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Institute of Theoretical
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F.S.Dzheparov, V.E.Shestopal

Asymptotically exactly soluble models of oscillations in random media and coherent potential method

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ASYMPTOTICALLY EXACTLY SOLUBLE MODELS OF OSCILLATIONS IN RANDOM MEDIA AND COHERENT POTENTIAL METHOD: Preprint ITEP 3-96/

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The recently proposed first asymptotically (at large t or small Laplace parameter λ) exactly soluble models of multidimensional oscillations in multidimensional random media are considered. In this report we formulate a respective version of the coherent potential method and show that it gives a true asymptotic expansion in λ and $1/t$ for averaged propagator of this models.

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1 Wave propagation

in disordered systems is a topical problem of physics. Among many processes of this kind we see at least two processes related to the neutron scattering. The first one is the dynamic neutron scattering itself. The second process is the phonon migration which can be observed by a neutron scattering. It is the subject of our work.

The mean-field-approach is one of the most popular in the theory of disorder. The coherent potential method (CPM) (see for example [1]) is one of the most flexible and profound version in this approach. Nevertheless this methods needs a permanent confirmation. The best way to check it is an application to exactly soluble models.

The theory of small oscillations in disordered media is not rich in models having exactly calculated long-time asymptotics of averaged solutions. As we know, the first essentially multidimensional nontrivial system has been proposed in [3].

Here we will consider the CPM in application to models of oscillations from [3]. We show the possibility to obtain long-time asymptotical expansion within this formalism.

2 Our models

have following description. "Molecules" of unit mass oscillate near the sites $x = (x_1, x_2)$ of regular lattice Z^2 with an even sum $x_1 + x_2$. Each molecule interacts with two nearest coordination sphere from the same sublattice. The displacements u_x^n, $n = 1, 2$ satisfy the equations:

$$d^2u/dt^2 = -\Phi^+\xi\Phi u, \qquad u = (u^1, u^2), \quad u(t=0) = v, \quad du(t=0)/dt = w \tag{1}$$

$$\Phi_{xy}^{mn} = \begin{pmatrix} a_1\nabla_{xy}^1 - \sigma b_1\nabla_{xy}^2 & a_2\nabla_{xy}^1 - \sigma b_2\nabla_{xy}^2 \\ b_1\nabla_{xy}^1 + \sigma a_1\nabla_{xy}^2 & b_2\nabla_{xy}^1 + \sigma a_2\nabla_{xy}^2 \end{pmatrix}$$

$$(\xi)_{xy}^{mn} = \xi_x^{mn}\delta_{xy}, \quad \nabla_{xy}^n = \delta_{x+e_n,y} - \delta_{x-e_n,y}, \quad e_1 = (1,0),\ e_2 = (0,1), \quad \sigma = \pm 1.$$

Here a, b are real vectors. The disorder is reflected in an array $\{\xi_x\ ;\ x \in Z^2, x_1+x_2 \text{ is odd}\}$ of random, independent, identically distributed, positive definite, real symmetric matrices which must have finite negative moments $< (\xi_x)^{-N} >$ for a sufficiently large N.

It is easy to specify the case of isotropic models: $a_1 = a_2, b_1 = -b_2, \sigma = 1$, and distribution of ξ_0 commutes with rotations of the lattice. Moreover, in this case the averaged (over $\{\xi_x\}$) long-time asymptotical solution of (1) in a continuum media approximation is invariant to two-dimensional rotation group $S(2, R)$.

The formal solution of (1) may be written in Laplace representation ($u(t) \to u(\lambda) = \int_0^{+\infty} dt e^{-t\lambda} u(t)$) as

$$u(\lambda) = (\lambda^2 - \Phi^+\xi\Phi)^{-1}(v + \lambda w). \tag{2}$$

The small λ asymptotics has been obtained in [3] by means of expansion of $u(\lambda)$ in powers of $\lambda^2(1/\xi - < 1/\xi >)$. The asymptotics of $u(\lambda)$ at $\lambda \to 0, Re\lambda > 0$ gives us the long-time asymptotic of $u(t)$.

