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RAISING OPERATORS FOR THE WHITTAKER WAVE FUNCTIONS OF THE TODA CHAIN AND INTERTWINING OPERATORS: Preprint ITEP 29-99/

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Intertwiners between representations of Lie groups can be used to obtain relations for matrix elements. We apply this technique to obtain different identities for the wave functions of the open Toda chain, in particular raising operators and bilinear relations for the wave functions at different energy levels. We also recall the group theory approach to the Toda chain: treating the wave functions as matrix elements in irreducible representations between the so-called Whittaker vectors, integral representations of the wave functions, etc.

Ref. - 25 name.

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1 Introduction

This paper is devoted to the application of the representation theory to studding integrable many-body systems. Interrelations between the theory of integrable systems and representation group theory proved to be fruitful for the both fields. The subject has become so vast and diverse that it looks impossible to pick up a few of surveys which would cover all the important questions. Some topics are covered in reviews [1, 2, 3] and books [4, 5]. Review [6] is closely related to our consideration. Usually it is possible to find a group theory interpretation for all the important structures of integrable systems. Phase space, symplectic structure, Hamiltonians, Lax pairs, etc. can be interpreted in terms of an underlying group symmetry for both classical and quantum integrable systems. However we will not discuss these questions here.

The main goal of this paper is to demonstrate how representation theory, and, more concretely, *intertwining operators, can be used to obtain relations between wave-functions at different energy levels.* We will work out only the simplest case of the open Toda chain with two and three particles, but the method can be applied to other integrable systems. The paper mainly follows the ideas of [7, 8].

Let us recall one of the main ideas of the group theory approach to integrable systems: *solutions of integrable equations usually are matrix elements or traces of the groups elements in irreducible representations.* Main examples we keep in mind are: the tau-function of the Kadomtsev-Petviashvili which is a matrix element in fundamental representations of the group GL_∞ (see [1, 6, 8]), and the wave functions of many-body problems like Toda or Calogero-Sutherland-Moser, which are matrix elements in infinite-dimensional irreducible representations of the semisimple groups (see for example [7]). Let us also mention the fruitful generalization of the considerations of matrix elements - the concept of vector-valued matrix elements and vector-valued characters developed by Etingof and Kirillov [9].

The idea to obtain different relations for wave functions is quite simple and goes as follows: let functions f_i (g_i) be solutions of some integrable system corresponding to some representation V_i (W_i respectively) of some group, (it can be the trace or matrix element, can be a wave function or a tau function, etc.) *If one has an intertwinning operator* $V_1 \otimes V_2 \otimes$

$...\otimes V_n \to W_1 \otimes W_2 \otimes ... \otimes W_m$ *then there 'usually'*[1] *exist the following relation:* $D(f_1 f_2 ... f_n) = \tilde{D}(g_1 g_2 ... g_m)$, where $D, \tilde{D}$ are some differential operators. Miraculously enough this simple idea leads to non-trivial relations. It was demonstrated in [8, 10] that in such way one can obtain famous bilinear identities for KP,KdV tau-functions, which corresponds to intertwiners between fundamental representations of the GL_∞, $\hat{SL}(n)$. In [11], it was shown, that, in this way, one can obtain the Baxter relation for transfer matrices of integrable lattice models. The above idea is also used in representation theory approach to Knizhnik-Zamolodchikov equation [13, 12, 5], and in the theory of special functions (see, for example [14, 15]).

In this paper, we show how to obtain different relations for the Toda wave-functions applying the above idea. The group theory approach to the classical Toda chain has been first proposed in [16], group theory approach to the quantum Toda chain has been proposed by Kostant and Kazhdan at late seventies (unpublished) and further elaborated in [17], [7]. Let us recall it following [7]. Consider the group $SL(n+1, \mathbb{R})$ with the standard Chevalley generators e_i, h_i, f_i, and let V_λ be irreducible representations with weight λ of the principal series with weight λ. Let us introduce the so called Whittaker vector $|w> \in V_\lambda$, i.e. the eigenvector of generators e_i. Then, one considers the Whittaker function, i.e. matrix element $W_\lambda(\phi_1, ..., \phi_n) =< w|exp(\phi_1 h_1 + \phi_2 h_2 + ... + \phi_n h_n)|w >$. The main observation is that *function* $W_\lambda(\phi_1, ..., \phi_n)$ *is a common eigenfunction of Toda Hamiltonians, which turn out to be Casimirs of the group* $SL(n+1, \mathbb{R})$, while the weight λ marks the corresponding energy level. (Existence and other issues on Whittaker vectors were deeply explored by Kostant [18]).

The relations between Whittaker functions $W_\lambda(\phi)$ for different V_λ will be obtained as follows: let us find the intertwinning operators $V_{\lambda_1} \otimes V_{fin} \to V_{\lambda_2}$ then we will show that there exists a relations $\mathcal{D}\ (W_{\lambda_1}(\phi_1, ..., \phi_n)) = \tilde{\mathcal{D}}\ (W_{\lambda_2}(\phi_1, ..., \phi_n))$ (see formulas 30,54) where $\mathcal{D}, \tilde{\mathcal{D}}$ are some differential operators, and V_{fin} is any finite-dimensional representation of the group $SL(n+1, \mathbb{R})$. This relation is an example of the so-called raising operators [19]. This idea was suggested to us by S. Kharchev to whom we are deeply indebted. Analogously we will construct bilinear relations from the intertwining operators $V_{\lambda_1} \otimes V_{\lambda_2} \to V_{\lambda_3} \otimes V_{\lambda_4}$. One can proceed this way to construct other relations like above, for example $V_{\lambda_1} \otimes V_{\lambda_2} = \oplus_i V_{\lambda_i}$ then one can obtain relation like $W_{\lambda_1} \otimes W_{\lambda_2} = \sum_i D_i(W_{\lambda_i})$ (see formula 37).

Let us also note that the existence of the bilinear identities in this situation is quite surprising, because it means that wave-function, which are the eigen-functions of Toda Hamiltonians, i.e. eigenfunctions of some *linear* differential operators, satisfy some system of *nonlinear* differential equations. One cannot separate some finite number of wave-functions from this system , only infinite set of them is closed nonlinear system. Nevertheless it is some system of nonlinear equations, but we do not know its sense, its hamiltonian description (if it is) or anything else, it seems to be interesting to clarify it.

Let us describe what kind of technical problems arise in proceeding to the concrete formulas. Let $|w>_{\lambda_1}$ be the Whittaker vector in representation V_{λ_1}, let us consider the intertwiner $\Phi : V_{\lambda_1} \to V_{\lambda_2} \otimes V_{fin}$ then $\Phi|w>_{\lambda_1} = \sum_i |w>_i \otimes |i>$ where $|i>$ is some basis in finite dimensional representation V_{fin}. *to construct the explicit relations one needs to find such polynomials* P_i *that* $|w>_i = P_i(f_i, h_i)|w>_{\lambda_2}$ *where* P_i *are some polynomials of generators*

[1] 'usually' means that there are no such relations for generic matrix elements, while they typically emerge for solutions of integrable systems.

f_i, h_i, i.e. one needs to express the components of the image of Whittaker vector in spaces V_{λ_1} as application of some operators to Whittaker in space V_{λ_2} . This is the main technical problem we treat in this paper, we solve it for the cases $SL(2,\mathbb{R})$ and $SL(3,\mathbb{R})$, but we were unable to find the general formula for the case of $SL(n,\mathbb{R})$.

Let us mention that the $SL(2)$ Toda equation reduces to the well-known textbook equation, whose solutions are Bessel and Mcdonald functions (depends on necessary asymptotics, we need Mcdonald one), and the relation from theorem 2.11 one can find, for example in [20].

Let us argue that the scheme like above can be applied to other integrable models. One knows that if instead of the Whittaker vector $|w>$ one considers a spherical vector $|s>$ i.e. a vector such that it is remains fixed under the action of the maximal compact subgroup then one can obtain wave function for Calodgero-Sutherland integrable system - zonal spherical function, obviously if one considers different intertwiners one can obtain different relations between wave functions. Also instead of finite-dimensional Lie group G one can takes affine group $\hat{G}$, which corresponds to periodic Toda chain, or quantum group $U_q(G)$, which leads to difference equations [21, 22]. Obviously, our scheme of obtaining different relations will work in this situations also. But of course explicit constructions of these relations seems to be not obvious.

We should also mention that raising operators considered here were widely explored recently [19], but all approaches are completely different from our approach, and very simple idea, that the origin of this relations is presence of the intertwining operators, was not present in the literature to our knowledge. Let us repeat that from our point of view one can consider all models in the same setup. Also we see that the group theory approach shows that such at first sight not related subjects such as Baxter relation important in lattice models [11] and conformal field theory [23], raising operators in integrable quantum many body problems [19] and bilinear relations in the theory of classical nonlinear equations [1, 6] have the same explanations from the point of view of intertwining operators.

Exposition is organized as follows: section 2 is devoted to the case of $SL(2)$ and the demonstration of the main ideas. In the section 3 we consider the case of $SL(3)$, which is mainly analogous to the $SL(2)$ case, but that is the case, where first arises the problems in writing the explicit formulas for the action of the intertwining operators on the Whittaker vectors. And we show how these problems can be solved for $SL(3)$. Section 4 is devoted to the $SL(N)$ case. We recall representation theory approach to the $SL(N)$ Toda chain and discuss the problem of obtaining explicit raising operators.

2 $SL(2)$ Toda Chain.

