2(r, \mu) =$$

$$= 2\pi \int_0^\infty dr \int_{-1}^1 d\mu \frac{1-\mu}{1+\mu} \left[1 - \frac{1+r}{(1-2r\mu+r^2)^{1/2}} \right]^2.$$

The above expression for V_{gl} must be taken as a limit $lim_{D\to\infty \Delta\to 0} V_{gl}(D, \Delta)$, where

$$V_{gl}(D, \Delta) = \int_0^D dr \ V'_{gl}(\Delta),$$

$$V'_{gl}(\Delta) = 2\pi \int_{-1+\Delta}^1 d\mu \frac{1-\mu}{1+\mu} \left[1 - \frac{1+r}{(1-2r\mu+r^2)^{1/2}} \right]^2.$$

Rewriting it as a sum of two terms

$$V_{gl}(D, \Delta) = V_{gl1}(D, \Delta) + V_{gl2}(D, \Delta),$$

$$V_{gl1}(D, \Delta) = \int_0^1 dr V'_{gl}(\Delta),$$

$$V_{gl2}(D, \Delta) = \int_1^D dr V'_{gl}(\Delta)$$

and taking corresponding limits, one finally has

$$V_{gl1} = 2\pi \left(11 - 12 \ln \ 2 - \pi^2/12 \right) \approx 11,7,$$

$$V_{gl2} = 2\pi \left(20 \ln \ 2 - 5 - \pi^2/4 \right) \approx 40,2,$$

$$\frac{V_{gl2}}{V_{gl1}} \approx 3,4, \ V_{gl} \approx 51,9$$

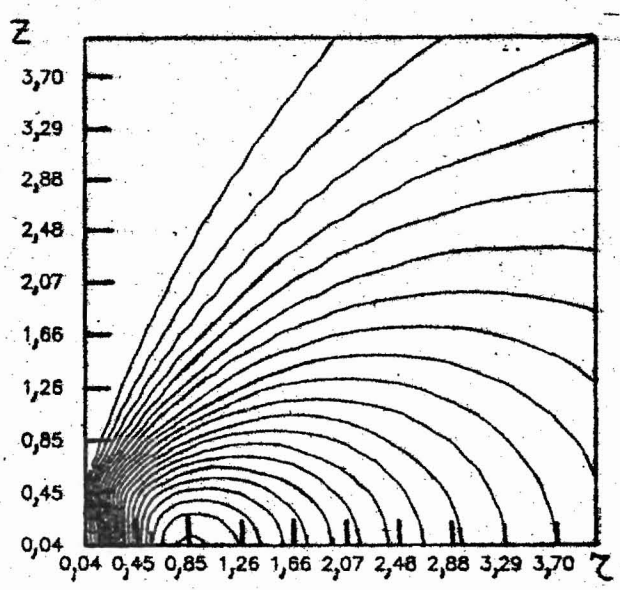


Fig. 1. Current lines of the first iteration j_θ.