The correctness of this solution may be proved by methods which are described in [2].

3 Coherent potential method

starts with decomposition:

$$\lambda^2 - \Phi^+\xi\Phi = A - \sum_z B^z, \tag{3}$$

$$A = \lambda^2 - \Phi^+\xi_{eff}\Phi, \quad B^z = \Phi^+(\xi_{eff} - \xi_z)\pi^z\Phi, \quad (\pi^z)^{mn}_{xy} = \delta_{mn}\delta_{xy}\delta_{yz};$$

here we see the "effective field" $(\xi_{eff})^{mn}_{xy} = (\xi_{eff})^{mn}\delta_{xy}$, which will be defined below.

For the propagator $\tilde{P}$ we have an expansion:

$$\tilde{P} = (\lambda^2 - \Phi^+\xi\Phi)^{-1} = G + G\sum_{n=1}^{N}\sum_{z_1 z_2 \ldots z_n}^{def} t^{z_1}Gt^{z_2}G\ldots t^{z_n}G + R_{N+1}, \tag{4}$$

$$t^z = B^z(1 - GB^z)^{-1}, \quad R_{N+1} = G\sum_{z_1 z_2 \ldots z_{N+1}}^{def} t^{z_1}Gt^{z_2}G\ldots t^{z_N}GB^{z_{N+1}}\tilde{P},$$

where $G = A^{-1}$ and $\sum\limits_{z_1 z_2 \ldots z_j}^{def}$ denotes the summation with the condition $z_i \neq z_{i+1}$, $i = 1 \div j-1$.

{For convenience we give a simple derivation of the chosen form (4) i.e. the main CPM decomposition. In an obvious relation:

$$(A - \sum_z B^z)^{-1} = G[1 + \sum_z B^z(A - \sum_q B^q)^{-1}],$$

the following equation must be iterated:

$$B^z(A - \sum_q B^q)^{-1} = B^z(A - B^z)^{-1}[1 + \sum_{q,\, q\neq z} B^q(A - \sum_r B^r)^{-1}],$$

and an evident relation $B^z(A - B^z)^{-1} = t^zG$ should be used.}

The next standard step of CPM is the definition of ξ_{eff} by means of equation:

$$< t^z >= \Phi^+ < (\xi_{eff} - \xi_z)\pi^z\Phi\,[1 - G\Phi^+(\xi_{eff} - \xi_z)\pi^z\Phi]^{-1} >= 0. \tag{5}$$

4 The following properties of CPM

are important and sufficient for the solution of system (1) for $\lambda \to \infty$ and $\lambda \to 0$.

A. After averaging of (4) we obtain:

$$P =< \tilde{P} >= G + G\sum_{n=4}^{N}\sum_{z_1 z_2 \ldots z_n}^{def} < t^{z_1}Gt^{z_2}G\ldots t^{z_n}G > + < R_{N+1} >, \tag{6}$$

i.e. decomposition begins with $n = 4$. It follows from the mutual independence of different ξ_z-es and (5). Properties of this expansion at $\lambda \to \infty$ (or $t \to 0$) are evident.

B. Due to the same independence only those terms contribute to the sum in right hand side of Eq. (6) in which each value z_i is repeated at least once in the string $(z_1, z_2, \ldots, z_n)$.