In this section we will show how to obtain the raising operators, bilinear, etc. identities for the wave functions of $SL(2)$ Toda. We will do it with the help of the intertwining operators between the tensor products of representations of $SL(2)$. Before doing this in subsections 1 and 2 we will recall representation theory interpretation of the Toda wave functions as matrix element in irreducible representation, here we closely follow [7]. Let us sketch the content: we consider Whittaker vector $|w>$ in irreducible representation V_λ of $SL(2)$, i.e. such vector that it is eigenvector for the generator e, consider Whittaker function $W(\phi) =< w|exp(\phi h)|w>$, we show that it is eigenfunction for Toda hamiltonian

$\frac{1}{2}\partial_\phi^2 + exp(-2\phi) + \partial_\phi$, the reason is that this Hamiltonian is simply the Casimir of $SL(2)$, then we consider intertwining operator $\Phi : V_{\lambda+1} \to V_\lambda \otimes V_1$, where V_1 is standard two-dimensional representation of $SL(2)$, and the main technical point is to show that $\Phi(|w>_{\lambda+1}) = \frac{1}{\lambda+1}\frac{1}{2}(h+\lambda+2)|w>_\lambda \otimes |0> + \mu_R|w>_\lambda \otimes |1>$, it is quite obvious for the case of $SL(2)$, because we know the representation of the Whittaker vector in terms of the action of the generator f on the vacuum vector in module V_λ, the fact that such representation for the case of $SL(N)$ is unreasonable is the main problem for writing the explicit formulas for the $SL(N)$ case . Then we proceed to the bilinear relations, to find them one should find the intertwining operator between some representations: $V_{\lambda_1} \otimes V_{\lambda_2} \to V_{\lambda_3} \otimes V_{\lambda_4}$. To find such intertwiners is not an easy problem, but there is a trick which is due to [8] that works as follows: if we have intertwiners : $V_{\lambda+1} \to V_\lambda \otimes V_1$ and $V_{\mu+1} \to V_\mu \otimes V_1$ we can consider their tensor product : $V_{\lambda+1} \otimes V_{\mu+1} \to V_\lambda \otimes V_1 \otimes V_\mu \otimes V_1$ then consider intertwiner: $V_1 \otimes V_1 \to \mathbb{C}$, then consider the composition: $V_{\lambda_1+1} \otimes V_{\mu_1+1} \to V_{\lambda_1} \otimes V_1 \otimes V_{\mu_1} \otimes V_1 \to V_{\lambda+1} \otimes V_{\mu+1}$, hence we get desirable intertwiner (consideration instead of V_1 other finite dimensional representations leads to various kinds of necessary intertwiners).

2.1 Notations:

Let us denote by (λ, n) the product $(\lambda)(\lambda-1)(\lambda-2)...(\lambda-n+1)$, we will often use this abbreviation.

The Lie algebra $SL(2)$ is defined by the generators e, f, h satisfying the following relations:

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

The fundamental representation of $SL(2)$ is defined as follows:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

The quadratic Casimir operator:

$$C = (ef + fe) + \frac{1}{2}h^2 = 2fe + h + \frac{1}{2}h^2. \tag{2}$$

Highest weight representation:
We consider the principal (spherical) series of representations, induced by the one-dimensional representations of the Borel subgroup. The space of representation V_λ is functions of one real variable x and matrix elements are defined by integrals with the flat measure. The action of the algebra is given by differential operators:

$$e = \frac{\partial}{\partial x}, \; h = -2x\frac{\partial}{\partial x} + \lambda, \; f = -x^2\frac{\partial}{\partial x} + \lambda x. \tag{3}$$

Obviously that $Constants$ are vacuum (i.e. the highest weight) vectors for such representation i.e. $e(Constant) = 0, \quad h(Constant) = \lambda(Constant)$

2.2 Whittaker function.

Definition 2.1: The vector $|w>_\lambda^\mu$ is called *Whittaker vector* in representation V_λ, if it is eigenvector for generator e i.e. $e|w>_\lambda^\mu = \mu|w>_\lambda^\mu$ for some constant μ.

Lemma 2.2: For given μ and irreducible representation V_λ $(\lambda \neq 1,2,3,...)$ there exists a unique (up to a scalar) Whittaker vector $|w>_\lambda^\mu \in V_\lambda$, which can be expressed as follows:

$$|w>_\lambda^\mu = exp(\mu x) = \sum_{i=0} \frac{\mu^n f^n}{n!(\lambda,n)}|vac> \tag{4}$$

where we denote $(\lambda,n) = \lambda(\lambda-1)...(\lambda-n+1)$ and $|vac>$ - vacuum (the highest weight) vector in V_λ.

The proof is obvious.

The dual Whittaker vector: ${}_\lambda^\mu < w|$ is given $\sum_{n=0}^{\infty} {}_\lambda < 0|\frac{\mu^n e^n}{n!(\lambda,n)} = x^{-\lambda-2}exp(-\mu x)$ (see [7]).

Definition 2.3: The Whittaker function $W_\lambda^{\mu_L,\mu_R}(\phi)$ is the function given by:

$$W_\lambda^{\mu_L,\mu_R}(\phi) = {}_\lambda^{\mu_L} < w|exp(\phi h)|w>_\lambda^{\mu_R} \tag{5}$$

Remark: we will sometimes omit indexes λ, μ of Whittaker vectors and Whittaker functions, if they are not important.

Proposition 2.4: $W_\lambda(\phi) = exp(\lambda\phi)\sum_{i=0}^{\infty}\frac{(\mu_R)^i(\mu_L)^n exp(-2i\phi)}{i!(\lambda,i)}$

The proof is obvious recalling (5) and $< vac|e^n|f^n|vac >= n!(\lambda,n)$

Remark: for the case $SL(2,\mathbb{R})$ Whittaker function coincides with the textbook Macdonald function (up to some exponents and scalars), this can be easy seen from the integral representation below, see also [7].

Proposition 2.5: The Whittaker function satisfies the equation:

$$\left[\frac{1}{2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial}{\partial\phi} - 2\mu_R\mu_L e^{-2\phi}\right] W_\lambda(\phi) = (\frac{1}{2}\lambda^2 + \lambda)W_\lambda(\phi) \tag{6}$$

Hence, the function $\Psi_\lambda(\phi) = exp(\phi)W_\lambda(\phi)$ satisfies two particle open Toda equation:

$$\left[\frac{1}{2}\frac{\partial^2}{\partial\phi^2} - 2\mu_R\mu_L e^{-2\phi}\right] \Psi_\lambda(\phi) = \frac{1}{2}(\lambda+1)^2\Psi_\lambda(\phi), \tag{7}$$

The proof is very simple, but we will reproduce it, since it is the same for all kinds of integrable many body problems. The main idea is that the Hamiltonian is a Casimir operators, hence it acts by scalar on every matrix element, that's how RHS of 6 appears, on the other hand it is some concrete operator - that's how LHS of 6 appears.

Proof: The Casimir operator $C = 2fe + h + \frac{1}{2}h^2$ acts as scalar $\frac{1}{2}\lambda^2 + \lambda$ on the representation V_λ. Hence: $< w|exp(\phi h)C|w >= (\frac{1}{2}\lambda^2 + \lambda)W_\lambda(\phi)$ On the other hand $< w|exp(\phi h)C|w >=< w|exp(\phi h)(\frac{1}{2}h^2 + h + 2fe)|w >= (\frac{1}{2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial}{\partial\phi})W_\lambda(\phi) + 2 < w|f\ exp(\phi(h-2))e|w >= (\frac{1}{2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial}{\partial\phi} + 2\mu_R\mu_l exp(-2\phi))W_\lambda(\phi)$ hence we obtain RHS of 6. Q.E.D.

Proposition 2.6 (Integral representation of the Whittaker function) :

$$W_\lambda(\phi) = \left(\frac{1}{\mu_R}\right)^{-(\lambda+1)} \int_0^\infty x^{-(\lambda+2)} e^{-\frac{\mu_L\mu_R e^{-2\phi}}{x} - x} dx = \tag{8}$$

$$= 2e^{(\lambda+1)\phi} \left(\sqrt{\frac{\mu_L}{\mu_R}}\right)^{-(\lambda+1)} K_{\lambda+1}(2\sqrt{\mu_L\mu_R}e^{-\phi})\, dx. \tag{9}$$

- where $K_\lambda(z)$ is the Macdonald function.

Remark: for the convergence we should require $Re\mu_L\mu_R > 0$.

The theorem is tautology after recalling that the invariant pairing $< v|u >$ is realized as integral $\int_0^\infty$, and dual Whittaker vector $< w|$ is realized as $x^{-2(\lambda+1)}e^{-\frac{\mu_L}{x}}$. The second equality is the textbook integral formula for the Macdonald function.

2.3 Intertwining operators

Denote by V_1 the fundamental representation of the $SL(2,\mathbb{R})$, denote $|0>$ the highest weight vector, $|1>$ the lowest weight vector, i.e.

$$|0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10}$$

hence $e|0>=0, f|0>=|1>, h|0>=|0>; e|1>=|0>, f|1>=0, h|1>=-|0>$.