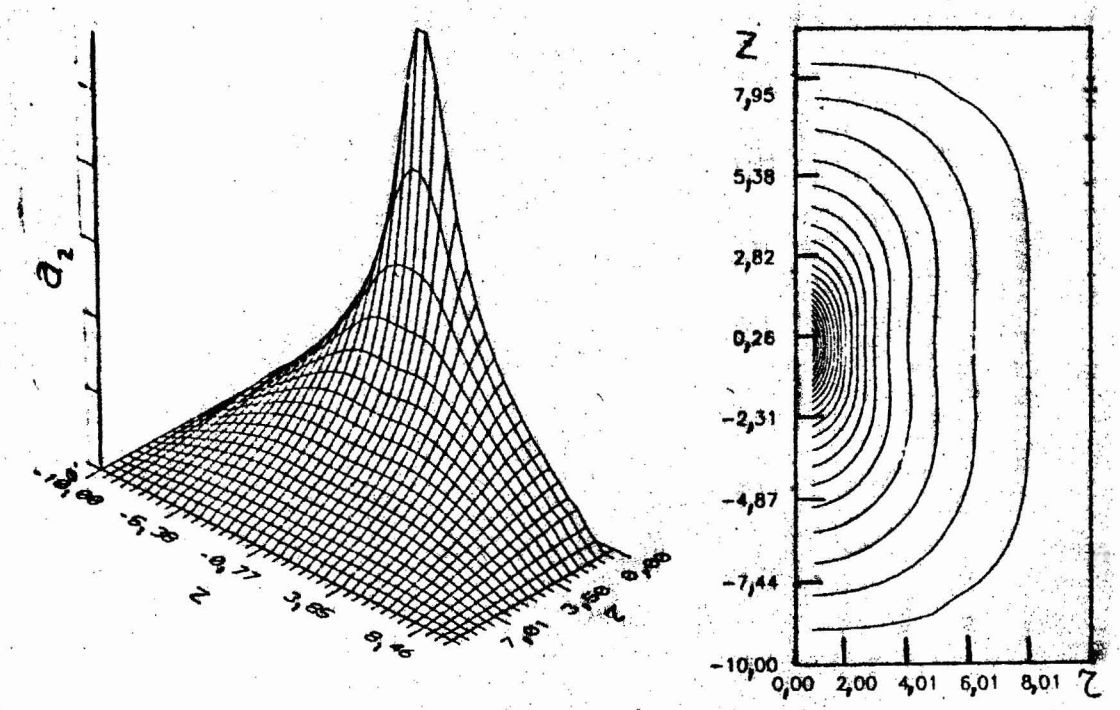


Fig. 2. The a_z component of vector field, with the grid lines.
$a_z = 0.1$, $\alpha = \frac{\pi}{2}$, $\hat{P} = \hat{Q} = 1$.

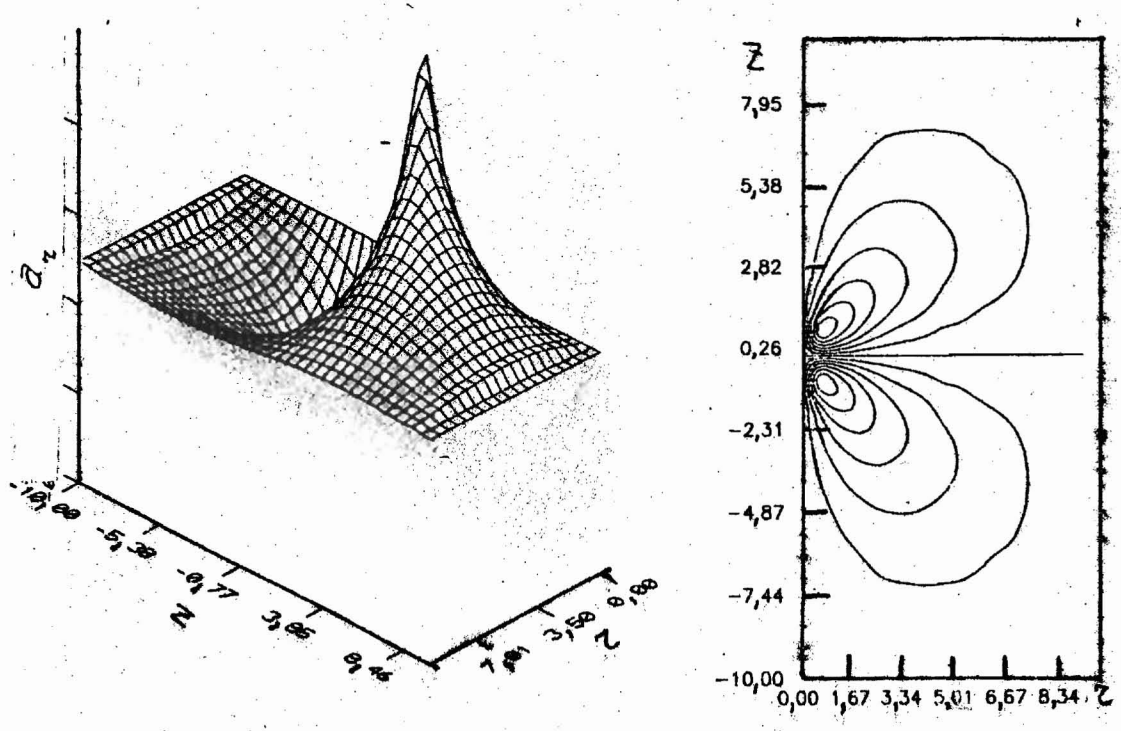


Fig. 3. The a_ρ component of vector field, with the grid lines.
$a_0 = 0.1$, $\alpha = \frac{\pi}{2}$, $\bar{P} = \bar{Q} = 1$.