C. The asymptotical smallness at $\lambda \to 0, Re\lambda > 0$ of averaged terms in (6) follows from point B. and relation:

$$t^zGt^q = \Phi^+D_z\pi^z\Phi[1 - G\Phi^+D_z\pi^z\Phi]^{-1}G\Phi^+\pi^qS^q = \Phi^+D_z\pi^z[1 - \Phi G\Phi^+D_z\pi^z]^{-1}\Phi G\Phi^+\pi^qS^q =$$

$$= \Phi^+ D_z [1 - (\Phi G \Phi^+)_{zz} D_z]^{-1} \pi^z \Phi G \Phi^+ \pi^q S^q, \tag{7}$$

where $D_z = \xi_{eff} - \xi_z$, $t^q = \Phi^+ \pi^q S^q$, $z \neq q$. It is evident that $\pi^z \pi^q = 0$, $(\pi^z)^2 = \pi^z$ and

$$\pi^z \Phi G \Phi^+ \pi^q = \pi^z \Phi [\lambda^2 - \Phi^+ \xi_{eff} \Phi]^{-1} \Phi^+ \pi^q =$$

$$= \pi^z \Phi \Phi^+ [\lambda^2 - \xi_{eff} \Phi \Phi^+]^{-1} \pi^q = \pi^z \xi_{eff}^{-1} \{1 + \lambda^2 [\lambda^2 - \xi_{eff} \Phi \Phi^+]^{-1}\} \pi^q =$$

$$= \lambda^2 \pi^z \xi_{eff}^{-1} [\lambda^2 - \xi_{eff} \Phi \Phi^+]^{-1} \pi^q.$$

D. Simplifying the equation (5) we have:

$$< (\xi_{eff} - \xi_z) \pi^z [1 - \pi^z \Phi G \Phi^+ (\xi_{eff} - \xi_z) \pi^z]^{-1} >=$$

$$= \pi^z < (\xi_{eff} - \xi_z)[1 - (\Phi G \Phi^+)_{00} (\xi_{eff} - \xi_z)]^{-1} >= 0, \tag{8}$$

(see the derivation of (7)). Taking into account that at small λ limit $\Phi G \Phi^+ \to \xi_{eff}{}^{-1}$ we obtain:

$$\xi_{eff} (\lambda \to 0)^{-1} =< \xi_z{}^{-1} > . \tag{9}$$

As a result we may check the main asymptotical term in (6) for the system (1):

$$P \approx G = [\lambda^2 - \Phi^+ \xi_{eff} \Phi]^{-1}. \tag{10}$$

The last relation coincides with result of [3].

But if we have exact or approximate solution of Eq. (9) and if this solution is exact at least for $\lambda \to \infty$ and $\lambda \to 0$ then Eq. (10) produces a uniform approximation for P which is exact both at $\lambda \to \infty (t \to 0)$ and $\lambda \to 0 (t \to \infty)$.

If the distribution of ξ_z is invariant to symmetries of Z^2 (rotations and reflections) we have: $< \xi_z{}^{-1} >^{mn} = \kappa^{-1} \delta_{mn}$. In this case the "effective" matrix of forces has the form:

$$\Phi^+ \xi_{eff} \Phi = \kappa \Phi^+ \Phi =$$

$$= \kappa \begin{pmatrix} a^2 (\nabla^1)^2 + b^2 (\nabla^2)^2 & (a^2 - b^2) \sigma \nabla^1 \nabla^2 \\ (a^2 - b^2) \sigma \nabla^2 \nabla^1 & b^2 (\nabla^1)^2 + a^2 (\nabla^2)^2 \end{pmatrix} + \kappa ab \begin{pmatrix} -2\sigma \nabla^1 \nabla^2 & (\nabla^1)^2 - (\nabla^2)^2 \\ (\nabla^1)^2 - (\nabla^2)^2 & 2\sigma \nabla^1 \nabla^2 \end{pmatrix}.$$

In the above mentioned isotropic case of model (1) when $\sigma = 1$ and $ab = 0$ we obtain the following effective matrix ($\Delta = (\nabla^1)^2 + (\nabla^2)^2$) :

$$\Phi^+ \Phi = b^2 \Delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (a^2 - b^2) \begin{pmatrix} (\nabla^1)^2 & \nabla^1 \nabla^2 \\ \nabla^2 \nabla^1 & (\nabla^2)^2 \end{pmatrix}.$$

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