Proposition 2.7: The isomorphism $\Phi_\lambda = \Phi_{\lambda,+} \oplus \Phi_{\lambda,-} : V_{\lambda+1} \oplus V_{\lambda-1} \to V_\lambda \otimes V_1$ is given by the formulas:

$$\Phi_{\lambda,+} : f^n|vac>_{\lambda+1} \to f^n|vac>_\lambda \otimes |0> + nf^{n-1}|vac>_\lambda \otimes |1>, \tag{11}$$

$$\Phi_{\lambda,+}^{-1} : f^n|vac>_\lambda \otimes |0> \to \frac{(\lambda+1-n)}{\lambda+1} f^n|vac>_{\lambda+1},$$

$$\Phi_{\lambda,+}^{-1} : f^n|vac>_\lambda \otimes |1> \to \frac{1}{\lambda+1} f^{n+1}|vac>_{\lambda+1}, \tag{12}$$

$$\Phi_{\lambda,-} : f^n|vac>_{\lambda-1} \to f^{n+1}|vac>_\lambda \otimes |0> + (n-\lambda) f^n|vac>_\lambda \otimes |1>, \tag{13}$$

$$\Phi_{\lambda,-}^{-1} : f^n|vac>_\lambda \otimes |0> \to \frac{n}{\lambda+1} f^{n-1}|vac>_{\lambda-1},$$

$$\Phi_{\lambda,-}^{-1} : f^n|vac>_\lambda \otimes |1> \to \frac{-1}{\lambda+1} f^n|vac>_{\lambda-1}. \tag{14}$$

The action of dual operators $\Phi^*, \Phi_1^*, \Phi_{-1}^*$ on the dual anti-representations are given by the same formulas (up to scalars) with the change of f to e , which follows from the Chevalley antiinvolution:

$$\Phi_1^* : {}_{\lambda+1}< vac|e^n \to {}_\lambda< vac|e^n \otimes < 0| + n\,{}_\lambda< vac|e^{n-1} \otimes < 1| \tag{15}$$

$$\Phi_1^{-1*} : {}_\lambda< vac|e^n \otimes < 0| \to \frac{(\lambda+1-n)}{\lambda+1}\ {}_{\lambda+1}< vac|e^n, \tag{16}$$

$$\Phi_1^{-1*} : {}_\lambda< vac|e^n \otimes < 1| \to \frac{1}{\lambda+1}\ {}_{\lambda+1}< vac|e^{n+1}, \tag{17}$$

$$\Phi^*_{-1} : {}_{\lambda-1} < vac|e^n \to \frac{1}{\lambda+1}({}_\lambda < vac|e^{n+1}\otimes < 0| + (n-\lambda)_\lambda < vac|e^n\otimes < 1|) \quad (18)$$

$$\Phi^{-1*}_{-1} : {}_\lambda < vac|e^n\otimes < 0| \to (n\lambda)\ {}_{\lambda-1} < vac|e^{n-1},$$

$$\Phi^{-1*}_{-1} : {}_\lambda < vac|e^n\otimes < 1| \to (-\lambda)\ {}_{\lambda-1} < vac|e^n. \quad (19)$$

The invariant pairing is given by: $< 0|0 >= 1, < 0|1 >= 0, < 1|0 >= 0, < 1|1 >= 1$.

The proof is obvious, let us only note that in order to find the intertwiners $\Phi^{-1}_{\pm 1}$ from $V_{\lambda\pm 1}\otimes V_1 \to V_\lambda$ it is more convinient to express them via the dual intertwiners: $V_{\lambda\pm 1} \to V_1^*\otimes V_\lambda$, which can be easily found.

Proposition 2.8: Intertwiners act on Whittaker vectors as follows:

$$|w >_{\lambda+1} \to \frac{(h+\lambda+2)}{2(\lambda+1)}|w > \otimes|0 > + \frac{\mu_R}{\lambda+1}|w > \otimes|1 >, \quad (20)$$

$$|w >_{\lambda-1} \to \frac{\lambda}{\mu_R}\frac{1}{2}(-h+\lambda)|w > \otimes|0 > -\lambda|w > \otimes|1 >, \quad (21)$$

$$|w >_\lambda \otimes|0 > \to |w >_{\lambda+1} + \frac{\mu_R}{\lambda(\lambda+1)}|w >_{\lambda-1}, \quad (22)$$

$$|w >_\lambda \otimes|1 > \to \frac{(\lambda+1-h)}{2\mu_R}|w >_{\lambda+1} - \frac{\lambda+1+h}{2\lambda}|w >_{\lambda-1}, \quad (23)$$

$${}_\lambda < w|\otimes < 0| \to \mu_L\ {}_{\lambda-1} < w| + {}_{\lambda+1} < w|. \quad (24)$$

$${}_\lambda < w|\otimes < 1| \to -\frac{h+\lambda+1}{2\lambda}\ {}_{\lambda-1} < w|\otimes < 1| + \frac{\lambda+1-h}{2\mu_L}\ {}_{\lambda+1} < w|. \quad (25)$$

Proof: The proof is simple, let us consider only the first equality. We will use the following trivial fact:

$(\lambda-n+1)f^n|vac >_\lambda = \frac{1}{2}(\lambda+2+h)f^n|vac >_\lambda$.

$\Phi_1(|w >_{\lambda+1}) = \Phi_1(\sum_{n=0}^\infty \frac{\mu_R^n}{n!(\lambda+1,n)}f^n|vac >_{\lambda+1}) =$

$= \sum_{n=0}^\infty \frac{\mu_R^n}{n!(\lambda+1,n)}(f^n|vac >_\lambda \otimes|0 > + nf^{n-1}|vac >_\lambda \otimes|1 >) =$

$= \frac{1}{\lambda+1}\sum_{n=0}^\infty \frac{\mu_R^n}{(n)!(\lambda,n)}(\lambda-n+1)f^n|vac >_\lambda \otimes|0 > + \frac{\mu_R}{\lambda+1}|w >_\lambda \otimes|1 >=$

$= \frac{(\lambda+2+h)}{2(\lambda+1)}|w >_\lambda \otimes|0 > + \frac{\mu_R}{\lambda+1}|w >_\lambda \otimes|1 >$.

Q.E.D.

Remark: let us note that such simple formulas like above cannot be true for an arbitrary vector $|v >_\lambda \in V_\lambda$ i.e. of course from the irredicibility of repsentations follows that there exist series $O(f,h,e)$ such that $\Phi_1(|v >_{\lambda+1}) = O(f,h,e)|v >_\lambda$, but in general such expression will be complicated.

Let us find the intertwiners between tensor product of infinite-dimensional representations.

Proposition 2.9: the isomorphism $\Phi = \oplus_{k=0}^\infty \Phi_k : V_\lambda \otimes V_\nu \to \oplus_{k=0}^\infty V_{\lambda+\nu-k}$ in realization (3) is given by the formulas:

$$\Phi_k : x^n \otimes x^m \to C_k \sum_{i=0}^k (-1)^i \binom{n}{k-i}\binom{m}{i}\frac{(\nu)_k}{(\nu)_i}\frac{(\lambda)_k}{(\lambda)_{k-i}}x^{n+m-k}, \quad (26)$$

$$where \quad C_k = \frac{\left(\prod_{i=1}^k (i(\nu^2(\lambda-i+1)+\lambda^2(\nu-i+1))\right)^{\frac{1}{2}}}{(\nu\lambda)^k(\sum_{i=0}^k \frac{(\nu)_k}{(\nu)_i}\frac{(\lambda)_k}{(\lambda)_{k-i}})}. \quad (27)$$

Or one can rewrite it as follows. Denote by mul_k operator $V_\lambda \otimes V_\nu \to V_{\lambda+\nu-k}$ which acts: $f(x) \otimes g(x) \to f(x)g(x)$. Then:

$$\Phi_k : f(x) \otimes g(x) \to C_k(mul_k) \times (\sum_{i=0}^{k} \frac{(-1)^i}{(k-i)!i!} \frac{(\nu)_k}{(\nu)_i} \frac{(\lambda)_k}{(\lambda)_{k-i}} \partial_x^{k-i} f(x) \otimes \partial_x^i g(x)). \quad (28)$$

The proof can be found in the literature, or proceeded as follows: from the equality: $\Phi_k * h = h * \Phi_k$ follows $\Phi_k(x^n \otimes x^m) = c_{n,m} x^{n+m-k}$, using $\Phi_k \times e = e \times \Phi_k$ one obtains the recurrence relation for the coefficients $c_{n,m}$. Solution of this relation can be guessed after considering several examples of small $k = 0, 1, 2$.

Proposition 2.10: The action of Φ_k on Whittaker vectors is given by:

$$\Phi_k : |w>_\lambda^{\mu_1} \otimes |w>_\nu^{\mu_2} \to C_k \sum_{i=0}^{k} \frac{(-1)^i}{(k-i)!i!} \frac{(\nu)_k}{(\nu)_i} \frac{(\lambda)_k}{(\lambda)_{k-i}} (\mu_1)^{k-i} (\mu_2)^i) |w>_{\lambda+\nu-k}^{\mu_1+\mu_2}. \quad (29)$$

This easily follows from 28.

2.4 Relations for the Whittaker wave functions.

Theorem (Raising operators) 2.11 The following relations holds:

$$W_{\lambda+1}(\phi) = exp(\phi) \frac{\partial_\phi + \lambda + 2}{2(\lambda+1)} W_\lambda(\phi), \ \mu_L W_{\lambda-1}(\phi) = \frac{\lambda}{\mu_R} exp(\phi) \frac{-\partial_\phi + \lambda}{2} W_\lambda(\phi). \quad (30)$$

Remark: easy to see that these relations are consistent: application of one then another gives that W_λ satisfy Toda equation.

Proof: We will calculate $_\lambda < w| \otimes < 0| \quad \Phi_1 exp(\phi h)|w>_{\lambda+1}$ in two ways, first we will apply Φ_1 to $exp(\phi h)|w>_{\lambda+1}$ and then take pairing, hence we will obtain the RHS of (30); to obtain the LHS of (30) we will apply Φ_1^* to $_\lambda < w| \otimes < 0|$ and consider the pairing after.

$_\lambda < w| \otimes < 0| \quad \Phi_1 exp(\phi h)|w>_{\lambda+1} = {}_\lambda < w| \otimes < 0| \quad exp(\phi h)\Phi_1|w>_{\lambda+1} = {}_\lambda < w| \otimes < 0| \quad exp(\phi h)(\frac{(h+\lambda+2)}{2(\lambda+1)}|w>_\lambda \otimes |0> + \frac{\mu_R}{\lambda+1}|w> \otimes |1>) = {}_\lambda < w| \otimes < 0| \quad \frac{(h+\lambda+2)}{2(\lambda+1)} exp(\phi h)|w>_\lambda \otimes exp(\phi h)|0> = exp(\phi) \frac{(\partial_\phi+\lambda+2)}{2(\lambda+1)} {}_\lambda < w| \otimes < 0| \quad exp(\phi h)|w>_\lambda$. Hence we get RHS of 30.