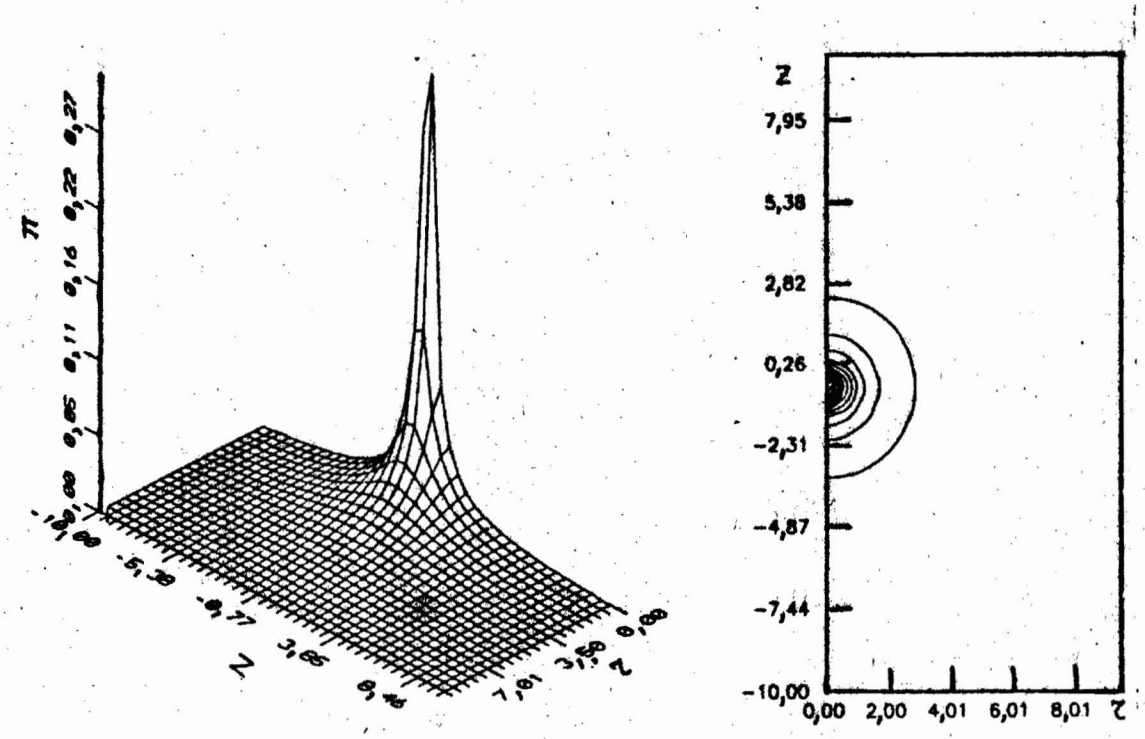


Fig 4. The potential π_1 with the grid lines.
$\alpha_s = 0.1$, $\alpha = \frac{\pi}{2}$, $\bar{P} = \bar{Q} = 1$.

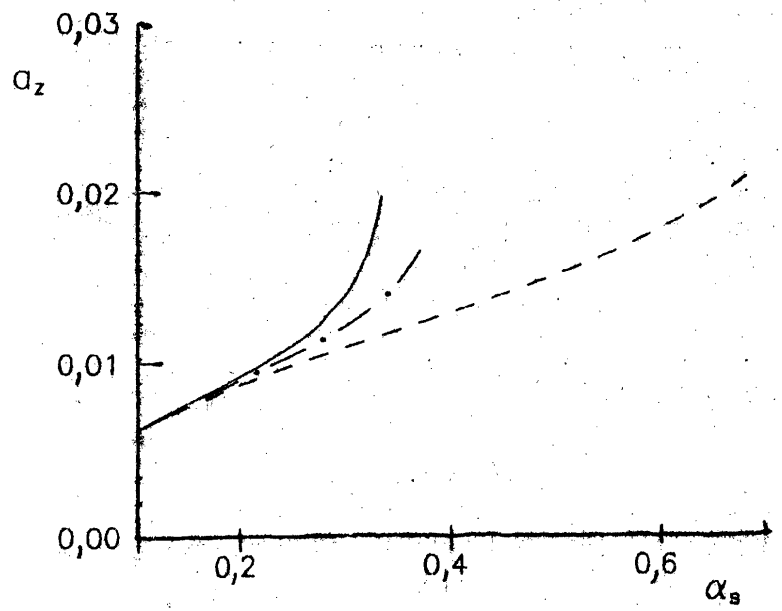
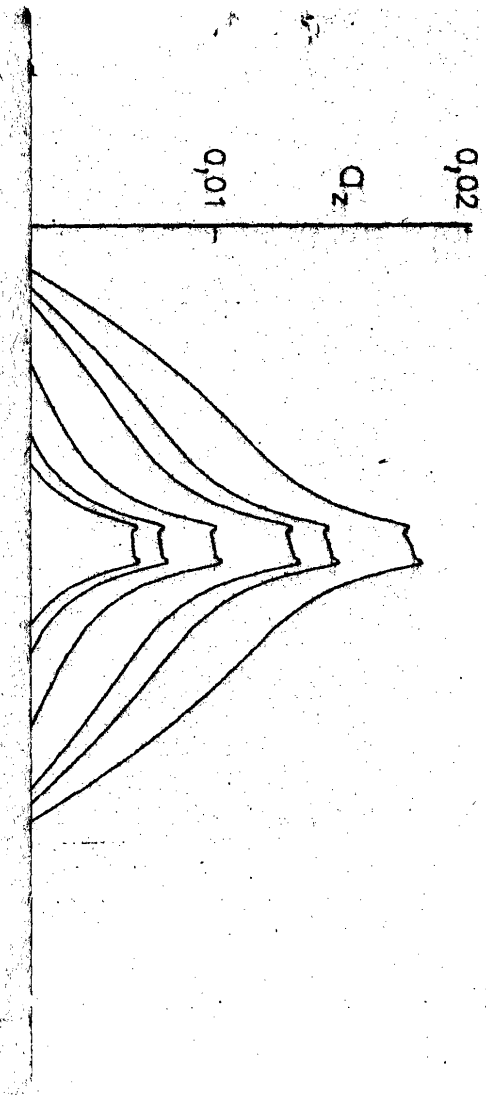


Fig. 5. The field on one of the charges as function α_s. Each curve corresponds to a given value of the angle α between the vectors $\vec{P}$ and $\vec{Q}$. Dashed lines: $\alpha = \pi/2$, point-dashed—$\alpha = \pi/4$, solid—$\alpha = 0$.



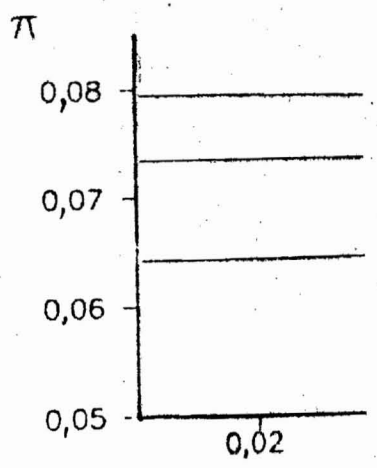
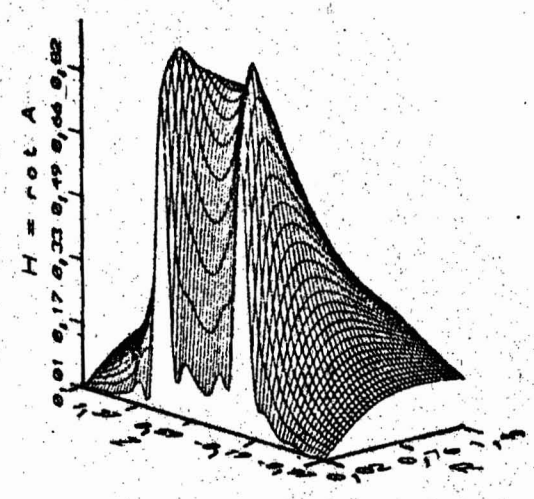