$_\lambda < w| \otimes < 0| \quad \Phi_1 exp(\phi h)|w>_{\lambda+1} = ({}_\lambda < w| \otimes < 0|\Phi_1^*) |exp(\phi h)|w>_{\lambda+1} =$
$= {}_{\lambda+1} < w|exp(\phi h)|w>_{\lambda+1} = W_{\lambda+1}(\phi)$. Hence we get the LHS of (30). Q.E.D.
The proof of the second equality in (30) is the same.

One can deduce directly from the (30) or deduce representation theoretically (as we will do for the demonstration of the idea) the following relations:

Theorem (Baxter like relations) 2.12:

$$exp(\phi) W_\lambda(\phi) = \frac{\mu_L \mu_R}{\lambda(\lambda+1)} W_{\lambda-1}(\phi) + W_{\lambda+1}(\phi),$$

$$exp(-\phi) W_\lambda(\phi) = \left(\frac{\partial_\phi + \lambda + 1}{2}\right)^2 W_{\lambda-1}(\phi) + \frac{1}{\mu_L \mu_R} \left(\frac{\lambda + 1 - \partial_\phi}{2}\right)^2 W_{\lambda+1}(\phi). \quad (31)$$

Remark: analogous considerations for the infinite-dimensional algebras leads to Baxter relations for the quantum transfer matrices in lattice models, see [11].

Proof: We will use isomorphism $\Phi : V_{\lambda+1} \oplus V_{\lambda-1} \to V_\lambda \otimes V_1$ and calculate the pairing $< 0| < w|exp(\phi h)|w > |0 >$ first directly, obtaining LHS, then applying isomorphism Φ^{-1} to $|w > |0 >$ and $^{-1*}$ to $< w| < 0|$, and obtain RHS.

$$
\begin{aligned}
& exp(\phi)W_\lambda(\phi) = {}_\lambda < w| \otimes < 0|exp(\phi h)|0 > \otimes |w >_\lambda = \\
& (\mu_L \ {}_{\lambda-1} < w| + {}_{\lambda+1} < w|)exp(\phi h)(\frac{\mu_R}{\lambda(\lambda+1)}|w >_{\lambda-1} + |w >_{\lambda+1}) = \\
& = \frac{\mu_L \mu_R}{\lambda(\lambda+1)} {}_{\lambda-1} < w|exp(\phi h)|w >_{\lambda-1} + {}_{\lambda+1} < w|exp(\phi h)|w >_{\lambda+1} = \\
& \frac{\mu_L \mu_R}{\lambda(\lambda+1)} W_{\lambda-1}(\phi) + W_{\lambda+1}(\phi).
\end{aligned}
\tag{32}
$$

Q.E.D.The proof of the second relation is the same, but one should consider ${}_\lambda < w| \otimes < 1|exp(\phi h)|1 > \otimes |w >_\lambda$ instead of ${}_\lambda < w| \otimes < 0|exp(\phi h)|0 > \otimes |w >_\lambda$. Obviously $0 = {}_\lambda < w| \otimes < 0|exp(\phi h)|1 > \otimes |w >_\lambda$, so one cannot obtain any more relations like above.

Theorem (Bilinear relations) 2.13:

$$
\begin{aligned}
& \left(-\frac{\mu_2^R exp(\phi_1 - \phi_2)}{2(\lambda+1)(\nu+1)}(\partial_{\phi_1} + \lambda + 2) + \frac{exp(\phi_2 - \phi_1)\mu_1^R}{2(\lambda+1)(\nu+1)}(\partial_{\phi_2} + \nu + 2) \right) W_\lambda^{\mu_1^L, \mu_1^R}(\phi_1) W_\nu^{\mu_2^L, \mu_2^R}(\phi_2) \\
& = \left(-\frac{\nu + 1 - \partial_{\phi_2}}{2\mu_2^L} + \frac{\lambda + 1 - \partial_{\phi_1}}{2\mu_1^L} \right) W_{\lambda+1}^{\mu_1^L, \mu_1^R}(\phi_1) W_{\nu+1}^{\mu_2^L, \mu_2^R}(\phi_2)
\end{aligned}
$$

Proof: let us denote by $\Phi_{\lambda,\nu}$ the intertwiner: $V_{\lambda+1} \otimes V_{\nu+1} \to V_\lambda \otimes V_\nu$. we will construct that intertwiner using a trick due to [8]: let us denote by S the intertwiner $V_1 \otimes V_1 \to \mathbb{C}$, than $\Phi_{\lambda,\nu}$ is given by composition: $(id \otimes S \otimes id)(\Phi_{\lambda,+} \otimes \Phi_{\nu,+})$.

Lemma 2.14: The action of the $\Phi_{\lambda,\nu}$ on Whittaker vectors is given by the formulas:

$$
\begin{aligned}
& \Phi_{\lambda,\nu} : exp(\phi_1 h)|w >_{\lambda+1}^{\mu_1^R} \otimes exp(\phi_2 h)|w >_{\nu+1}^{\mu_2^R} \to \\
& -exp(\phi_1 - \phi_2)\frac{h + \lambda + 2}{2(\lambda+1)} exp(\phi_1 h)|w >_\lambda^{\mu_1^R} \otimes \frac{\mu_2^R exp(\phi_2 h)}{\nu + 1} exp(\phi_2 h)|w >_\nu^{\mu_2^R} + \qquad (33) \\
& + exp(\phi_2 - \phi_1)\frac{\mu_1^R exp(\phi_1 h)}{(\lambda+1)}|w >_\lambda^{\mu_1^R} \otimes \frac{(h + \nu + 2)}{2(\nu+1)} exp(\phi_2 h)|w >_\nu^{\mu_2^R}, \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
& \Phi^*_{\lambda,\nu} : {}^{\mu_1^L}{}_\lambda < w| \otimes {}^{\mu_2^L}{}_\nu < w| \to -{}^{\mu_1^L}{}_{\lambda+1} < w| \otimes {}^{\mu_2^L}{}_{\nu+1} < w| \frac{\nu + 1 - h}{2\mu_2^L} + \\
& {}^{\mu_1^L}{}_{\lambda+1} < w| \frac{\lambda + 1 - h}{2\mu_1^L} \otimes {}^{\mu_2^L}{}_{\nu+1} < w|. \qquad (35)
\end{aligned}
$$

The proof of the lemma follows from (11) and recalling the fact that action of S is given by: $S(|0 > |1 >) = 0$, $S(|1 > |0 >) = 0$, $S(|1 > |0 >) = 1$, $S(|0 > |1 >) = -1$ and respectively: $S^* : \mathbb{C} \to V_1 \otimes V_1$ acts as follows: $1\ S^* = - < 0| < 1| + < 1| < 0|$.

The proof of the theorem easily follows from the lemma:

${}_\lambda < w| \otimes {}_\nu < w| (\Phi_{\lambda,\nu} exp(\phi_1 h)|w >_{\lambda+1} \otimes exp(\phi_2 h)|w >_{\nu+1})$ equals to LHS of 33 and $\left({}_\lambda < w| \otimes {}_\nu < w| \Phi^*_{\lambda,\nu}\right) exp(\phi_1 h)|w >_{\lambda+1} \otimes exp(\phi_2 h)|w >_{\nu+1}$ equals to RHS of 33.

And obviously ${}_\lambda < w| \otimes {}_\nu < w| \left(\Phi_{\lambda,\nu} exp(\phi_1 h)|w>_{\lambda+1} \otimes exp(\phi_2 h)|w>_{\nu+1}\right) =$
$= \left({}_\lambda < w| \otimes {}_\nu < w|\Phi^*_{\lambda,\nu}\right) exp(\phi_1 h)|w>_{\lambda+1} \otimes exp(\phi_2 h)|w>_{\nu+1}$. Q.E.D.

Theorem 2.15 (Nonlinear equations):

$$\frac{(\partial_\phi + \lambda + 2)}{2(\lambda+1)} W_\lambda(\phi) \frac{\mu_2^R(1-\partial_\phi)}{\nu+1} W_\nu(\phi) + \frac{\mu_1^R}{\lambda+1} W_\lambda(\phi) \frac{(\partial_\phi + \lambda + 2)}{\nu+1}(1+\partial_\phi) W_\nu(\phi) =$$
$$= -W_{\lambda+1}(\phi) \frac{\nu+1-\partial_\phi}{2\mu_1^L} \partial_\phi W_{\nu+1}(\phi) + \frac{\lambda+1-\partial_\phi}{2\mu_1^L} W_{\lambda+1}(\phi) \partial_\phi W_{\nu+1}(\phi). \tag{36}$$

The proof is the standard procedure of deducing nonlinear equations from the bilinear ones: one should substitute $\phi_2 = \phi_1 + \delta$ and consider the Taylor expansion of the $W(\phi_2)$ in δ and necessary relations are obtained as equalities in different powers δ^k, the above one is for $k = 1$. In this way one obtains Kadomtsev-Petviashvili hierarchy from the Hirota bilinear relations.

Theorem 2.16 (Product formula):

$$W_\lambda^{\mu_L,\mu_R}(\phi) W_\nu^{\mu'_L,\mu'_R}(\phi) = \sum_{k=0}^{\infty} C_k W_{\lambda+\nu-k}^{\mu_L+\mu'_L,\mu_R+\mu'_R}(\phi) \sum_{i=0}^{k} \frac{(-1)^i \mu_R^i (\mu'_R)^{(k-i)}}{(k-i)!i!} \times \tag{37}$$
$$\times \binom{n}{k-i} \binom{m}{i} \sum_{i=0}^{k} \frac{(-1)^i \mu_L^i (\mu'_L)^{(k-i)}}{(k-i)!i!} \binom{n}{k-i} \binom{m}{i}$$

To prove the theorem above one should note that LHS in 37 equals to ${}_\lambda < w| \otimes {}_\nu < w|exp(\phi h)|w>_\lambda \otimes |w>_\nu$, on the other hand one can compute this expression by use of isomorphism from proposition 2.9 and obtain RHS of 37. The calculation is very similar to the one in proof of the theorem 2.12 (Baxter relation).