Fig. 6. The values of π_i on
ing values of $\dot{\pi}$ for $\alpha_0 = 0,3$ a



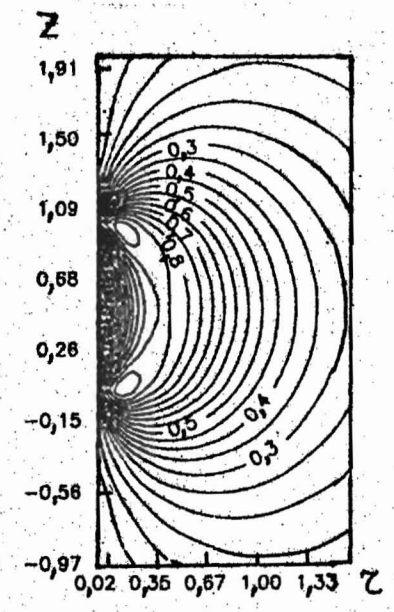


Fig. 8. "Vulcáno pit". The field $H_\varphi = rot\ a$ with the grid lines.

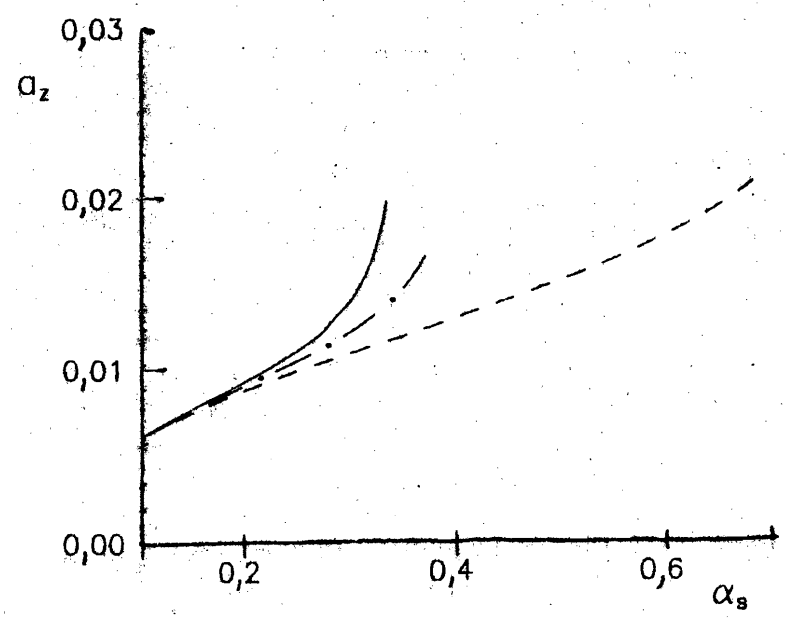


Fig. 5. The field on one of the charges as function α_s. Each curve corresponds to a given value of the angle α between the vectors $\tilde{P}$ and $\tilde{Q}$. Dashed lines: $\alpha = \pi/2$, point-dashed—$\alpha = \pi/4$, solid—$\alpha = 0$.

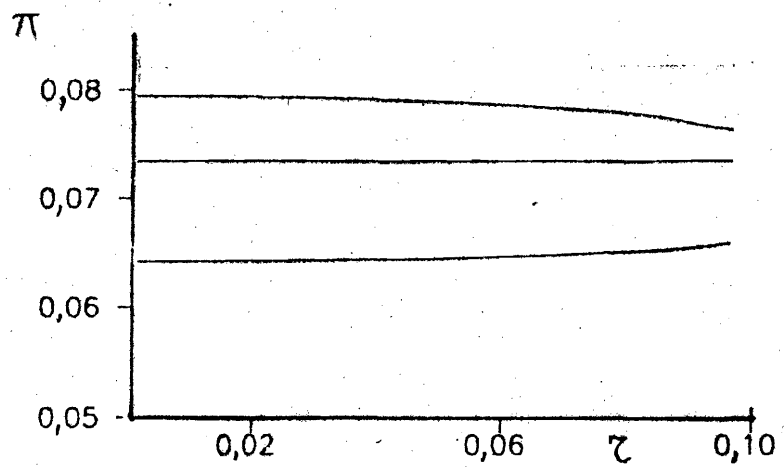


Fig. 6. The values of π_i on the surface "small" cylinder and corresponding values of $\overset{*}{\pi}$ for $\alpha_s = 0,3$ at zero angle between $\bar{P}$ and $\bar{Q}$.

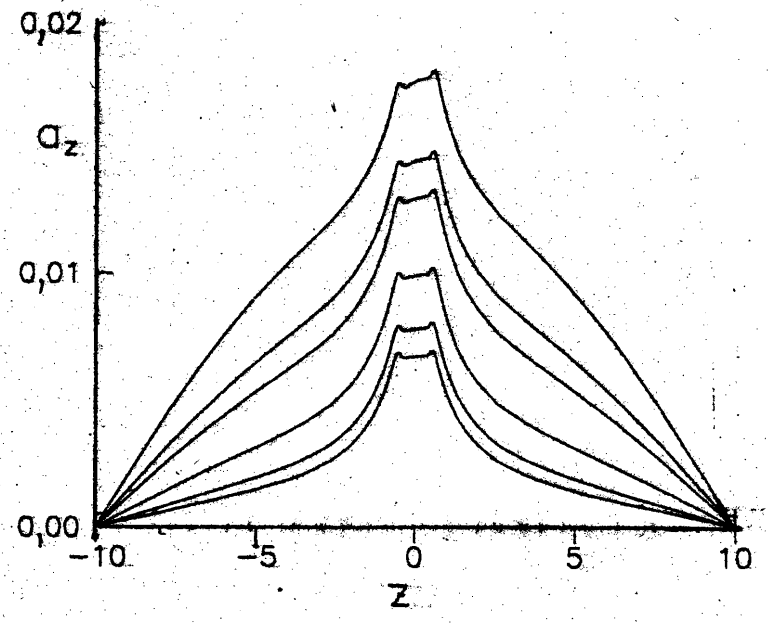


Fig. 7. The vector field component A_z at the axis of the system for different values of a coupling constant.

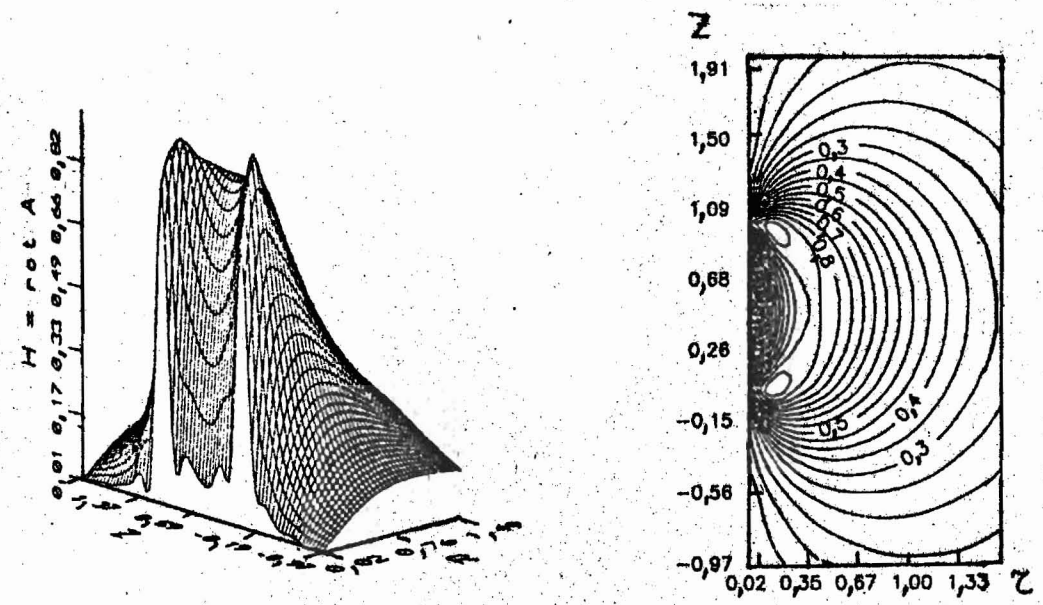


Fig. 8. "Vulcáno pit". The field $H_\varphi = rot\ a$ with the grid lines.