So in this section we demonstrated how to deduce different relations for the Whittaker wave function of Toda by use of intertwiners.

3 $SL(3)$ Toda Chain.

In this section we will show how to obtain raising operators and the bilinear identities for the wave functions of $SL(3)$ Toda. The idea is completely analogous to the case of $SL(2)$: the wave functions are matrix elements of $SL(3)$ in irreducible representations, so considerations of the different intertwiners: $V_\lambda \otimes V_{finite-dim} \to V_{\lambda\pm1}$, $V_{\lambda_1} \otimes V_{\lambda_2} \to V_{\nu_1} \otimes V_{\nu_2}$, etc. gives different relations for the wave functions. The main technical problem here is to find the explicit expression for the image of the Whittaker vector $|w>_\lambda$ through the Whittaker vector $|w>_{\lambda\pm1}$: find $P(f,h)$, such that $\Phi(|w>_\lambda) = P(f,h)|w>_{\lambda+1}$, pay attention that we need expression of the operator P in terms of generators of the $SL(3)$, (expression in operators $x_{i,j}, \partial_{x_{i,j}}$ in Borel-Weil realization one can easily find). In the case of $SL(2)$ such problem was easily solved by the fact that there was an explicit realization of the vector $|w>_\lambda$: $|w>_\lambda = \sum_{i=0} \frac{\mu^n f^n}{n!(\lambda,n)}|vac>_\lambda$, but in case of $SL(3)$ such formula is not reasonable or useable. So one need to find the main formula: $\Phi(|w>_\lambda) = P(f,h)|w>_{\lambda+1}$ using more or less the only information about vector $|w>_\lambda$ that it is eigenvector of the generators e_i. So it is the main difference with the case of $SL(2)$ and we will mostly pay attention to this question and we will be brief in questions analogous to the case of $SL(2)$.

3.1 Notations:

The Lie algebra $SL(3)$ is defined by the generators $e_1, e_2, e_{12}, h_1, h_2, f_1, f_2, f_{12}$. Sometimes we will refer e_{12} as e_3, and f_{12} as f_3. The commuation relations can be found, for example, in the next section where we consider the general $SL(N)$ case.

Quadratic Casimir operator:

$$C_2 = \sum_{i=1}^{3}(e_i f_i + f_i e_i) + \frac{2}{3}\left(\sum_{i=1}^{2} h_i^2 + h_1 h_2\right) = \tag{38}$$

$$= 2\sum_{i=1}^{3} f_i e_i + 2\sum_{i=1}^{2} h_i + \frac{2}{3}\left(\sum_{i=1}^{2} h_i^2 + h_1 h_2\right). \tag{39}$$

Highest weight representation:
We consider the principal (spherical) series of representations, induced by the one-dimensional representations of the Borel subgroup. Let us describe Borel-Weil realization. The space of representation V_λ $\lambda = (\lambda_1, \lambda_1)$ is functions of three real variable x_1, x_2, x_{12}, which are matrix elements of the $3*3$ upper triangular matrices, i.e. coordinates on the biggest Schubert cell of the $SL(3)$ flag variety G/B. Matrix elements are defined by integrals with the flat measure. The action of the algebra is given by the differential operators:

$$e_1 = \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_{12}}, \quad e_{12} = \frac{\partial}{\partial x_{12}},$$

$$h_1 = -2x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_{12}\frac{\partial}{\partial x_{12}} + \lambda_1, \quad h_2 = x_1\frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_2} - x_{12}\frac{\partial}{\partial x_{12}} + \lambda_2,$$

$$f_1 = \lambda_1 x_1 - x_1^2\frac{\partial}{\partial x_1} - x_1 x_{12}\frac{\partial}{\partial x_{12}} - (x_{12} - x_1 x_2)\frac{\partial}{\partial x_2},$$

$$f_2 = \lambda_2 x_2 + x_{12}\frac{\partial}{\partial x_1} - x_2^2\frac{\partial}{\partial x_2},$$

$$f_{12} = \lambda_1 x_{12} - \lambda_2(x_1 x_2 - x_{12}) - x_1 x_{12}\frac{\partial}{\partial x_1} + x_2(x_1 x_2 - x_{12})\frac{\partial}{\partial x_2} - x_{12}^2\frac{\partial}{\partial x_{12}}. \tag{40}$$

Obviously that *Constants* are vacuum (i.e. the highest weight) vectors in such representation, i.e. $e_i(Constant) = 0, \quad h_i(Constant) = \lambda_i(Constant)$.

3.2 Whittaker function.

Definition 3.1: The vector $|w>_\lambda^{\mu_1,\mu_2}$ is called *Whittaker vector* in representation V_λ, if it is eigenvector for generators e_1, e_2 i.e. $e_1|w>_\lambda^{\mu_1,\mu_2} = \mu_1|w>_\lambda^{\mu_1,\mu_2}$ and $e_2|w>_\lambda^{\mu_1,\mu_2} = \mu_2|w>_\lambda^{\mu_1,\mu_2}$ for some constants μ_i. Obviously $e_3|w>_\lambda^{\mu_1,\mu_2} = 0$. We will sometimes omit indexes λ, μ, if they remains unchanged during the calculations.

Lemma 3.2: For given μ_i and the irreducible representation V_λ $(\lambda_i \neq 1, 2, 3, ...)$ there exist a unique (up to scalar) Whittaker vector $|w> \in V_\lambda$, which can be expressed as follows:

$$|w> = exp(\mu_1 x_1 + \mu_2 x_2) \tag{41}$$

The proof is obvious.

Remark: in case of $sl(2)$ we also gave an expression for the Whittaker vector $|w>$ in terms of the generator f, i.e. $|w>=\sum_{i=0}\frac{\mu^n f^n}{n!(\lambda,n)}|vac>$ one can see that such expression in case of $sl(3)$ is rather complicated and unusable.

Lemma 3.3: The dual Whittaker vector: $<w|$ is given by the formula:

$$<w| = (x_{12}-x_1x_2)^{-(\lambda_2+2)}x_{12}^{-(\lambda_1+2)}e^{-\mu_2^L\frac{x_1}{x_{12}}-\mu_1^L\frac{x_2}{x_1x_2-x_{12}}}. \tag{42}$$

see [7].

Definition 3.4: The Whittaker function $W_\lambda^{\mu_L,\mu_R}(\phi_1,\phi_2)$ is the function given by:

$$W_\lambda^{\mu_L,\mu_R}(\phi_1,\phi_2) =< w|exp(\phi_1h_1+\phi_2h_2)|w> \tag{43}$$

We will sometimes omit indexes λ,μ.

Proposition 3.5: The Whittaker function satisfies the equation:

$$\left(\frac{2}{3}\left(\frac{\partial^2}{\partial\phi_1^2}+\frac{\partial^2}{\partial\phi_2^2}+\frac{\partial^2}{\partial\phi_1\partial\phi_2}\right)+\frac{\partial}{\partial\phi_1}+\frac{\partial}{\partial\phi_2}-\mu_1^L\mu_1^Re^{\phi_2-2\phi_1}-\mu_2^L\mu_2^Re^{\phi_1-2\phi_2}\right)W_\lambda(\phi_1,\phi_2)$$
$$=(2\lambda_1+2\lambda_2+\frac{2}{3}(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2))W_\lambda(\phi_1,\phi_2). \tag{44}$$

Remark: the proof is based on idea that this hamiltonian is the second order Casimir operator for the $SL(3)$, and completely analogous to the $SL(2)$ case. The function $W_\lambda(\phi_1,\phi_2)$ is also eigenfunction for the third order hamiltonian, which is the image of the third order Casimir for the $SL(3)$. So we see that Whittaker function is the wave function for the $SL(3)$ quantum Toda chain. These facts are standard so we are very brief.

Proposition 3.6 (Integral representation of the Whittaker function) ([7]) :

$$W_\lambda(\phi_1,\phi_2)=e^{(\lambda_1+1)\phi_1+(\lambda_2+1)\phi_2}\int dx_1dx_2dx_{12}\,(x_{12}-x_1x_2)^{(-\lambda_2-2)}x_{12}^{(-\lambda_1-2)}\times \tag{45}$$
$$\times e^{-\mu_2^L\frac{x_1}{x_{12}}-\mu_1^L\frac{x_2}{x_1x_2-x_{12}}-\mu_1^Rx_1e^{\phi_2-2\phi_1}-\mu_2^Rx_2e^{\phi_1-2\phi_2}}. \tag{46}$$

The theorem is simple corollary of the facts that the invariant pairing $<v|u>$ is realized as integral f with the flat measure, and dual Whittaker vector $<w|$ is realized as $(x_{12}-x_1x_2)^{(-\lambda_2-2)}x_{12}^{(-\lambda_1-2)}exp(-\mu_2^L\frac{x_1}{x_{12}}-\mu_1^L\frac{x_2}{x_1x_2-x_{12}})$.

3.3 Intertwining operators

Denote by $V_{(1,0)}$ the first fundamental representation of the $SL(3,\mathbb{R})$, denote by $|0>$ the highest weight vector, by $|1>$ the vector $f_1|0>$, by $|2>$ the vector $f_2f_1|0>=f_3|0>$.