References

[1] V.V. Goloviznin, S.V. Molodtsov, A.M. Snigirev, Preprint ITEP -136, Moscow, 1990, [in Russian]; Sov. J. Nucl. Phys. 55, P. 1117 (1992).

[2] I.B. Khriplovich//JETP. V. 74, P. 37(1978).

[3] P. Sikive, N. Weiss//Phys. Rev. D 18, P. 3809 (1978); Phys. Rev. Lett. 40, P. 1411 (1978).

[4] J. Mandula//Phys. Rev. D 14, P. 3497 (1976); Phys. Lett. 67B, P. 175 (1977); Phys. Lett. 69B, P. 495 (1977).

[5] R. Jackiw, L. Jacobs, C. Rebbi//Phys. Rev. D 20, P. 474 (1979).

[6] M.P. Isidro Filho, A.K. Kerman, Howard D. Trottier// Phys. Rev. D 40, P. 4142 (1989).

[7] G.Z. Basseyan, S.G. Matinyan, G.K. Savvidi// JETP Lett. V. 29, P. 641 (1979); S.G. Matinyan, G.K. Savvidi, N.G. Ter-Arutyunyan-Savvidi// JETP. V. 80, P. 830 (1981); JETP Lett. V. 34, P. 613 (1981) B.V. Medvedev//Teor.Mat.Phys. V. 60, P. 224 (1984); Teor.Mat.Phys. V. 73, P. 404 (1989).

[8] V.V. Goloviznin, S.V. Molodtsov, A.M. Snigirev, Preprint ITEP-62, Moscow, 1991, [in Russian].

[9] I.G. Petrovsky, Lection of the Theory of Integral Equations. [in Russian] (Moscow. MSU. 1984); D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, (Springer, Berlin, Heidelberg, New York 1977).

[10] A.F. Matveev, S.V. Molodtsov, Preprint ITEP—69, Moscow, 1992, [in Russian].

[11] V.V. Goloviznin, S.V. Molodtsov, A.M. Snigirev, Preprint ITEP—85, Moscow, 1991, [in Russian].

[12] L.D. Landau, E.M. Lifschitz, Quantum Mechanics-Nonrelativistic Theory. 2 nd ed. (Pergamon, New York, 1965).

[13] A.A. Grib, S.G. Mamaev, V.M. Mostepanenko, The Quantum Effect in the Intensive External Field. Moscow: Atomizdat, 1980 (in Russian).
A.M. Perelomov, Generalized Coherent States and its Applications. [in Moscow: Nauka, 1987 (in Russian).

[14] A.A. Samarsky, E.S. Nikolaev, The Methods of Resolving the Set Equations. Moscow: Nauka, 1978 (in Russian).

[15] V.V. Goloviznin, S.V. Molodtsov, A.M. Snigirev, Preprint ITEP—70, Moscow, 1992, [in Russian].

[16] S.K. Wong/Nuovo Cim. **65A**, P. 689 (1970).

[17] F Rohrlich, Classical Charged Particles. (Addison-Wesley. Reading, Mass., 1965).

[18] B.P. Kosyakov/Teor.Mat.Phys. V. **87**, P. 422 (1991); Usp. Fiz. Nauk. V. **162**, P. 161 (1992).

[19] A.I. Alekseev, B.A. Arbusov/Teor.Mat.Phys. V. **65**, P. 202 (1985); A.I. Alekseev/Teor.Mat.Phys. V. **77**, P. 389 (1988); A.I. Alekseev, B.A. Arbusov/Phys. Lett. **103B**. P. 242 (1990).

[20] *Simonov Yu.A.*//Sov. J. Nucl.Phys. V. **54**, P. 192 (1991); Materials of XXII LIYAP School.[in Russian] Leningrad, 1987.

В.В.Головизнин и др.
Приближение классики в задаче двух тел, взаимодействующих посредством неабелевого калибровочного поля.

Подписано к печати 01.06.93 Формат 60x90 1/16 Офсетн.печ.
Усл.-печ.л.2,75. Уч.-изд.л.2,0. Тираж 130 экз. Заказ 42.
Индекс 3649

Отпечатано в ИТЭФ, 117259, Москва, Б.Черемушкинская, 25