Proposition 3.7: Let us denote by $\Phi_\lambda=\Phi_{\lambda,(+1,0)}\oplus\Phi_{\lambda,(-1,1)}\oplus\Phi_{\lambda,(0,-1)}$ the ismorphism $V_{\lambda+(1,0)}\oplus V_{\lambda+(-1,1)}\oplus V_{\lambda+(0,-1)}\to V_\lambda\otimes V_1$. The operators $\Phi_{\lambda,(+1,0)}$ and $\Phi_{\lambda,(+1,0)}{}^{-1}$ in Borel-Weil realization are given by the formulas:

$$\Phi_{\lambda,(+1,0)}: x_1^{n_1}x_2^{n_2}x_3^{n_3}|0>_\lambda\otimes|0>\to x_1^{n_1}x_2^{n_2}x_3^{n_3}|0>_{\lambda+(1,0)}, \tag{47}$$
$$\Phi_{\lambda,(+1,0)}: x_1^{n_1}x_2^{n_2}x_3^{n_3}|0>_\lambda\otimes|1>\to x_1^{n_1+1}x_2^{n_2}x_3^{n_3}|0>_{\lambda+(1,0)}, \tag{48}$$
$$\Phi_{\lambda,(+1,0)}: x_1^{n_1}x_2^{n_2}x_3^{n_3}|0>_\lambda\otimes|2>\to x_1^{n_1}x_2^{n_2}x_3^{n_3+1}|0>_{\lambda+(1,0)}. \tag{49}$$

$$(\Phi_{\lambda,(+1,0)})^{-1} : x_1^{n_1}x_2^{n_2}x_3^{n_3}|0>_{\lambda+(1,0)} \to \frac{1}{(\lambda_1+1)(\lambda_1+\lambda_2+2)}$$
$$((n_1n_2x_3 + ((\lambda_1+2)n_3 + n_2n_3)x_1x_2)x_1^{n_1-1}x_2^{n_2-1}x_3^{n_3-1}|0>_\lambda \otimes |2> +$$
$$(n_1(\lambda_1+\lambda_2+2-n_2)x_3 + n_3(\lambda_2-n_2)n_3x_2)x_1^{n_1-1}x_2^{n_2}x_3^{n_3-1})|0>_\lambda \otimes |1> +$$
$$(-n_1n_2x_3^2 + n_3(-\lambda_2+n_2)x_1^2x_2^2 + +((\lambda_1+1)(\lambda_1+\lambda_2+2) - n_1(\lambda_1+\lambda_2+2) +$$
$$+n_1n_2 - n_3(\lambda_1+2) - n_2n_3)x_1x_2x_3)x_1^{n_1-1}x_2n_2 - 1x_3^{n_3-1}|0>_\lambda \otimes |0>). \quad (50)$$

This proposition is auxillary for us. We use it to fix the constants in the formulas from the next proposition, which is really crucial for us.

To prove the formula 49 one should only note that the representation $V_{(1,0)}$ also have Borel-Weil realization and the above intertwiner is given by the multiplication of functions from the two representations. The proof of the formula 50 is more complicated, but it is standard representation theory reasoning, and should be available in the literature, but we were not able to find it. So we sketch the proof. Let us denote the three components of the operator $(\Phi_{\lambda,(+1,0)})^{-1}$ by Ψ_0, Ψ_1, Ψ_2, i.e. $(\Phi_{\lambda,(+1,0)})^{-1}|v>= \Psi_0(|v>)\otimes|0> + \Psi_1(|v>)\otimes|1> + \Psi_2(|v>) \otimes |2>$, first simple, but useful step in proving 50 is that all the three operators Ψ_i can be easily expressed one throw another, so it's enough to find only one of them. Actually it is easy to see that, for example: $\Psi_1 = [\Psi_2, f_2]$, $\Psi_0 = [\Psi_2, f_3]$. Looking at the formula 50 it is clear that operator Ψ_2 is much simpler than the other two, so it is rather useful that we can express them through the operator Ψ_2. Second step is to find Ψ_2. It goes as follows, first let us note that: $\Psi_2(x_1^{k_1}x_2^{k_2}x_3^{k_3}) = \alpha_{k_1,k_2,k_3}x_1^{k_1-1}x_2^{k_2-1}x_3^{k_3} + \beta_{k_1,k_2,k_3}x_1^{k_1}x_2^{k_2}x_3^{k_3-1}$. That easily follows from the equalities: $\Psi_2h_1 = h_1\Psi_2, \Psi_2h_2 = h_1\Psi_2$ and $e_1^{k_1+1}\Psi_2(x_1^{k_1}x_2^{k_2}x_3^{k_3}) = 0, e_3^{k_1+1}\Psi_2(x_1^{k_1}x_2^{k_2}x_3^{k_3}) = 0$. the second point is to use the commutation relation of e_2 and Ψ_2 and to find recurrence relations for $\alpha_{k_1,k_2,k_3}, \beta_{k_1,k_2,k_3}$, they can be solved directly after some work. Hence we find Ψ_2 and as explained above it follows that we find Φ_1, Φ_0.

Remark: Let us note that it is rather easy to find the formulas for the intertwiners in Verma realization, not Borel-Weil, but the problem is that expression of the vector $x_1^{k_1}x_2^{k_2}x_3^{k_3}$ through the vectors $f_1^{n_1}f_2^{n_2}f_3^{n_3}$ is rather complicated and untreatable.

Proposition 3.8: The action of the intertwiner $(\Phi_{\lambda,(+1,0)})$ on the Whittaker vector can be expressed as follows:

$$(\Phi_{\lambda,(+1,0)})|w>_\lambda^{\mu_1,\mu_2} \otimes |0> \to |w>_{\lambda+(1,0)}^{\mu_1,\mu_2} = exp(\mu_1x_1 + \mu_2x_2)|0>_{\lambda+(1,0)}, \quad (51)$$
$$(\Phi_{\lambda,(+1,0)})|w>_\lambda^{\mu_1,\mu_2} \otimes |1> \to \bar{\Lambda}_1|w>_{\lambda+(1,0)}^{\mu_1,\mu_2} = x_1exp(\mu_1x_1 + \mu_2x_2)|0>_{\lambda+(1,0)}, \quad (52)$$

$$(\Phi_{\lambda,(+1,0)})|w>_\lambda^{\mu_1,\mu_2} \otimes |2> \to \frac{1}{\mu_1}(f_2 + \bar{\Lambda}_2^2\mu_2 - \lambda_2\bar{\Lambda}_2)|w>_{\lambda+(1,0)}^{\mu_1,\mu_2} = x_{12}exp(\mu_1x_1 + \mu_2x_2).$$

where $\bar{\Lambda}_1 = \frac{2(\lambda_1+1)+\lambda_2-2h_1-h_2}{3\mu_1}$ and $\bar{\Lambda}_2 = \frac{(\lambda_1+1)+2\lambda_2-h_1-2h_2}{3\mu_2}$.

The action of the intertwiner $(\Phi_{\lambda,(+1,0)})^{-1}$ on the Whittaker vector can be expressed as follows:

$$(\Phi_{\lambda,(+1,0)})^{-1}|w>_{\lambda+1}^{\mu_1,\mu_2} \to \frac{1}{(\lambda_1+1)(\lambda_1+\lambda_2+2)}(\mu_1\mu_2|w>_\lambda^{\mu_1,\mu_2} \otimes |2> +$$
$$\frac{1}{\mu_2}(\tilde{\Lambda}_2+2)|w>_\lambda^{\mu_1,\mu_2} \otimes |1> + \frac{1}{\mu_2}(f_1 + \frac{1}{\mu_2}(\tilde{\Lambda}_1^2 - (\lambda_2-3)\tilde{\Lambda}_1))|w>_\lambda^{\mu_1,\mu_2} \otimes |0>). \quad (53)$$

where $\tilde{\Lambda}_1 = \frac{2h_1+h_2+\lambda_1+2\lambda_2}{3}$ and $\tilde{\Lambda}_2 = \frac{2h_2+h_1+\lambda_2+2\lambda_1}{3}$

This proposition is crucial in this section. The relations for the Whittaker wave functions (54,55) easily follows from it. Let us sketch the proof and explain the difference with $SL(2)$ case, which is much more simple. We discuss the analogous reasonings with more details in the next section in the case of $SL(n)$. Let us denote by P_0, P_1, P_2 the operators such that: $(\Phi_{\lambda,(+1,0)})^{-1}|w>_{\lambda+1} = P_0|w>_\lambda \otimes|0> + P_1|w>_\lambda \otimes|1> + P_2|w>_\lambda \otimes|2>$. We will look for them as some polynomials $P_i(h_i, f_i)$ from the generators e_i. From the commutation relations with the operators e_i and uniqueness of the Whittaker vector it's easy to see that the operator P_2 is some constant depending only on μ_i. To fix this constant we look on the formulas 50, 41 and see that $P_2 = \mu_1\mu_2$. Also from the commutation relations with operators e_i follows the following relations: $[P_1, e_1] = 0$, $[P_1, e_2] = e_2 + Constant$ so A_1 equals $\frac{1}{\mu_2}\Lambda_2 + C_1$, where Λ_i are fundamental coroots, i.e. elements of Cartan subalgebra such that $[\Lambda_i, e_j] = \delta_{i,j}$, the constant C_1 can also be fixed by the formulas 50,41. Analogously A_0 satisfy the identities: $[P_0, e_2] = 0$, $[P_0, e_1] = \frac{\alpha}{\mu_1}A_1e_1 + \beta P_1$, where $\alpha + \beta = 1$. the solution of this equation is given by the ansatz $P_0 = \frac{1}{\mu_2}(f_1 + \frac{1}{\mu_1}(\Lambda_1^2 + C_0\Lambda_1))$ the constants c_0, α, β also can be fixed by the formulas 50,41. So the necessary formulas for P_i are obtained.

Let us one more time emphasize the difference with the $SL(2)$ case, where we had the formula for the Whittaker vector $|w> = \sum_{i=0} \frac{\mu^n f^n}{n!(\lambda,n)}|vac>$. Hence the proposition 2.8, which is analogous to the proposition 3.8 above immediately follows from the formula for the action of the intertwiner on the vector $f^n|vac>$. In the present case the formula for $|w>$ in terms of the generators f_i is too complicated to extract something from it, so the proof of the proposition above was based mostly on the property of the Whittaker vector: $e_i|w> = \mu_i|w>$.

3.4 Relations for the Whittaker wave functions.

Theorem (Raising operators) 3.9: The following relations holds:

$$W_{\lambda+1}(\phi_1,\phi_2) = \frac{exp(\phi_1)}{\mu_1^R(\lambda_1+1)(\lambda_1+\lambda_2+2)}(\mu_1^R exp(-2\phi_1+\phi_2) + \frac{1}{\mu_2^R}((\frac{2\partial_{\phi_1}++\partial_{\phi_2}+\lambda_1+2\lambda_2}{3})^2 -(\lambda_2-3)\frac{2\partial_{\phi_1}+\partial_{\phi_2}+\lambda_1+2\lambda_2}{3} - 2(\lambda_2-1)))W_\lambda(\phi_1,\phi_2). \tag{54}$$

This proposition follows from the 51,53 in a way completely analogous to the $SL(2)$ case (proposition 2.11), so we omit the details.

Theorem (Bilinear relations) 3.10:

$$((\frac{1}{\mu_2'^L}(\mu_2'^R exp(-\phi_2+2\phi_1) + ((\left(\frac{-\partial_{\phi_2'} - 2\partial_{\phi_1'} + (\nu_2+1) + 2\nu_1}{3\mu_1'^L}\right)^2 -(\nu_1)\frac{-\partial_{\phi_2'} - 2\partial_{\phi_1'} + (\nu_2+1) + 2\nu_1}{3\mu_1'^L}) -$$

$$
\begin{aligned}
&-\frac{-2\partial_{\phi_1}-\partial_{\phi_2}+2(\lambda_1+1)+\lambda_2}{3\mu_1^L}\frac{-2\partial_{\phi_2'}-\partial_{\phi_1'}+2(\nu_2+1)+\nu_1}{3\mu_2'^L}\\
&+(\frac{1}{\mu_1^L}(\mu_1^R exp(-\phi_1+2\phi_2)+(\left(\frac{-\partial_{\phi_1}-2\partial_{\phi_2}+(\nu_1+1)+2\nu_2}{3\mu_2^L}\right)^2\\
&-(\nu_2)\frac{-\partial_{\phi_1}-2\partial_{\phi_2}+(\nu_1+1)+2\nu_2}{3\mu_2^L}))W_{\lambda+1}^{\mu_1^L,\mu_2^L,\mu_1^R,\mu_2^R}(\phi_1,\phi_2)W_{\nu+1}^{\mu_1'^L,\mu_2'^L,\mu_1'^R,\mu_2'^R}(\phi_1',\phi_2')=\\
&\frac{1}{(\lambda_1+1)(\lambda_1+\lambda_2+2)}\frac{1}{(\nu_2+1)(\nu_1+\nu_2+2)}\\
&((\frac{1}{\mu_1^R}(\mu_1^L exp(-2\phi_1+\phi_2)+\frac{1}{\mu_2^R}(\left(\frac{2\partial_{\phi_1}+\partial_{\phi_2}+\lambda_1+2\lambda_2}{3}\right)^2\\
&-(\lambda_2-3)\frac{2\partial_{\phi_1}+\partial_{\phi_2}+\lambda_1+2\lambda_2}{3}-2(\lambda_2-1))))\mu_1'^R\mu_2'^R+\\
&\frac{(\partial_{\phi_1}+2\partial_{\phi_2}+2\lambda_1+\lambda_2)}{3\mu_2}\frac{(\partial_{\phi_1'}+2\partial_{\phi_1}+2\nu_1+\nu_1)}{3\mu_1}-\\
&-\mu_1\mu_2(\frac{1}{\mu_2'^R}(\mu_2'^L exp(-2\phi_2'+\phi_1')+\frac{1}{\mu_1'^R}(\left(\frac{2\partial_{\phi_2'}+\partial_{\phi_1'}+\nu_2+2\nu_1}{3}\right)^2\\
&-(\nu_1-3)\frac{2\partial_{\phi_2'}+\partial_{\phi_1'}+\nu_2+2\nu_1}{3}-2(\nu_1-1))))W_{\lambda}^{\mu_1^L,\mu_2^L,\mu_1^R,\mu_2^R}(\phi_1,\phi_2)W_{\nu}^{\mu_1'^L,\mu_2'^L,\mu_1'^R,\mu_2'^R}(\phi_1',\phi_2').
\end{aligned}
$$

The proof is also completely analogous to the $SL(2)$ case, so we refer the reader to the previous section.

One can obtain the nonlinear equations from this bilinear one, as it was done in the $SL(2)$ case, but we omit these considerations due to their length.

4 $SL(n)$ Toda Chain.

4.1 Notations.

Algebra $sl(n)$ is completely given by the generators $e_i, h_i, f_i, i=1,...,n-1$ and commutation relations:

$$[e_i,h_j]=-A_{ij}e_{\pm,i},\quad [f_i,h_j]=A_{ij}e_{\pm,i},\quad [e_i,f_j]=\delta_{ij}h_j,\quad i,j=1,\ldots,N-1, \tag{55}$$

and the Serre relations

$$\mathrm{ad}_{e_i}^{1-A_{ij}}(e_j)=0,\qquad \mathrm{ad}_{f_i}^{1-A_{ij}}(f_j)=0, \tag{56}$$

where $\mathrm{ad}_x^k(y)\equiv\underbrace{[x,[x,...,[x,y]..]]}_{k\ \text{times}}$.

Where A_{ij} is Cartan matrix, which is for the algebra $sl(N)$ is equal to

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Quadratic Casimir operator is

$$C_2 = \sum_{\alpha\in\Delta} e_\alpha f_\alpha + f_\alpha e_\alpha + \sum_{ij}^{N-1} A_{ij}^{-1} h_i h_j, \tag{57}$$

where the first sum goes over all positive roots.

We consider the standard Borel-Weil realization of highest weight repsentations on the flag manyfold $B\backslash G$, the biggest cell of which, in turn, may be identified with the strictly upper-triangular matrices N_+. Hence we obtain realization of the representation of the $SL(n)$ in the space of functions of $x_{i,j}$, $1 \le i < j \le n$, where $x_{i,j}$ are the matrix elements of $n * n$ upper-triangular matrices N_+. The explicit formulas can be found for example in [24, 25].

4.2 Whittaker function.

Definition 4.1: The vector $|w>_\lambda^{\mu_1,\dots,\mu_{n-1}}$ is called *Whittaker vector* in representation V_λ, if it is eigenvector for generators $e_1,\dots,e_{n-1}$ i.e. $e_i|w>_\lambda = \mu_i|w>_\lambda$ for some constants μ_i. Obviously $[e_i,e_j]|w>_\lambda^{\mu_1,\mu_2} = 0$. We will sometimes omit indexes λ,μ, if they remains unchanged during the calculations.

Lemma 4.2: For given μ_i and irreducible representation V_λ there exist a unique (up to scalar) Whittaker vector $|w>\in V_\lambda$, which can be expressed as follows:

$$|w> = exp(\mu_1 x_{1,2} + \mu_2 x_{2,3} + \dots + \mu_{n-1} x_{n-1,n}). \tag{58}$$

The proof is obvious due to explicit formulas for generators e_i in Borel-Weil realization. Generators e_i acts on such vector as $\partial_{x_{i,i+1}}$, see [7] for details.

Definition 4.3: The Whittaker function $W_\lambda^{\mu_i^L,\mu_j^R}(\phi_1,\phi_2,\dots,\phi_{n-1})$ is the function given by:

$$W_\lambda^{\mu_i^L,\mu_j^R}(\phi_1,\phi_2) = < w|exp(\phi_1 h_1 + \phi_2 h_2 + \dots + \phi_{n-1} h_{n-1})|w> \tag{59}$$

We will sometimes omit indexes λ,μ.

Proposition 4.4: The Whittaker function satisfies the equation:

$$\left(\sum_{ij} A_{ij}^{-1}\partial_{\phi_i}\partial_{\phi_j} + 2\sum_{ij} A_{ij}^{-1}\partial_{\phi_j} - 2\sum_i \mu_i^L\mu_i^R exp(\sum_j A_{i,j}\phi_j)\right) W_\lambda^{\mu_i^L,\mu_j^R}(\phi_i) =$$
$$= (\lambda^2 - \rho^2) W_\lambda^{\mu_i^L,\mu_j^R}(\phi_i). \tag{60}$$

where ρ is one half of sum of positive roots.

Remark: the proof is based on the idea, that this hamiltonian is the second order Casimir operator for the $SL(n)$, and completely analogous to the $SL(2)$ case. The function $W_\lambda(\phi_i)$ is also eigenfunction for the higher order hamiltonians, which are the images of the higher order Casimirs for the $SL(n)$. Hence, we see that Toda is completely integrable system and the Whittaker function is the wave function for all Hamiltonians of the $SL(n)$ quantum Toda chain. These facts are standard so we are very brief (see [7]).

Proposition 4.5 (Integral representation of the Whittaker function) ([7]) :

$$W_\lambda(\phi_i) = e^{-\sum_i \lambda_i \phi_i} \int_{X=B\backslash G} \prod_{i<j} dx_{ij} \prod_{i=1}^{N-1} \Delta_i^{-(\sum_k A_{k,i}\lambda_k+1)}(xS^{-1}) \times$$
$$\times e^{\mu_i^R x_{i,i+1} e^{(\sum_k A_{k,i}\lambda_k)} - \mu_{N-i}^L \frac{\Delta_{i,i+1}(xS^{-1})}{\Delta_i(xS^{-1})}}. \tag{61}$$

where Δ_i - is i-th principal minor of matrix $x_{i,j}$, $\Delta_{i,i+1}$ - determinant $(n-1)*(n-1)$ submatrix, which obtained by interchanging $n-1$ and n column. Matrix S is given by $S_{i,j} = \delta_{i+j,n}$.

The theorem is corollary of the facts that the invariant pairing $< v|u >$ is realized as integral with the flat measure, and the realizations of the Whittaker and the dual Whittaker vectors. The most nontrivial part is to find dual Whittaker vector. We refer to [7] for detailed exposition.

4.3 Intertwining operators

Denote by $V_{(1,0,\ldots,0)}$ the first fundamental representation of the $SL(n,\mathbb{R})$, which is just the standard action of the $n*n$ matrices on $\mathbb{C}^n$, denote by $|0>$ the highest weight vector, by $|1>$ the vector $f_1|0>$, by $|2>$ the vector $f_2f_1|0>= f_3|0>$, etc.

To construct relations for the wave functions, one needs to express the action of the intertwiner on Whittaker vector, in terms of the generators f_i, h_i. For example the crucial formula for us in the case of $SL(3)$ was:

$$|w>_{\lambda+1}^{\mu_1,\mu_2} \to \frac{1}{(\lambda_1+1)(\lambda_1+\lambda_2+2)}(\mu_1\mu_2|w>_\lambda^{\mu_1,\mu_2} \otimes |2> + \tag{62}$$

$$\frac{1}{\mu_2}(\tilde{\Lambda}_2+2)|w>_\lambda^{\mu_1,\mu_2} \otimes |1> + \frac{1}{\mu_2}(f_1 + \frac{1}{\mu_2}(\tilde{\Lambda}_1^2 - (\lambda_2-3)\tilde{\Lambda}_1))|w>_\lambda^{\mu_1,\mu_2} \otimes |0>). \tag{63}$$

Where $\tilde{\Lambda}_i$ were fundamental coweights minus some constants.

We need analogous formulas for the case of $SL(n)$. We are unable to find them. We present only the equations for the components of the desired intertwiners and first steps towards their solution. As a corollary we obtain higher degree terms of the raising operator for the $SL(4)$.

Let us consider the most simple intertwiner $\Phi : V_{\lambda+(1,0,\ldots,0)} \to V_\lambda \otimes V_{(1,0,\ldots,0)}$. We need to find such polynomials $P_j(f_i,h_i)$, $0 \le j < n$ that:

$$\Phi|w>_{\lambda+(1,0,\ldots,0)} = \sum_{j=0}^{n-1} P_j(f_i,h_i)|w>_\lambda \otimes |j> \tag{64}$$

Proposition 4.6 The following relations holds :

$$e_k P_j(f_i, h_i)|w>_\lambda + \delta_{j,k-1} P_k(f,h)|w>_\lambda = \mu_k P_j(f,h)|w>_\lambda \tag{65}$$

The proof is obvious.

Corollary 4.7: One can look for $P_j(f,h)$ as an elements of the $U(sl(n))$, satisfying the following relations:

$$[e_k, P_j(f,h)] = H_{k,j}(f,h)(\frac{e_k}{\mu_k} - 1) \text{ for } k \neq j+2 \tag{66}$$

$$[e_k, P_{k-2}(f,h)] = \alpha_k P_{k-1}(f,h)\frac{e_k}{\mu_k} + \beta_k P_{k-1}(f,h), \text{ where } \alpha_k + \beta_k = 1. \tag{67}$$

where $H_{k,j}(f,h)$ are some polynomials.

This allows one to organize itterative search of polynomials P_j: if P_j already found then P_{j-1} is defined by conditions: $[e_k, P_{j-1}(f,h)] = H_{k,j-1}(f,h)(\frac{e_k}{\mu_k} - 1)$ for $k \neq j+1$ and $[e_{j+1}, P_{j-1}(f,h)] = \alpha_k P_{k-1}(f,h)e_k + \beta_k P_{k-1}(f,h)$. Despite some unknown polynomial $H_{k,j-1}(f,h)$ and parameters α, β enters the equations, they are restrictive enough, and even overdetermined in general, to define the polynomial P_{j-1}.

We can conclude from the considered examples that the highest degree part of the P_{j-1} can be defined uniquely from this relations. The problem is that we do not know the way of solving such recurrence relations. Let us present the first 4 polynomials P_j found by hands.

Proposition 4.8: Polynomials $P_{n-1}, P_{n-2}, P_{n-3}, P_{n-4}$ are given by the formulas:

$$P_{n-1} = \mu_1\mu_2...\mu_{n-1}, \quad P_{n-2} = (\Lambda_{n-1} + Constant)\mu_1\mu_2...\mu_{n-2}, \tag{68}$$

$$\begin{aligned} P_{n-3} &= (\mu_{n-2}f_{n-2} + \mu_{n-3}f_{n-3} + ... + \mu_1 f_1 + \\ &+ (\Lambda_{n-2}^2 - \Lambda_{n-2}\Lambda_{n-3}) + (\Lambda_{n-3}^2 - \Lambda_{n-3}\Lambda_{n-4}) + ... + (\Lambda_1)^2)\mu_1\mu_2...\mu_{n-3} + F_1(\Lambda_i), \end{aligned} \tag{69}$$

$$\begin{aligned} P_{n-4} &= f_{n-3}(\Lambda_{n-2} + \Lambda_{n-3} - \Lambda_{n-4}) + f_{n-4}(\Lambda_{n-3}\frac{2\mu_{n-4}}{\mu_{n-3}} + \Lambda_{n-5}\frac{-\mu_{n-4}}{\mu_{n-3}}) + \\ &+ f_{n-5}(\Lambda_{n-3}\frac{\mu_{n-5}}{\mu_{n-3}} + \Lambda_{n-4}\frac{\mu_{n-5}}{\mu_{n-3}} - \Lambda_{n-6}\frac{\mu_{n-5}}{\mu_{n-3}}) + f_{n-6}(\Lambda_{n-3}\frac{\mu_{n-6}}{\mu_{n-3}} + \Lambda_{n-5}\frac{\mu_{n-6}}{\mu_{n-3}} - \\ &- \Lambda_{n-7}\frac{\mu_{n-6}}{\mu_{n-3}}) + ... + f_1(\Lambda_{n-3}\frac{\mu_1}{\mu_{n-3}} + \Lambda_2\frac{\mu_1}{\mu_n - 3} + \\ &+ [f_{n-2}, f_{n-3}]\mu_{n-2} + [f_{n-3}, f_{n-4}]\mu_{n-4} + [f_{n-4}, f_{n-5}]\frac{\mu_{n-5}\mu_{n-4}}{\mu_{n-3}} + ... + [f_2, f_1]\frac{\mu_1\mu_2}{\mu_{n-3}} + \\ &+ \frac{1}{\mu_{n-3}}(\Lambda_{n-3}^3 - 2\Lambda_{n-3}^2\Lambda_{n-4} + 2\Lambda_{n-3}\Lambda_{n-4}^2) + \\ &+ \frac{\Lambda_{n-3}}{\mu_{n-3}}(-\Lambda_{n-4}\Lambda_{n-5} + (\Lambda_{n-5}^2 - \Lambda_{n-5}\Lambda_{n-6}) + (\Lambda_{n-6}^2 - \Lambda_{n-6}\Lambda_{n-7}) + ... + \\ &+ (\Lambda_2^2 - \Lambda_2\Lambda_1) + \Lambda_1^2) + + \frac{1}{\mu_{n-3}}(\Lambda_{n-4}\Lambda_{n-5}^2 - \Lambda_{n-4}^2\Lambda_{n-5}) + \\ &+ \frac{1}{\mu_{n-3}}(\Lambda_{n-5}\Lambda_{n-6}^2 - \Lambda_{n-5}^2\Lambda_{n-6}) + ... + \frac{1}{\mu_{n-3}}(\Lambda_2\Lambda_1^2 - \Lambda_2^2\Lambda_1) + F_2(\Lambda, f). \end{aligned} \tag{70}$$

where Λ_i - fundamental coweights, F_i are some polynomials of degree i.

The proof is straightforward check that given polynomials satisfy relations (66) and one can see that they are the only with this property. Let us note that the system for the P_{n-4} is overdetermined, but nevertheless it has solutions.

So we see polynomials P_{n-1-j} are of degree j. It's easy to guess that it also holds true for $j > 3$. It's easy to understand that to write the simplest raising relation one should find P_0, which should be some polynomial of degree $n-1$. Our first hope to do that was to find some recurrence formula: $P_{j-1} = A_j P_j$, but we did not succeed, looking at 69 it seems that such formula may not exist. Another idea is that may be P_j are somehow related to the Casimirs of subalgebras $SL(n-j)$, which one can guess looking at 69 and at commutation relation: $[e_k, P_j] = H_{k,j}(\frac{e_k}{\mu_k} - 1)$, which roughly means that P_j and e_k almost commute for $k \neq j+2$.

Let us also note that it is possible to write intertwiner $V_{\lambda+(1,0,\dots,0)} \to V_\lambda \otimes V_{(1,0,\dots,0)}$ in Borel-Weil realization so it is possible to find the expression for P_j as some polynomial of operators of multiplication on monomials $x_{i,j}$ but the problem is then express such operators via the polynomial of the f_i, h_i.

As a corollary of the formula 70 we present the highest degree terms of the raising operator for the $SL(4)$.

Corollary 4.9: Highest degree terms of the raising operator for the Whittaker functions of the $SL(4)$ are given by the formula:

$$W_{\lambda+(1,0,0)}(\phi_1,\phi_2,\phi_3) = exp(\phi_1)(\frac{1}{\mu_1}(\frac{3\partial_{\phi_1} + 2\partial_{\phi_2} + \partial_{\phi_3}}{4})^4 +$$
$$+\mu_1^R exp(-2\phi_1 + \phi_2)(\frac{5\partial_{\phi_1} + 6\partial_{\phi_2} + 3\partial_{\phi_3}}{4}) - \mu_2^L \mu_2^R \mu_1^R exp(-\phi_1 - \phi_2)))W_\lambda(\phi_1,\phi_2,\phi_3) +$$
$$+(F_2(\phi,\partial_\phi))W_\lambda(\phi_1,\phi_2,\phi_3). \tag{71}$$

where $F_2(\phi,\partial_\phi)$ is polynomial which contains lower degree terms in ϕ, ∂_ϕ.

The proof is the same as in the $SL(2)$ and $SL(3)$ cases.

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